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THE STRENGTH OF MATERIALS

A TEXT-BOOK FOR ENGINEERS AND ARCHITECTS

BY

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WITH NUMEROUS ILLUSTRATIONS, TABLES
AND WORKED EXAMPLES

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PREFACE

The advance in the application of scientific methods to architectural and engineering problems has made increasing demands upon the theoretical knowledge required by architects and engineers; it is the aim of the present book to present in as simple a method as is consistent with accuracy, the principles which underlie the design of machines and structures from the standpoint of their strength.

The subjects commonly called respectively the Strength of Materials and the Theory of Structures have much in common; much of the subject matter contained in the author's books upon the latter subject has, therefore, been incorporated, the same general method involving the use and application of graphical methods in preference to purely mathematical methods having been adopted in the other branches of the subject. An attempt has been made to present more clearly than is general the various theories as to the cause of failure in materials and the effect of these theories upon design.

Although the author hopes that the book will be especially useful for students reading for the Assoc. M. Inst. C. E., and University degree examinations in Engineering, he has attempted to present the subject in sufficiently practical form for it to be of greater assistance in practical design than is the case with an ordinary class book; with this in view many diagrams and tables have been incorporated for enabling the formulæ to be applied with a minimum of time and trouble.

A large number of numerical examples are worked out and further exercises are given; the student is recommended to work for himself all such examples and to pay particular attention to the assumptions which are made in deriving the various formulæ. Nearly all engineering formulæ are only approximately correct; in the present branch of the subject this is chiefly because there is no material known which conforms exactly to the simple laws of elasticity upon which the subject is based. We cannot condemn too strongly the blind application to a particular practical problem of formulæ which were never intended to be so applied; the unfortunate distrust which practical engineers so often have to "theory" is to some extent brought about by the fact that the theories that they see employed are often inapplicable. It is essential for us to acknowledge the limits of theoretical methods and not to attempt to express our results to a greater degree of accuracy than the nature of the problem will allow.

The author's thanks are due to Mr. J. H. Wardley, A.M.I.C.E., for much assistance and valuable criticism in the reading of the proofs; to Mr. W. Mason, D.Sc., for the photograph from which Fig. 24 was made; and to the various firms who have courteously assisted by supplying illustrations and descriptions of the various testing machines and apparatus with which their names are associated in the text. The author's indebtedness should also be recorded to the many text-books and periodicals that have been consulted and are referred to in the various portions of the book.

The author will be grateful for the notification of clerical and other errors that may be found in the book.

London,
May, 1915.

EWART S. ANDREWS

PREFACE TO THE SECOND EDITION

As the text-book in its present form appears to fulfil the needs of the engineers, architects and students for whom it was intended, the author has not thought it necessary to alter the treatment in the book, but has taken the opportunity of correcting printers' errors and of including the properties of the New British Standard Sections.

EWART S. ANDREWS.

329 *High Holborn,*
London, W.C. 1,
March, 1925.

CONTENTS

CHAPTER I

PAGE
STRESS, STRAIN AND ELASTICITY 1

Kinds of stresses and strains—Hooke's Law—Poisson's ratio—Stress-strain diagrams—Elastic moduli and the relations between them—Principal stresses—Ellipse of stress—Maximum strain due to combined strains—Resilience—Impact and temperature stresses—Stresses in heterogeneous bars.

CHAPTER II

THE BEHAVIOUR OF VARIOUS MATERIALS UNDER TEST . 41

Properties other than elastic—The cause of failure of material—Cast iron—Steel, wrought iron and other ductile metals—Real and apparent tensile strength—Lüder's Lines—Overstrain—Timber—Stone, concrete, cement and like brittle materials—Effect of relative breadth and height of compression specimens—Tensile and shear strength of concrete—Adhesion between concrete and steel.

CHAPTER III

REPETITION OF STRESSES; WORKING STRESSES . 84

Wöhler's experiments—Natural elastic limit—Effect of speed upon results—Fatigue of metals—Sudden change of section—The conflict between theory and practice—Working stresses and factor of safety—Allowance for live loads.

CHAPTER IV

RIVETED JOINTS; THIN PIPES 102

Forms of rivet heads and joints—Methods of failure—Efficiency of joint—Practical considerations—Thin pipes and cylinders—The collapse of thin pipes under external pressure.

CHAPTER V

BENDING MOMENTS AND SHEARING FORCES ON BEAMS . 121

Cantilevers and simply supported beams; standard cases—Graphical construction—Relation between load, shear and B.M. diagrams—A template for B.M. diagrams—B.M. and shear diagrams for inclined loads.

CHAPTER VI

PAGE
GEOMETRICAL PROPERTIES OF SECTIONS 161

The determination of areas—Sum curves—Centroid—Moment of inertia and radius of gyration—Momental ellipse or ellipse of inertia—Polar moment of inertia—Graphical constructions—Formulæ for standard cases.

CHAPTER VII

STRESSES IN BEAMS 192

Neutral axis—Assumptions in ordinary beam theory—Moment of resistance—Unit section modulus; beam factor—Discrepancies between theoretical and actual strengths of beams—Diagonal square sections—Influence of shearing force on stresses—Moment of resistance in general case.

CHAPTER VIII

STRESSES IN BEAMS (*continued*) 215

Reinforced concrete beams—Straight line no-tension method—General no-tension method—T beams—Combined bending and direct stresses—Beams with oblique loading.

CHAPTER IX

DEFLECTIONS OF BEAMS 248

General relation in bending—Curvature—Mohr's Theorem and graphical investigation of general and standard cases—Mathematical investigation of standard cases—Resilience of bending.

CHAPTER X

COLUMNS, STANCHIONS AND STRUTS 279

Buckling factor and slenderness ratio—Methods of fixing ends, Euler's, Rankine, Straight line, Johnson, Gordon, Fidler and Lilly formulæ—Reinforced concrete columns—Braced columns—Eccentric loading of columns.

CHAPTER XI

TORSION AND TWISTING OF SHAFTS 311

Stresses in a shaft coupling—General case of grouped bolts or rivets—Circular shafts; horse-power transmitted; solid and hollow shafts—Combined bending and torsion—Shafts with axial pull or thrust—Torsional resilience—Torsion of non-circular shafts—Effect of keyways in shafts.

CHAPTER XII

PAGE
SPRINGS 336

Time of vibration—Close-coiled helical springs—Safe loads on springs—Open-coiled helical springs—Leaf or plate springs—Piston rings—Plane spiral springs—Close-coiled helical springs under bending stress.

CHAPTER XIII

THE TESTING OF MATERIALS 364

Testing machines of various types — Calibration of testing machines—Grips and forms of test piece—Extensometers—Autographic recorders — Torsion testing machines — Torsion meters — Repetition stress machines.

CHAPTER XIV

THE TESTING OF MATERIALS (*continued*) . . . 396

Impact, ductility and hardness testing—Arnold; repeated impact and Izod's impact machines—Brinell hardness test—Testing cement and concrete—Tension tests; compression tests; special gravity, fineness, soundness and setting tests for Portland cement—Thermal and optical methods of testing materials.

CHAPTER XV

FIXED AND CONTINUOUS BEAMS 416

Effect of fixing and continuity—Fixed beams with uniform, central and uniformly increasing loads—Fixed beams with asymmetrical loading—Continuous beams of two equal spans—Effect of lowering central support—Theorem of Three Moments—Table of results for equal spans and uniform loads—Graphical treatment of continuous beams—Beams fixed at one end and supported at the other.

CHAPTER XVI

DISTRIBUTION OF SHEARING STRESSES IN BEAMS . . 470

Horizontal shear—Distribution of shear in standard cases—Graphical treatment—Deflection due to shear-strain of cross section of a beam due to shear.

CHAPTER XVII

FLAT PLATES AND SLABS 488

Slab coefficients—Bach's theory for circular slabs—Grashof's results—Grashof-Rankine theory for square and rectangular slabs—Bach's theory for square and rectangular slabs with various modifications.

CHAPTER XVIII

PAGE
THICK PIPES 510

Lamé's Theory; variation of hoop and radial stress; maximum shear stress; maximum stress equivalent to strain; diagram for designing thick tubes—Shrinkage stresses in a compound tube, and strengthening by initial compression.

CHAPTER XIX

CURVED BEAMS 529

General analysis of stresses—Winkler's formula; graphical treatment; Résal construction, special cases of rectangular and circular section; correction coefficients—Andrews-Pearson formula—Rings and chain links.

CHAPTER XX

ROTATING DRUMS, DISKS AND SHAFTS 552

Thin rotating drum or ring—Revolving disk; disk of uniform strength—Whirling of shafts; critical speeds; centrally-loaded and unloaded shaft; Dunkerley's approximate formulæ.

EXERCISES 571

APPENDIX—TABLES OF PROPERTIES OF OLD AND NEW BRITISH STANDARD SECTIONS 587

MATHEMATICAL TABLES 600

INDEX 605

THE STRENGTH OF MATERIALS

Note.—Portions marked with an asterisk may be omitted on the first reading.

CHAPTER I

STRAIN, STRESS, AND ELASTICITY

Strain may be defined as the change in shape or form of a body caused by the application of external forces.

Stress may be defined as the force between the molecules of a body brought into play by the strain.

An elastic body is one in which for a given strain there is always induced a definite stress, the stress and strain being independent of the duration of the external force causing them, and disappearing when such force is removed. A body in which the strain does not disappear when the force is removed is said to have a *permanent set,* and such body is called a *plastic body.*

When an elastic body is in equilibrium, the resultant of all the stresses over any given section of the body must neutralise all the external forces acting over that section. When the external forces are applied, the body becomes in a state of strain, and such strain increases until the stresses induced by it are sufficient to neutralise the external forces.

For a substance to be useful as a material of construction, it must be elastic within the limits of the strain to which it will be subjected. Most solid materials are elastic to some extent, and after a certain strain is exceeded they become plastic.

Hooke's Law—enunciated by Hooke in 1676—states that in an elastic body the *strain is proportional to the stress.* Thus, according to this law, if it takes a certain weight to stretch a rod a given amount, it will take twice that weight to stretch the rod twice that amount; if a certain weight is required to make a beam deflect to a given extent, it will take twice that weight to deflect the beam to twice that extent.

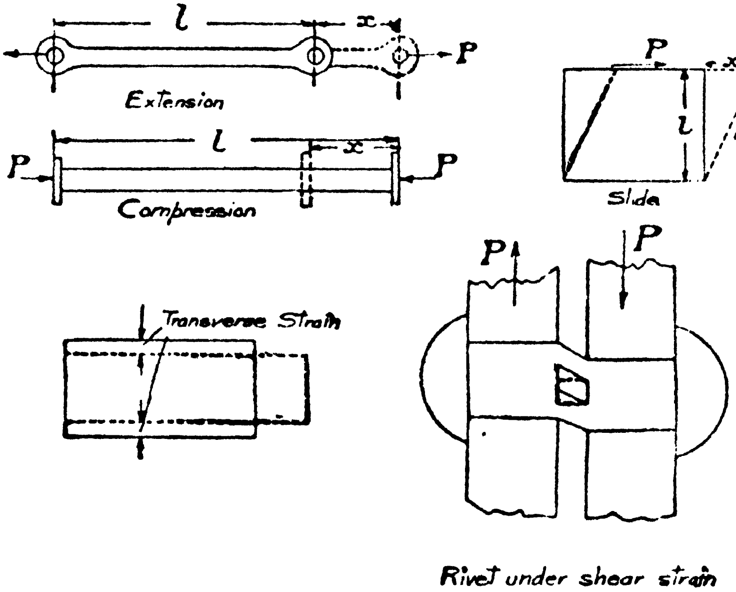

FIG. 1.—Kinds of Strain.

Kinds of Strain and Stress.—Strains may be divided into three kinds, viz. (1) an *extension;* (2) a *compression;* (3) a *slide.* Corresponding to these strains we have (1) *tensile* stress; (2) *compressive* stress; (3) *shear* stress.

A body that is subjected to only one of these, is said to be in a state of *simple* strain, while if it is subjected to more than one, it is said to be in a state of *complex* strain.

Examples of simple strains are to be found in the cases of a tie bar; a column with a central load; a rivet. The best example of a body under complex strain is that of a beam in which, as we shall show later, there exist all the kinds of strain.

INTENSITY OF STRESS.—Imagine a small area a situated at a point X in the cross section of a body under strain, then if S is the resultant of all the molecular forces across the small area, $\frac{S}{a}$ is called the *intensity of stress* at the point X. In the case of bodies under complex strain, the intensity of stress will be different at different points of the cross section, while in a body subjected to a simple strain, the stress will be the same over each point of the cross section, so that in this case if A is the area of the whole cross section and P is the whole force acting over the cross section, the intensity of stress will be equal to $\frac{P}{A}$. In future, unless it is stated to the contrary, we shall use the word "stress" to mean the "intensity of stress."

UNITAL STRAIN.—The unital strain is the strain per unit length of the material. In the case of extension and compression, the total strain is proportional to the original length of the body. Thus, a rod 2 ft. long will stretch twice as much as a rod 1 ft. long for the same load. In Fig. 1, if l is the unstrained length of the rods under tension and compression and x the extension or compression, the unital strain is $\frac{x}{l}$.

In the case of slide strain, the angle but not the length of the body is altered, and this angle β is a measure of the unital strain. If the angle is small, as it always will be in practice with materials of construction, then it will be nearly equal to $\frac{x}{l}$, where x and l are the quantities shown on the figure.

Poisson's Ratio—Transverse Strain.—When a body is extended or compressed, there is a transverse strain tending to prevent change of volume of the body. The amount of transverse strain bears a certain ratio to the longitudinal strain.

This ratio $= \dfrac{\text{transverse strain}}{\text{longitudinal strain}} = \eta$ varies from $\frac{1}{3}$ to $\frac{1}{4}$ for most materials, and is called *Poisson's ratio.*

According to one school of elasticians, the value of this ratio η should be $\frac{1}{4}$, but experimental evidence does not quite support this view, although it is very nearly true for some materials. The ratio is very difficult to measure directly, its value being best obtained by working backwards from the elastic moduli in shear and tension in the manner which will be explained later.

Stress-strain Diagrams.—If a material be tested in tension or compression, and the strain at each stress be measured, and such strains be plotted on a diagram against the stresses, a diagram called the *stress-strain diagram* is obtained. If a material obeys Hooke's Law, such diagram will be a straight line. For most metals, the stress-strain diagram will be a straight line until a certain point is reached, called the *elastic limit,* after which the strain increases more quickly than the stress, until a point called the *yield point* is reached, when there is a sudden comparatively large increase in strain. After the yield point is reached, the metal becomes in a plastic state, and the strains go on increasing rapidly until fracture occurs.

Fig. 2 shows the stress-strain diagram for a tension specimen of mild steel, such as is suitable for structural work.

The portion A B of the diagram is a straight line, and represents the period over which the material obeys Hooke's Law. At the point C, the yield point is reached, and the strain then increases to such an extent that the first portion of the diagram is re-drawn to a considerably smaller scale, such portion being shown as A B_1 C_1. The strain then increases in the form shown until the point D is reached, the curve between C_1 and D being approximately a parabola. When the point D is reached, the maximum stress has been reached, and the specimen begins to pull out and thin down at one section, and if the stress is sustained, fracture will then occur. The portion D E, shown dotted, represents increase of strain with apparent diminution of stress. This diminution is only apparent because the area of the specimen beyond the point rapidly gets smaller, so that the *load* may be decreased, and still keep the *stress* the same. In practice, it is very difficult to diminish the load so as to keep pace with the decrease in area, so that this last portion of the curve is very seldom accurate, and has, moreover, little practical importance in commercial testing because the

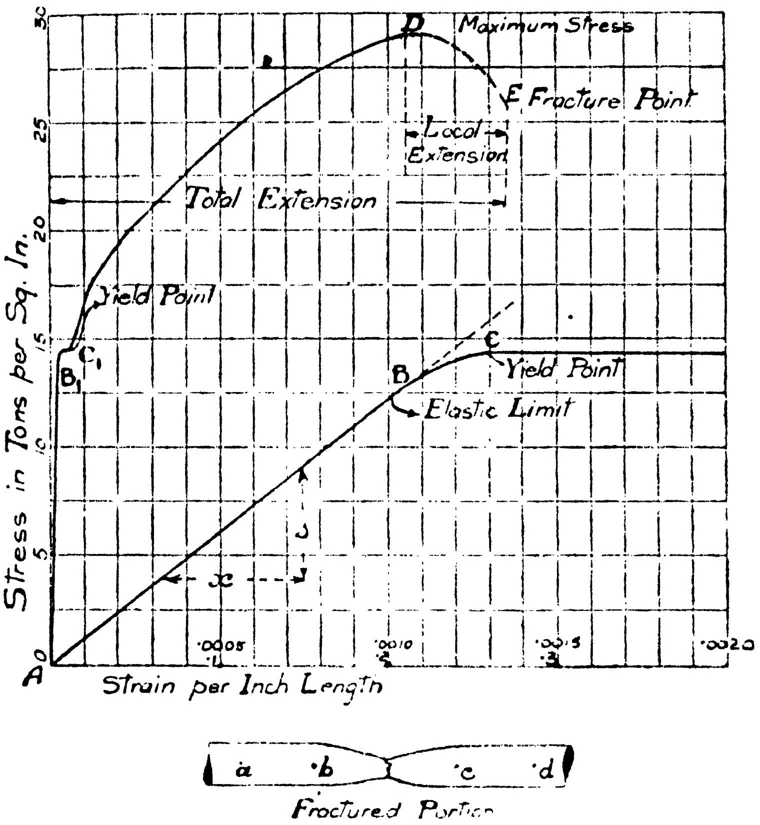

FIG. 2.—Stress-strain Diagram for Mild Steel.

maximum stress is always taken as that given at D (see also p. 52).

The specimen draws down at the point of fracture in the manner shown in the diagram. Before the test, it is customary to make centre-punch marks at equal distances apart along the length of the specimen. The distance apart of these points after fracture of the specimen indicates the distribution of the elongation at different points along the length. Four such marks, *a*, *b*, *c*, *d*, are shown on the figure. The greatest extension occurs at the point of fracture, so that on a specimen short length, the percentage total extension will be greater than on a longer specimen. We deal further with this point on p. 55.

The stress-strain diagrams in compression and shear for mild steel are very similar to that for tension. In compression it is difficult to get the whole diagram, because failure occurs by *buckling*, except on very short lengths, where it is very difficult to measure the strains, and in shear the test is best made by torsion, because it is almost impossible to eliminate the bending effect. Now, in torsion, the shear stress is not uniform, so that the metal at the exterior of the round bar reaches its yield point before the material in the centre, and this has the effect of raising the apparent yield point. We shall see later that the same occurs in testing for compression or tension by means of beams.

The importance of the elastic limit has been overlooked to a great extent by designers of machines and structures; but inasmuch as the theory, on which most of the formulæ for obtaining the strength of beams are based, assumes that the stress is proportional to the strain, it must be remembered that our calculations are true only so long as Hooke's Law is true, so that the elastic limit of the material is a very important quantity. We shall deal further with this question in discussing working stresses (Chap. III.).

CONFUSION BETWEEN ELASTIC LIMIT AND YIELD POINT.—In commercial testing, it is quite common to use no accurate means for measuring the strains (instruments for such measurements are called *extensometers*, see pp. 371–379). The load on the steelyard of the machine is run out until the steelyard suddenly drops down on to its stops. The "steelyard-drop" happens when the yield point is reached, but many people call this the elastic limit; it is also sometimes called the "apparent elastic limit." As will be seen from the diagram, Fig. 2, there is no appreciable error made by this confusion in tension testing, but in cross bending the difference is much more marked, and gives rise to confused ideas. We shall deal further with this point on pp. 207–209.

The Elastic Constants or Moduli.—If a material is truly elastic, *i.e.* if the strain is proportional to the stress, then it follows that the intensity of stress is always a certain number of times the unital strain, or that the ratio $\frac{\text{intensity of stress}}{\text{unital strain}}$ is constant. Now this stress-strain ratio is called a *modulus*. That for tension and compression is generally known as *Young's modulus*, and is given the letter E; that for shear is called the *shear*, or *rigidity modulus* (G). There is an additional modulus called the *bulk* or *volume modulus* (K), which represents the ratio between the intensity of pressure or tension and the unital change in volume on a cube of material subjected to pressure or tension on all faces.

Young's modulus is the one with which we shall be most concerned. Suppose that a tension member (a *tie* as it is called) or a compression member (a *strut*), of length l and cross-sectional area A, is subjected to a pull or thrust P, and that the extension or compression is x, Fig. 1. Then the intensity of stress is $\frac{\mathrm{P}}{\mathrm{A}}$, and the unital strain is $\frac{x}{l}$

$$\therefore \text{ Young's modulus} = \mathrm{E} = \frac{\mathrm{P}}{\mathrm{A}} \div \frac{x}{l} = \frac{\mathrm{P}l}{\mathrm{A}x}$$

NUMERICAL EXAMPLE.—*A mild-steel tie bar,* 12 *ins. long and of* $1\frac{1}{2}$ *ins. diameter, is subjected to a pull of* 18 *tons. If the extension is* ·0094 *in., find Young's modulus.*

Area of section of $1\frac{1}{2}$ ins. diam. = 1·767 sq. ins.

$$\therefore \text{Stress per sq. in.} = \frac{18}{1·767} = 10·19 \text{ tons per sq. in.}$$

$$\text{Unital strain} = \frac{·0094}{12} = ·000783$$

$$\therefore \text{Young's modulus} = \frac{10·19}{·000783} = 13{,}000 \text{ tons per sq. in.}$$

The value of Young's modulus can be found from the stress-strain diagram. Thus, in that for mild steel, Fig. 2,

$$E = \frac{s}{x}$$

Now in the relation $E = \frac{\text{stress}}{\text{strain}}$, if the strain is equal to 1, *i.e.* if the bar is pulled to twice its original length, we have that E = stress, and this accounts for the definition of Young's modulus that some writers have given, viz.: Young's modulus is the stress that is necessary to pull a bar to twice its original length. Some students find this definition more clear than the one previously given, but it must be remembered that no material of construction will pull out to twice its original length without fracture.

Relation between Elastic Constants.—For an elastic material there will be certain relations between the elastic moduli E, G, K, and Poisson's ratio η. These relations can be found as follows—

To first find a relation between E and K consider a cube of unit side subjected to a pull f_t, Fig. 3 (*a*).

Let the elongation along the axis be a, and let the transverse contraction be b.

$$\begin{aligned} \text{Then original volume of cube} &= 1 \\ \text{strained volume of cube} &= (1 + a)\,(1 - b)^2 \\ &= 1 - 2b + b^2 + a - 2ab + ab^2 \\ &= 1 + a - 2b \text{ (nearly)} \end{aligned}$$

because as the strains are very small their product may be neglected.

$$\begin{aligned} \therefore \text{Increase in volume} &= (1 + a - 2b) - 1 \\ &= (a - 2b) \end{aligned}$$

Now apply a pull to each side of the cube. There will now be three pulls, each producing an increase of volume equal nearly to $(a - 2b)$.

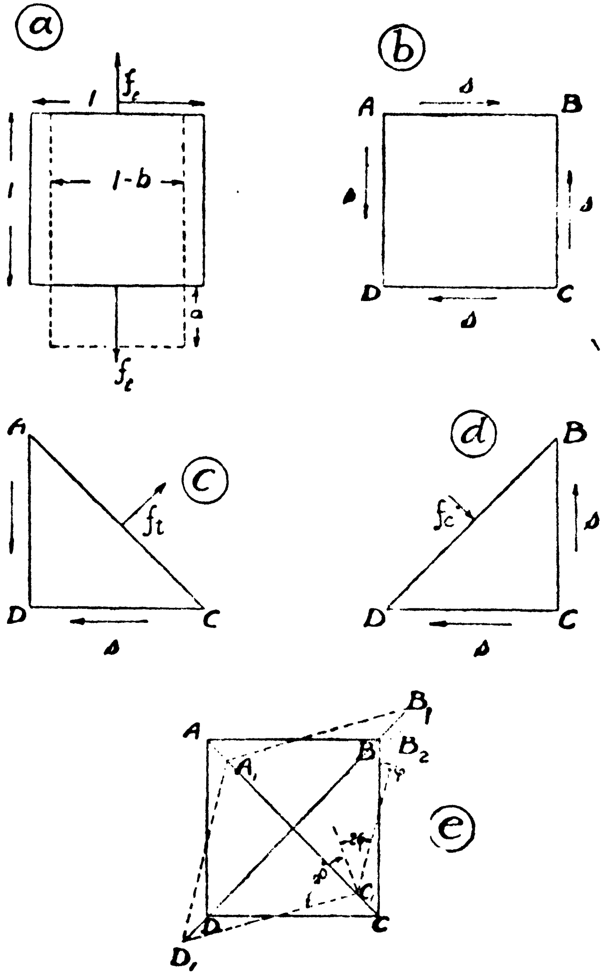

FIG. 3.

$\therefore$ Total increase in volume is nearly equal to $3(a - 2b)$

$$= 3a\left(1 - \frac{2b}{a}\right)$$

Now $\frac{b}{a} = \frac{\text{transverse strain}}{\text{longitudinal strain}} = \eta$

$\therefore$ Increase in volume $= 3a(1 - 2\eta)$

$\therefore$ Since original volume $= 1$

$$\frac{\text{increase in volume}}{\text{original volume}} = \text{volume unital strain} = 3a(1 - 2\eta)$$

$$\text{Now } K = \frac{\text{intensity of pull}}{\text{unital strain}} = \frac{f_t}{3a(1 - 2\eta)}$$

$$\text{and } \frac{f_t}{a} = \frac{\text{tensile intensity of stress}}{\text{unital tensile strain}} = \text{Young's modulus}$$

$$= E$$

$$\therefore K = \frac{E}{3(1 - 2\eta)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now find the relation between E *and* G *as follows*—

Suppose that two shearing forces of intensity s are applied to the faces of a unit cube A B C D, Fig. 3 (*b*). Now consider the equilibrium of the portion A D C, Fig. 3 (*c*). To balance the forces s there must be a force pulling f_t across the diagonal A C, and the value of f_t must be $\sqrt{2} \times s$. Now the area over which this force acts will be $\sqrt{2}$ since the cube is of unit side, so that there will be a tensile stress of $\frac{\sqrt{2}\,s}{\sqrt{2}} = s$. Similarly considering the portion B C D, Fig. 3 (*d*) there must be a compressing force f_c along the diagonal B D, and the compressive stress will be $= \frac{\sqrt{2}\,s}{\sqrt{2}} = s$. Therefore we see that: *Two shear stresses on planes at right angles to each other are equivalent to tensile and compressive stresses of intensity equal to that of the shear stress at right angles to each other, and at an angle of 45° to the shear stresses.*

Now the cube will be deformed to the shape $A_1 B_1 C_1 D_1$, Fig. 3 (*e*).

The unital shear strain is measured by the angle of distortion 2ϕ. Since the strains are very small, this is practically equal to $\frac{2\,BB_2}{\frac{1}{2}\,BC} = \frac{2\,BB_2}{\frac{1}{2}}$ (since $BC = 1$) $= 4\,BB_2$.

Let the unital strain due to the tension along the diagonal B D be a. Then there will also be a strain along this diagonal due to the transverse strain from the compression stress in A C. This will be equal to $\eta \times a$. ∴ Total unital tensile strain along diagonal $= a(1+\eta)$. Then $\mathrm{B\,B_1}$ = unital strain along diagonal × ½ B D, since $\mathrm{B\,B_1}$ is equal to $\mathrm{D\,D_1}$

$$\therefore \mathrm{B\,B_1} = a(1+\eta) \times \tfrac{1}{2}\mathrm{B\,D} = \frac{\sqrt{2}\,a}{2}(1+\eta).$$

Because the strains are in reality very small, $\mathrm{B\,B_2\,B_1}$ is very nearly a right-angled triangle.

$$\therefore \mathrm{B\,B_1} = \sqrt{2} \times \mathrm{B\,B_2}$$

$$\text{or } \mathrm{B\,B_2} = \frac{\mathrm{B\,B_1}}{\sqrt{2}} = \frac{a(1+\eta)}{2}$$

$$\text{But } \frac{\text{intensity of tensile stress}}{\text{unital tensile strain}} = \frac{f_t}{a} = \mathrm{E}$$

$$\text{and } \frac{\text{intensity of shear stress}}{\text{unital shear strain}} = \frac{s}{4\,\mathrm{B\,B_2}} = \mathrm{G}$$

Since we have proved that the tensile stress along the diagonal is equal in intensity to the shear stress.

$$\text{Therefore } f_t = a\,\mathrm{E} = \mathrm{G} \times 4\,\mathrm{B\,B_2}$$

$$\therefore \frac{\mathrm{E}}{\mathrm{G}} = \frac{4\,\mathrm{B\,B_2}}{a} = \frac{4a(1+\eta)}{2a}$$

$$\therefore \frac{\mathrm{E}}{\mathrm{G}} = 2(1+\eta) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Now we have already shown in (1) that

$$\frac{\mathrm{E}}{\mathrm{K}} = 3(1-2\eta) \ldots\ldots\ldots\ldots (3)$$

$$\text{From (2) } \eta = \frac{\mathrm{E}}{2\mathrm{G}} - 1$$

$$\text{From (3) } \eta = \frac{1}{2} - \frac{\mathrm{E}}{6\mathrm{K}}$$

$$\therefore \frac{\mathrm{E}}{2\mathrm{G}} - 1 = \frac{1}{2} - \frac{\mathrm{E}}{6\mathrm{K}}$$

$$\therefore \frac{\mathrm{E}}{2}\left(\frac{1}{\mathrm{G}} + \frac{1}{3\mathrm{K}}\right) = \frac{3}{2}$$

$$\therefore \frac{1}{\mathrm{G}} + \frac{1}{3\mathrm{K}} = \frac{3}{\mathrm{E}}$$

$$\text{or } \frac{9}{\mathrm{E}} = \frac{3}{\mathrm{G}} + \frac{1}{\mathrm{K}} \quad \ldots\ldots\ldots\ldots (4)$$

This expresses the relation between the constants in its simplest form.

It will be noted that if $\eta = \frac{1}{4}$, as some authorities state, then $\frac{E}{G} = 2{\cdot}5$; this may be taken as true if the value of G for the material is not known.

Strains in different Directions.—Suppose that the stresses in a material acting normally to, *i. e.* at right angles to, three planes at right angles to each other are f_x, f_y, f_z. Then the unital strain in the first direction is made up of the direct strain due to f_x and the transverse strains due to the other two.

$$\therefore \text{unital strain in first direction} = s_x = \frac{f_x}{E} - \frac{\eta\,(f_y + f_z)}{E}$$

$$\text{unital strain in second direction} = s_y = \frac{f_y}{E} - \frac{\eta\,(f_x + f_z)}{E}$$

$$\text{unital strain in third direction} = s_z = \frac{f_z}{E} - \frac{\eta\,(f_x + f_y)}{E}$$

Lateral Strain prevented in one Direction.—Now take the case of a piece of material which is free to dilate or expand in one direction, but is prevented from doing so in another at right angles, a compression stress f_x being applied in the third direction.

Let the z direction be that in which strain is prevented.

We then have

$$s_z = 0$$

and let the x direction be the one with the stress f_x.

Then if f_z is the stress caused in the z axis we have, since $f_y = 0$, because the material can expand freely in the y direction—

$$E\,s_x = f_x - \eta\,f_z$$

$$E\,s_y = -\,\eta\,(f_x + f_z)$$

$$0 = f_z - \eta\,f_x$$

$$\therefore f_z = \eta\,f_x$$

$$\therefore E\,s_y = -\,\eta\,(1 + \eta)\,f_x;\ E\,s_x = f_x\,(1 - \eta^2)\ \therefore \frac{f_x}{s_x} = \frac{E}{(1 - \eta^2)}.$$

This result is interesting as showing that the ordinary definition of E only holds when lateral strain is not prevented.

* **Complex Stress.**—PRINCIPAL STRESSES.—It can be shown that when a body is under a complex system of stresses, such stresses will be the same as those due to the combination of three simple tensile or compressive stresses in planes at right angles to each other. Such simple stresses are called the *principal stresses.*

Consider the case of a block of material subjected to pulls P and Q, Fig. 4, in two directions at right angles, and let the pull per square inch of the sectional area in each direction be p and q, respectively, these being principal stresses.

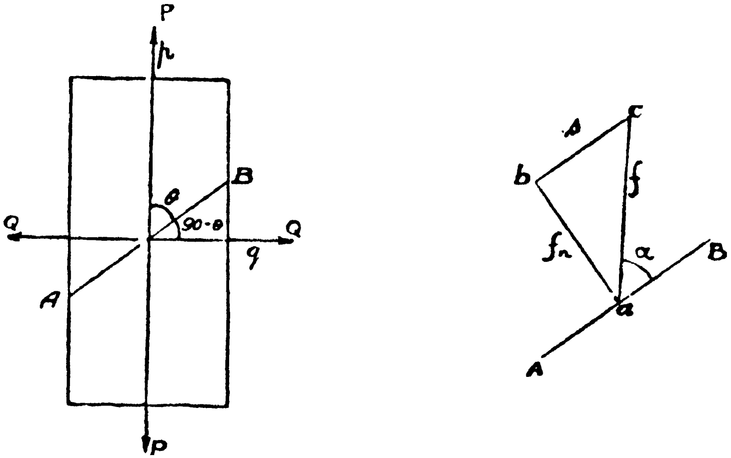

FIG. 4.—Principal Stresses.

Consider the stresses on a plane A B inclined at an angle θ to the force P.

The stress p can be resolved perpendicularly and along A B, *i.e.* normally and tangentially to A B.

Now consider 1 sq. in. of area perpendicular to p. The corresponding area along A B will be $\frac{1}{\sin \theta}$.

Now the component of p perpendicular to A B will be p

$p \sin\theta$, and the component along A B will be $p \cos\theta$, but stress = component of force ÷ area.

∴ Normal or perpendicular component of stress p along A B

$$= p \sin\theta \div \frac{1}{\sin\theta} = p \sin^2\theta$$

tangential or shear component of stress p along A B

$$= p \cos\theta \div \frac{1}{\sin\theta} = p \sin\theta \cos\theta$$

Now considering stress q, its tangential component to A B will be opposite in direction to that of p, and since in this case the area is $\frac{1}{\sin(90-\theta)} = \frac{1}{\cos\theta}$ and the normal and tangential components of q are respectively $q \cos\theta$ and $q \sin\theta$, the normal component of stress will be $q \cos^2\theta$, and the tangential component of stress will be $- q \sin\theta \cos\theta$, since in this case the tangential components are not in the same direction.

∴ Total normal component $= f_n = p \sin^2\theta + q \cos^2\theta$ ..(1)

Total tangential component $= s = (p - q) \sin\theta \cos\theta$ ..(2)

Now the resultant of the stresses f_n and s, which we will call f, will be given by $a\ c$.

$$\begin{aligned}\text{i.e. } f &= \sqrt{f_n^2 + s^2} \\ &= \sqrt{(p \sin^2\theta + q \cos^2\theta)^2 + (p - q)^2 \sin^2\theta \cos^2\theta} \\ &= \sqrt{p^2(\sin^4\theta + \sin^2\theta\cos^2\theta) + q^2(\cos^2\sin^2\theta + \cos^4\theta) + 2pq(\cos^2\theta\sin^2\theta - \cos^2\theta\sin^2\theta)} \\ &= \sqrt{p^2\sin^2\theta(\cos^2\theta + \sin^2\theta) + q^2\cos^2\theta(\sin^2\theta + \cos^2\theta) + 0} \\ &= \sqrt{p^2\sin^2\theta + q^2\cos^2\theta} \quad \ldots\ldots\ldots\ldots\ldots(3)\end{aligned}$$

because $\cos^2\theta + \sin^2\theta = 1$.

The inclination a of this stress is given by

$$\tan a = \frac{f_n}{s} = \frac{p \sin^2\theta + q \cos^2\theta}{(p - q) \sin\theta \cos\theta}$$

$$= \frac{p \tan^2\theta + q}{(p-q) \tan\theta} \quad \ldots\ldots\ldots\ldots\ldots(4)$$

If ϕ is the angle between *a c* and the direction of p,
Then $\phi = (\alpha - \theta)$

$$\tan\phi = \tan(\alpha - \theta) = \frac{\tan\alpha - \tan\theta}{1 + \tan\alpha \,.\, \tan\theta}$$

$$= \frac{\dfrac{p\tan^2\theta + q}{(p-q)\tan\theta} - \tan\theta}{1 + \dfrac{p\tan^2\theta + q}{(p-q)\tan\theta}\,.\,\tan\theta}$$

$$= \frac{p\tan^2\theta + q - (p-q)\tan^2\theta}{(p-q)\tan\theta + \tan\theta\,(p\tan^2\theta + q)}$$

$$= \frac{q\,(1 + \tan^2\theta)}{p\tan\theta\,(1 + \tan^2\theta)} = \frac{q}{p\tan\theta}$$

$$= \frac{q}{p}\cot\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Unlike Stresses.—If the stresses are unlike, *i.e.*, one tension and one compression, we shall have by similar reasoning

$$f_n = p\sin^2\theta - q\cos^2\theta$$
$$s = (p + q)\sin\theta\cos\theta$$

It will be noted that for $p = q$ and $\theta = 45°$ we have $f_n = 0$ and $s = p$.

We have, therefore, a pure shear as equivalent to equal tensile and compressive stresses at right angles to each other, at 45° to the shear stress and equal in intensity to the shear stress (cf. p. 10).

The Ellipse of Stress.—Draw circles of radius O X and O Y, Fig. 5, equal to q and p respectively, and let O R be drawn at angle θ to O Y.

Draw a radius O F to the larger circle at right angles to O R and cutting the smaller circle in E.

Draw F H at right angles to O Y, and E G at right angles to F H, and join O G.

$$\text{Now OH} = \text{OF}\cos(90 - \theta) = p\sin\theta$$
$$\text{and GH} = \text{EK} = \text{OE}\sin(90 - \theta) = q\cos\theta$$
$$\therefore \text{OG} = \sqrt{\text{OH}^2 + \text{HG}^2} = \sqrt{p^2\sin^2\theta + q^2\cos^2\theta}$$
$$\therefore \text{by equation (3) OG} = f$$

$$\text{Now } \tan \text{HOG} = \frac{\text{HG}}{\text{OH}} = \frac{q \cos \theta}{p \sin \theta} = \frac{q}{p} \cot \theta = \tan \phi$$

$$\therefore \angle \text{HOG} = \phi \text{ and, since } \alpha = \theta + \phi,\ \angle \text{GOR} = \alpha.$$

Now the locus of the point G is an ellipse of major axis 2 p and minor axis 2 q, and such ellipse is called the *Ellipse of Stress.*

We see, therefore, that by drawing a line O F from the centre O, at right angles to a given direction to the outer circle, and drawing F H horizontal to meet the ellipse of stress in G, then O G gives the resultant stress on a plane in the given direction, and the angle G O R = α gives the angle between such resultant stress and the plane.

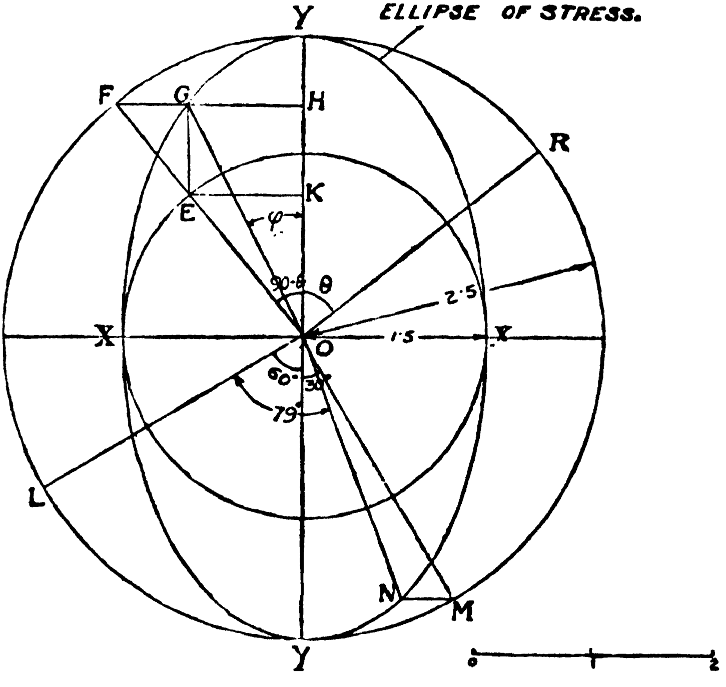

FIG. 5.—Ellipse of Stress.

NUMERICAL EXAMPLE.—*Suppose a square bar of* 2 *ins. side and* 4 *ins. long is subjected to pulls of* 10 *and* 12 *tons respectively in axial and transverse directions. Find the resultant stress on* *a plane inclined at 60 degrees to the axis, and find the inclination of the stress to that plane.*

$$\text{In this case } p = \frac{10}{4} = 2{\cdot}5 \text{ tons per square inch,}$$

$$\text{and } q = \frac{12}{8} = 1{\cdot}5 \text{ tons per square inch.}$$

Then Fig. 5 shows the ellipse of stress drawn to scale.

Draw O L at 60° to O Y and draw O M at right angles to O L to cut the outer circle in M; drawing M N horizontal to meet the ellipse of stress in N, then O N gives the resultant stress

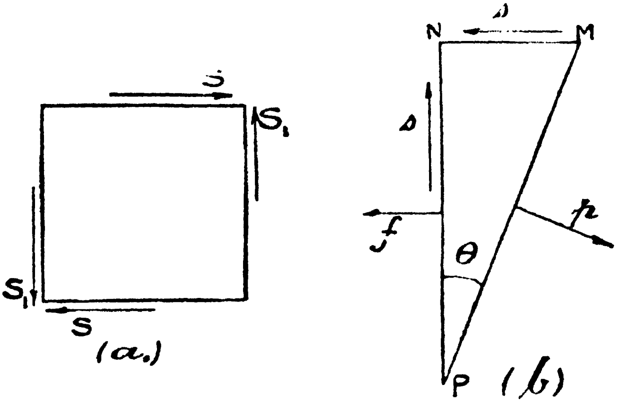

FIG. 6.—Combined Normal and Shear Stress.

and ∠ L O N gives its inclination to the plane. O N will be found to be 2·29 tons per square inch, and ∠ L O N to be 79°.

Now considering again the stresses p and q and the normal and tangential stresses f_n and s at an inclination θ to p we see that p and q are the *principal stresses* corresponding to the stresses f_n and s. Now in practice we often require to find the magnitude and inclination of the principal stresses, because one of these stresses will be that of maximum intensity of stress. This is clear from the figure of the ellipse of stress, since O Y is obviously the maximum radius vector of the ellipse. We will now therefore find the principal stresses due to a normal stress f_n and a shear or tangential stress s at right angles to each other.

*** Combined Normal and Shear Stress.**—To investigate this problem we must first prove that a shear stress must always be accompanied by an equal shear stress at right angles to it. Take for example a unit cube, Fig. 6 (*a*), subjected to shearing forces S along opposite sides. These forces S form a couple, and the cube can be kept in equilibrium only by another couple of equal moment and opposite sense, which couple is given by shearing forces S_1 at right angles to S.

Now consider the case of a complex system of stress consisting of a normal stress f and a shear or tangential stress s.

Let P N, Fig. 6 (*b*), represent a portion of the plane on which the stresses f and s act.

Let one of the planes of principal stress be represented by P M, and let this principal stress be p. Then along M N there acts a shear stress also of intensity s.

Then the resolved portions of the forces due to p and to the stresses f and s must be equal in the directions P N and M N.

Therefore we have

$$f \cdot \mathrm{PN} + s \cdot \mathrm{MN} = p \cdot \mathrm{PM} \cos\theta \ \ldots\ldots\ldots\ldots(1)$$

$$\text{also } s \cdot \mathrm{PN} = p \cdot \mathrm{PM} \sin\theta \ \ldots\ldots\ldots\ldots(2)$$

$$\therefore \text{From (1) } f\frac{\mathrm{PN}}{\mathrm{PM}} + s\frac{\mathrm{MN}}{\mathrm{PM}} = p\cos\theta$$

$$\text{i.e. } f\cos\theta + s\sin\theta = p\cos\theta$$

$$\therefore (p-f)\cos\theta = s\sin\theta \ \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\text{From (2)} \quad s\frac{\mathrm{PN}}{\mathrm{PM}} = p\sin\theta$$

$$\therefore s\cos\theta = p\sin\theta \ \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$\therefore$ Dividing (3) and (4) we have

$$\frac{p-f}{s} = \frac{s}{p}$$

$$p(p-f) = s^2$$

$$p^2 - pf - s^2 = 0$$

$$p = \frac{f}{2} \pm \frac{1}{2}\sqrt{f^2 + 4s^2}$$

$$\text{or } p = \frac{f}{2}\left(1 \pm \sqrt{1 + \frac{4s^2}{f^2}}\right) \ \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

The minus sign corresponds to the second principal stress q, which will be in compression; as we are concerned only with the maximum stress, we will take the positive value, viz.—

$$\underline{p = \frac{f}{2}\left(1 + \sqrt{1 + \frac{4s^2}{f^2}}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

The direction of the plane at which this stress occurs is given by θ. This is found as follows—

From (3) $p \cos \theta - f \cos \theta = s \sin \theta$

From (4) $p \sin \theta = s \cos \theta$

$$\therefore p = \frac{s \cos \theta}{\sin \theta}$$

$$\therefore \frac{s \cos^2 \theta}{\sin \theta} - f \cos \theta = s \sin \theta \quad \ldots\ldots\ldots\ldots\ldots (7)$$

$$\therefore s(\cos^2 \theta - \sin^2 \theta) = f \sin \theta \cos \theta$$

$$\therefore s \cos 2\theta = f \frac{\sin 2\theta}{2}$$

$$\text{or } \tan 2\theta = \frac{2s}{f} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

This will give two values of θ, 90° apart, and so gives the inclination of both planes of principal stress.

Maximum Shear Stress.—Returning to the consideration of the principal stresses p and q, we saw that the tangential component on a plane at angle θ to p was given by $(p - q) \sin \theta \cos \theta$. (See p. 14, equation (2).) Now this will be a maximum when $\sin \theta \cos \theta$ is a maximum, *i.e.* when $\frac{\sin 2\theta}{2}$ is a maximum, or when $\theta = 45°$. Therefore we see that the maximum shear stress occurs at 45° to the principal stresses, and is equal to $\frac{(p - q)}{2}$.

In the problem that we are considering, we have proved that

$$p = \frac{f}{2}\left(1 + \sqrt{1 + \frac{4s^2}{f^2}}\right) \text{ and that } q = \frac{f}{2}\left(1 - \sqrt{1 + \frac{4s^2}{f^2}}\right)$$

$$\therefore \frac{p - q}{2} = \frac{f}{2}\sqrt{1 + \frac{4s^2}{f^2}}$$

$$\therefore \text{ Maximum shear stress} = \frac{f}{2}\sqrt{1 + \frac{4s^2}{f^2}} \quad \ldots\ldots\ldots\ldots(9)$$

$$\text{or} = \sqrt{\frac{f^2}{4} + s^2} \quad \ldots\ldots\ldots\ldots(10)$$

The latter form is more convenient because in the case when $f = 0$, the former gives an indeterminate result.

NUMERICAL EXAMPLE.—*A steel bolt, 1 in. in diameter, is subjected to a direct pull of 3000 lbs. and to a shearing force of 1 ton. Find the maximum tensile and shearing stresses in lbs. per square inch, and the inclinations of the directions of the stresses to the longitudinal axis of the bolt.*

$$\text{In this case } f = \frac{3000}{\text{area of 1 in. bolt}} = \frac{3000}{\cdot 7854}$$

$$= 3819 \text{ lb. per sq. in.}$$

$$s = \frac{2240}{\cdot 7854} = 2852 \text{ lb. per sq. in.}$$

$$\therefore \text{ Maximum tensile stress} = p = \frac{f}{2}\left(1 + \sqrt{1 + \frac{4s^2}{f^2}}\right)$$

$$= \frac{3819}{2}\left(1 + \sqrt{1 + \frac{4 \times 2852^2}{3819^2}}\right)$$

$$= \frac{3819}{2}\left(1 + \sqrt{1 + 2 \cdot 23}\right)$$

$$= \frac{3819}{2}\left(1 + 1 \cdot 797\right)$$

$$= \underline{5342 \text{ lb. per square inch.}}$$

Inclination of principal plane to plane perpendicular to axis is given by

$$\tan 2\theta = \frac{2s}{f} = \frac{2 \times 2852}{3819}$$

$$= 1 \cdot 494$$

$$\therefore 2\theta = 56^\circ\ 12' \text{ nearly}$$

$$\therefore \theta = 28^\circ\ 6'$$

$$\therefore \text{ Inclination to longitudinal axis} = 90 - 28^\circ\ 6'$$

$$= \underline{61^\circ\ 54'}$$

$$\text{Maximum shear stress} = \sqrt{\frac{f^2}{4} + s^2}$$
$$= \sqrt{\frac{3819^2}{4} + 2852^2}$$
$$= 2852\sqrt{1 + \frac{3819^2}{4 \times 2852^2}}$$
$$= 2852\sqrt{1 + \cdot448}$$
$$= 2852 \times 1\cdot203$$
$$= 3428 \text{ lb. per square inch.}$$

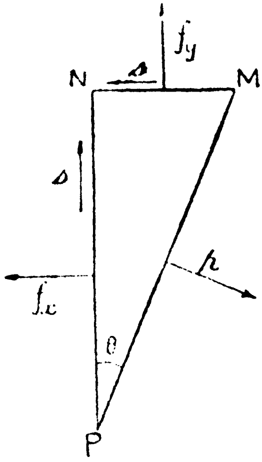

FIG. 7.

This stress will occur at 45° to the direction of principal stress, *i.e.* at 61° 54′ − 45° = 16° 54′ with longitudinal axis, or else at 90° to this, *i.e.* at 73° 6′ with longitudinal axis.

* Combined Shear Stress and two Normal Stresses at Right Angles to each other.

Next consider the case of a shear stress s combined with normal stresses f_x, f_y at right angles to each other (Fig. 7)

Then resolving as before we have

$$f_x \cdot \text{PN} + s\,\text{MN} = p \cdot \text{PM} \cos\theta \ldots\ldots\ldots\ldots(11)$$
$$f_y\,\text{NM} + s\,\text{PN} = p \cdot \text{PM} \sin\theta \ldots\ldots\ldots\ldots(12)$$

From (11) $f_x \cos\theta + s \sin\theta = p \cos\theta$

From (12) $f_y \sin\theta + s \cos\theta = p \sin\theta$

$$\text{i.e. } (p - f_x)\cos\theta = s \sin\theta \quad \ldots\ldots\ldots\ldots\ldots(13)$$

$$(p - f_y)\sin\theta = s \cos\theta \quad \ldots\ldots\ldots\ldots\ldots(14)$$

We therefore have by multiplying and cancelling $\sin\theta \cos\theta$

$$(p - f_x)(p - f_y) = s^2$$

$$\text{i.e. } p^2 - p(f_x + f_y) + f_x f_y = s^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

Solving this quadratic we get

$$p = \tfrac{1}{2}\left\{(f_x + f_y) \pm \sqrt{(f_x + f_y)^2 - 4(f_x f_y - s^2)}\right\}$$

$$= \tfrac{1}{2}\left\{(f_x + f_y) \pm \sqrt{(f_x - f_y)^2 + 4s^2}\right\}$$

$$= \frac{(f_x + f_y)}{2} \pm \sqrt{\frac{(f_x - f_y)^2}{4} + s^2} \ldots\ldots\ldots\ldots\ldots(16)$$

In this case as in the previous one we usually take the positive sign.

To get the direction of the principal stresses we have from (13) and (14)

$$(p - f_x) = s \tan\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots(17)$$

$$(p - f_y) = s \cot\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots(18)$$

$\therefore$ subtracting

$$(f_x - f_y) = s(\cot\theta - \tan\theta).$$

$$\text{Now } \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2}{\dfrac{1}{\tan\theta} - \tan\theta} = \frac{2}{\cot\theta - \tan\theta}$$

$$\textit{i.e. } \cot\theta - \tan\theta = \frac{2}{\tan 2\theta}$$

$$\therefore (f_x - f_y) = \frac{2s}{\tan 2\theta}$$

$$\textit{i.e. } \tan 2\theta = \frac{2s}{(f_x - f_y)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(19)$$

The maximum shear stress is as before equal to $\dfrac{(p - q)}{2}$

$$\therefore \text{Maximum shear stress} = \sqrt{\frac{(f_x - f_y)^2}{4} + s^2}$$

Graphical Representation of Results.—The following graphical construction, due we believe to Professor R. H. Smith, solves these equations.

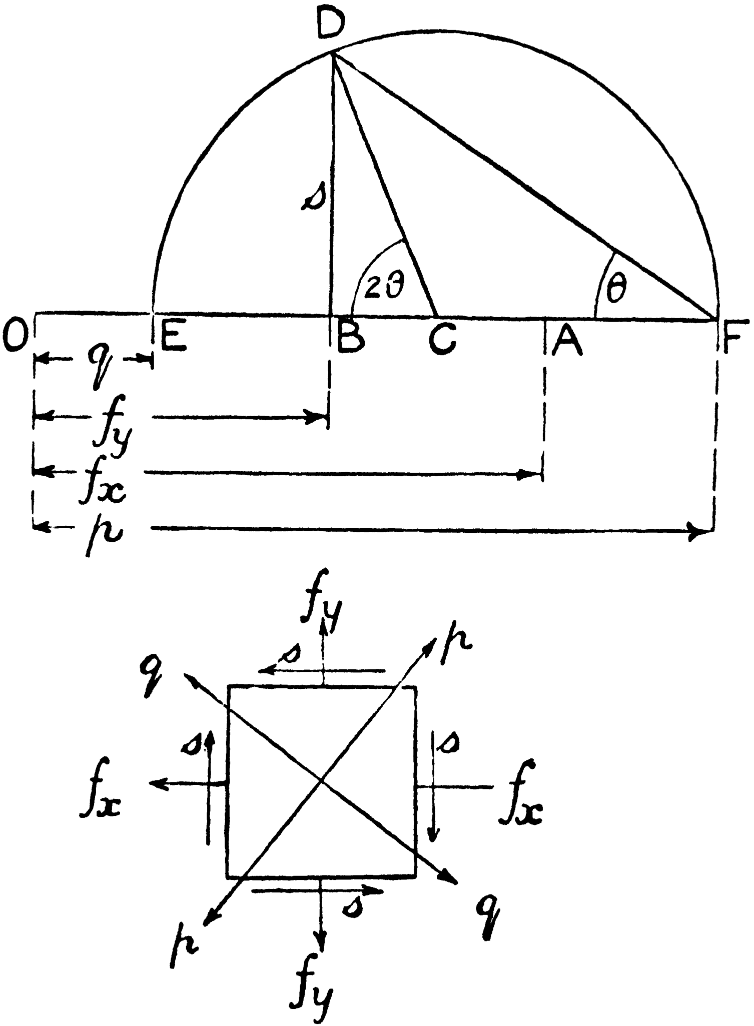

Fig. 8.—Graphical Construction for Combined Stresses.

Set out O A, Fig. 8, to represent f_x to a convenient scale and O B to represent f_y to a convenient scale; if f_x and f_y are opposite in sign they should be set out in opposite directions.

Bisect A B in C and at B set up B D at right angles to O A to represent the shear stress s; then with centre C and C D as radius draw a semicircle, cutting O A in F and E.

Then O F $= p$ and O E $= q$.

If E comes on the other side of O, the stress is negative.

Join D F, then $\underline{/\text{D F B}} = \theta$, the angle of the principal stresses.

Also maximum shear stress = C E.

Proof.— $$\text{B C} = \frac{\text{B A}}{2} = \frac{f_x - f_y}{2}$$

$$\text{O C} = \text{O B} + \frac{\text{B A}}{2} = f_y + \frac{f_x - f_y}{2} = \frac{f_x + f_y}{2}$$

$$\text{C D} = \sqrt{\text{B C}^2 + \text{B D}^2} = \sqrt{\frac{(f_x - f_y)^2}{4} + s^2}$$

$$\therefore \text{O F} = \text{O C} + \text{C F} = \text{O C} + \text{C D}$$

$$= \frac{(f_x + f_y)}{2} + \sqrt{\frac{(f_x - f_y)^2}{4} + s^2} = p$$

$$\text{O E} = \text{O C} - \text{C E} = \text{O C} - \text{C D}$$

$$= \frac{(f_x + f_y)}{2} - \sqrt{\frac{(f_x - f_y)^2}{4} + s^2} = q$$

$$\text{C E} = \text{C D} = \sqrt{\frac{(f_x - f_y)^2}{4} + s^2} = \text{maximum shear stress.}$$

Now $\underline{/\text{B C D}}$ = angle at centre $= 2\,\underline{/\text{D F B}}$

$$\tan \underline{\text{B C D}} = \frac{\text{B D}}{\text{B C}} = \frac{s}{\frac{(f_x - f_y)}{2}} = \frac{2s}{(f_x - f_y)}.$$

$\therefore$ From equation (19) B C D $= 2\theta$. $\therefore$ $\underline{/\text{D F B}} = \theta$.

APPLICATION TO A SINGLE NORMAL STRESS.—In this case, Fig. 9, B and O coincide so that O C $= \frac{1}{2}$ O A $= \frac{f}{2}$, and the construction comes as shown.

This figure has been drawn for $f = 3819$ and $s = 2852$ as in the numerical example of p. 20.

*** Maximum Strain compared with Maximum Stress.**—In questions involving complex stresses it is necessary to remember that the maximum strain does not occur on the same plane as the maximum stress. There is some considerable divergence among elasticians (a term suggested by Professor Karl Pearson, F.R.S.) as to whether the ultimate criterion of strength of a material depends on the tensile or compressive stress exceeding a certain value, or the shear stress exceeding a certain value, or on the strain exceeding a certain value. This is dealt with on pp. 42–48.

We have considered the cases of maximum tensile or compressive and shear stresses. We will now consider the question of maximum strain.

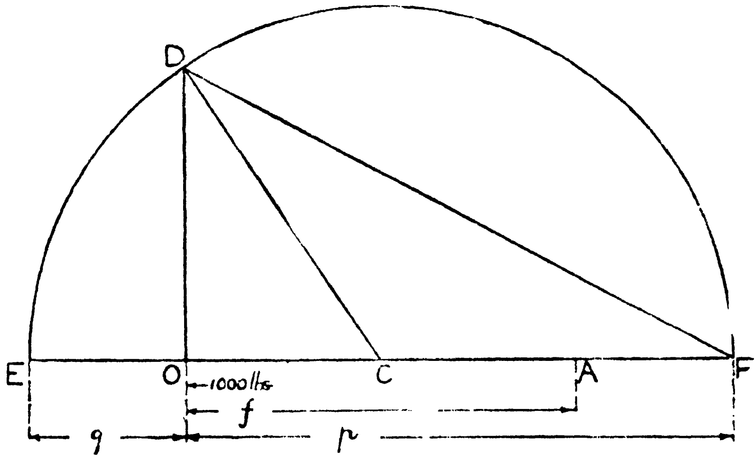

FIG. 9.

The given stresses are equivalent to simple stresses p, q at right angles to each other; we will assume p to be greater than q

$$\therefore \text{ Strain in direction of } p = \frac{p}{E} - \frac{\eta q}{E} \quad (\eta = \text{Poisson's ratio}).$$

Simple stress in direction of p to cause same strain = Equivalent direct stress $p_e = p - \eta q$

$$= \frac{f}{2}\left(1 + \sqrt{1 + \frac{4s^2}{f^2}}\right) - \frac{\eta f}{2}\left(1 - \sqrt{1 + \frac{4s^2}{f^2}}\right)$$

$$= \frac{f}{2}\left\{(1 - \eta) + (1 + \eta)\sqrt{1 + \frac{4s^2}{f^2}}\right\} \quad \text{..........(11)}$$

In the direction at right angles there will be an equivalent stress equal to $q - \eta p$ which comes equal to

$$\frac{f}{2}\left\{(1-\eta) - (1+\eta)\sqrt{1+\frac{4s^2}{f^2}}\right\} \quad \ldots\ldots\ldots(12)$$

These formulæ may also be derived from first principles as follows.

Suppose a rectangular block A B C D receives two tensile strains at right angles and a slide strain in the same plane. Under the combined strain the block assumes the position A D_2 C_2 B_1. Then, if A B $= x$, B C $= y$, and A C $= r$, and $x_1\, y_1\, r_1$ are the strained lengths, and $\underline{/D_1 A D_2} = \beta$

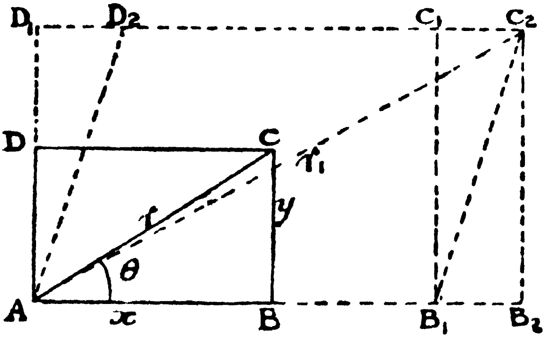

FIG. 10.—Combined Strains.

$$\text{Unital strain in direction } x = s_x = \frac{x_1 - x}{x}$$

$$\text{,, \quad ,, \quad ,, } \quad y = s_y = \frac{y_1 - y}{y}$$

$$\text{,, \quad ,, \quad ,, } \quad r = s_r = \frac{r_1 - r}{r}$$

$$\therefore \text{ We have } x_1 = x\,(1 + s_x) \ldots\ldots\ldots\ldots(1)$$

$$y_1 = y\,(1 + s_y) \ldots\ldots\ldots\ldots(2)$$

$$r_1 = r\,(1 + s_r) \ldots\ldots\ldots\ldots(3)$$

$$\therefore r_1^2 = r^2(1 + 2s_r + s_r^2)$$

$$= r^2(1 + 2s_r) \ldots\ldots\ldots\ldots(4)$$

Since squares of strains may be neglected.

$$\begin{aligned} \text{Now } r_1^2 = \mathrm{A\,C}_2^2 &= \mathrm{A\,B}_2^2 + \mathrm{C}_2\,\mathrm{B}_2^2 \\ &= (\mathrm{A\,B}_1 + \mathrm{B}_1\,\mathrm{B}_2)^2 + \mathrm{A\,D}_1^2 \\ &= (\mathrm{A\,B}_1 + \mathrm{D}_1\,\mathrm{D}_2)^2 + \mathrm{A\,D}_1^2 \end{aligned}$$

$$\begin{aligned} \text{Now } \mathrm{A\,B}_1 = x_1 &= x\ (1 + s_x) \\ \mathrm{A\,D}_1 = y_1 &= y\ (1 + s_y) \\ \mathrm{D}_1\,\mathrm{D}_2 = y_1\beta &= \beta\, y\,(1 + s_y) = \beta y \end{aligned}$$

since β is small and therefore $\beta \times s_y$ is of second order and therefore negligible.

$$\begin{aligned} \therefore\ r_1^2 &= \{x\,(1 + s_x) + \beta\, y\}^2 + \{y\,(1 + s_y)\}^2 \\ &= x^2\,(1 + 2\,s_x) + 2\,x\,y\,\beta + y^2\,(1 + 2\,s_y) \quad \ldots\ldots(5) \end{aligned}$$

neglecting all second powers of strains,

$$\text{but } r^2 = x^2 + y^2$$

$$\therefore\ r_1^2 = r^2 + 2\,x^2\,s_x + 2\,y^2\,s_y + 2\,x\,y\,\beta \quad \ldots\ldots\ldots(6)$$

∴ From (4)

$$r^2\,(1 + 2\,s_r) = r^2 + 2\,x^2\,s_x + 2\,y^2\,s_y + 2\,x\,y\,\beta$$

$$\text{or } s_r = \left(\frac{x}{r}\right)^2 s_x + \left(\frac{y}{r}\right)^2 s_y + \frac{x\,y}{r^2}\,\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Expressing this in terms of the angle θ we get

$$\underline{s_\theta = s_x \cos^2\theta + s_y \sin^2\theta + \beta \sin\theta\cos\theta} \quad \ldots\ldots(8)$$

Our next problem is to find the value of θ, for which the resultant unital strain s_θ is a maximum.

This occurs when $\dfrac{d\,s_\theta}{d\,\theta} = 0$

i.e. when

$$s_x\,.\,2\cos\theta\,(-\sin\theta) + s_y\,2\sin\theta\cos\theta + \beta\,(\cos\theta\cos\theta + \sin\theta\,[-\sin\theta]) = 0$$

i.e. when $-\,s_x \sin 2\,\theta + s_y \sin 2\,\theta + \beta\cos 2\,\theta = 0$

$$\sin 2\,\theta\,(s_x - s_y) = \beta\cos 2\,\theta$$

$$\text{or } \tan 2\,\theta = \frac{\beta}{s_x - s_y} \quad ..(9)$$

This gives two values of θ at right angles, and so we see that the directions of maximum strain are at right angles.

Now consider equation (8), reuniting and putting $1 = \cos^2\theta + \sin^2\theta$, we get

$$s_\theta\,(\cos^2\theta + \sin^2\theta) = s_x \cos^2\theta + s_y \sin^2\theta + \beta\sin\theta\cos\theta.$$

Dividing by $\cos^2\theta$, we get

$$s_\theta(1+\tan^2\theta) = s_x + s_y\tan^2\theta + \beta\tan\theta$$

$$\text{or } \tan^2\theta\,(s_y - s_\theta) + \beta\tan\theta + s_x - s_\theta = 0$$

$$\text{i.e. } \tan\theta = \frac{-\beta \pm \sqrt{\beta^2 - 4(s_y - s_\theta)(s_x - s_\theta)}}{2(s_y - s_\theta)}$$

For this to be real,

$$\beta^2 \text{ must be not } < 4(s_y - s_\theta)(s_x - s_\theta)$$

Now as s_θ increases, $4(s_y - s_\theta)(s_x - s_\theta)$ will increase, since the latter expression is equal to $4(s_\theta - s_x)(s_\theta - s_y)$

∴ The greatest value s_θ can have is such as to make

$$\beta^2 = 4(s_y - s_\theta)(s_x - s_\theta)$$

$$\text{i.e. } s_\theta^2 - s_\theta(s_x + s_y) + s_x s_y - \frac{\beta^2}{4} = 0$$

$$\text{i.e. } s_\theta = \frac{s_x + s_y \pm \sqrt{(s_x - s_y)^2 + \beta^2}}{2} \quad \text{......(10)}$$

Now consider the first case for which we have worked out the principal stress, viz. the combined stress due to a tensile or compressive stress f and a shear stress s. (NOTE.—This shear *stress* s must not be confused with the *strains* s_x, etc.) In this case if s_x = strain due to stress f, the only strain in direction y is the transverse strain due to s_x, i.e. $s_y = -\eta s_x$ (negative because the transverse strain is compressive).

Considering only the positive value in equation (10)

$$\therefore s_\theta = \frac{s_x(1-\eta) + \sqrt{s_x^2(1+\eta)^2 + \beta^2}}{2}$$

$$\text{Now } s_x = \frac{f}{E} \text{ and } \beta = \frac{s}{G}$$

Also $s_\theta = \frac{p_e}{E}$, where p_e is the equivalent principal stress due to considering the maximum strain, E and G being the Young's and shear moduli.

$$\therefore \frac{p_e}{E} = \frac{f(1-\eta)}{2E} + \frac{1}{2}\sqrt{\frac{f^2}{E^2}(1+\eta)^2 + \frac{s^2}{G^2}}$$

$$= \frac{f}{2E}\left\{(1-\eta) + \sqrt{(1+\eta)^2 + \frac{s^2 \cdot E^2}{f^2 \cdot G^2}}\right\}$$

$$\text{but } \frac{E}{G} = 2(1+\eta)$$

$$\therefore p_e = \frac{f}{2}\left\{(1-\eta) + (1+\eta)\sqrt{1 + \frac{4s^2}{f^2}}\right\} \quad \text{......(11)}$$

Now η is very nearly $\frac{1}{4}$ for steel.

$\therefore$ taking this value, we get

$$\underline{p_e = \frac{f}{2}\left(\frac{3}{4} + \frac{5}{4}\sqrt{1 + \frac{4s^2}{f^2}}\right)} \quad \ldots\ldots\ldots\ldots(12)$$

Comparing this with the corresponding equation (6) (p. 19), from considering the stress we see clearly the difference between the results from the two points of view.

NUMERICAL EXAMPLE.—*Consider the same problem as worked on p. 20.*

In that case $f = 3819$ lbs. per square inch.

$s = 2852$ „ „ „

$$\therefore p_e = \frac{3819}{2}\left(\frac{3}{4} + \frac{5}{4}\sqrt{1 + \frac{4 \times 2852^2}{3819^2}}\right)$$

$$= \frac{3819}{2}\left(\frac{3}{4} + \frac{5}{4}\sqrt{3 \cdot 23}\right)$$

$$= \frac{3819}{2}(\cdot 75 + 2 \cdot 246)$$

$$= \frac{3819}{2} \times 2 \cdot 996$$

$= 5722$ lbs. per square inch.

To get the inclination at which the maximum strain occurs return to equation (9) by which

$$\tan 2\theta = \frac{\beta}{s_x - s_y}$$

In this case we get

$$\tan 2\theta = \frac{\frac{s}{G}}{s_e(1+\eta)} = \frac{\frac{s}{G}}{\frac{f(1+\eta)}{E}} = \frac{s \cdot E}{f(1+\eta) \cdot G}$$

$$= \frac{s \cdot 2(1+\eta)}{f(1+\eta)} = \frac{2s}{f}$$

This is the same as in the case considering the principal stress, and so θ has the value as given before.

Similarly for normal stresses f_x and f_y at right angles we have

$$s_x = \frac{f_x}{E} - \frac{\eta f_y}{E}$$

$$s_y = \frac{f_y}{E} - \frac{\eta f_x}{E}$$

$$\beta = \frac{s}{G}$$

$$\therefore\ s_x + s_y = \frac{(1 - \eta)}{E}(f_x + f_y)$$

$$s_x - s_y = \frac{(1 + \eta)}{E}(f_x - f_y)$$

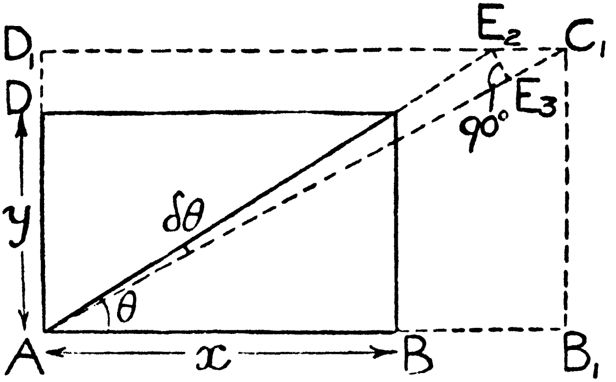

FIG. 11.

$\therefore$ From equation (10)

$$s_\theta = \frac{p_r}{E} = \frac{(f_x + f_y)(1 - \eta)}{2E} \pm \sqrt{\frac{(f_x - f_y)^2 (1 + \eta)^2}{4E^2} + \frac{s^2}{4G^2}}$$

$$= \frac{1}{2E}\left\{(f_x + f_y)(1 - \eta) \pm \sqrt{(f_x - f_y)^2 (1 + \eta)^2 + \frac{s^2 E^2}{G^2}}\right\}$$

$$= \frac{1}{2E}\{(f_x + f_y)(1 - \eta) \pm (1 + \eta)\sqrt{(f_x - f_y)^2 + 4s^2}\}$$

$$p_r = \frac{1}{2}\{(f_x + f_y)(1 - \eta) \pm (1 + \eta)\sqrt{(f_x - f_y)^2 + 4s^2}\} \quad ..(13)$$

* **Shear Strain equivalent to Two Direct Strains at Right Angles.**—We will now consider from first principles

in a similar manner the shear strain equivalent to two direct strains at right angles to each other.

Let a rectangular block A B C D, Fig. 11, become strained to the form A B_1 C_1 D_1, the extensions being shown to a greatly exaggerated scale, then we have

$$A\,D_1 = y\,(1 + s_y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

$$A\,B_1 = x\,(1 + s_x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

∴ Neglecting squares of strains

$$\begin{aligned} A\,C_1{}^2 &= A\,B_1{}^2 + A\,D_1{}^2 \\ &= x^2\,(1 + s_x)^2 + y^2\,(1 + s_y)^2 \\ &= x^2 + y^2 + 2\,x^2\,s_x + 2\,y^2\,s_y \quad \ldots\ldots\ldots(16) \end{aligned}$$

$$\begin{aligned} A\,E_2{}^2 &= A\,D_1{}^2 + D_1 E_2{}^2 = y^2\,(1 + s_y)^2 + \left\{\frac{x\,y\,(1 + s_y)}{y}\right\}^2 \\ &= (x^2 + y^2)\,(1 + 2\,s_y) \\ &= x^2 + y^2 + 2\,x^2\,s_y + 2\,y^2\,s_y \quad \ldots\ldots\ldots(17) \end{aligned}$$

$$\therefore A\,C_1{}^2 - A\,E_2{}^2 = 2\,x^2\,(s_x - s_y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(18)$$

$$\begin{aligned} \text{Now } A\,C_1{}^2 - A\,E_2{}^2 &= (A\,C_1 + A\,E_2)\,(A\,C_1 - A\,E_2) \\ &= 2\,A\,E_2\,.\,C_1\,E_3 \text{ approx.} \end{aligned}$$

$$\therefore C_1\,E_3 = \frac{2\,x^2\,(s_x - s_y)}{2\,A\,E_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

Further $E_2\,E_3 = C_1\,E_3 \tan\theta$ (very nearly; strictly

$\tan\theta - \delta\theta) = C_1\,E_3\,\frac{y}{x}$

$$\therefore E_2\,E_3 = \frac{x\,y\,(s_x - s_y)}{A\,E_2}$$

$$\begin{aligned} \text{but } \delta\theta &= \frac{E_2\,E_3}{A\,E_2}\left(= \frac{\text{arc}}{\text{radius}} \text{ approx.}\right) \\ &= \frac{x\,y\,(s_x - s_y)}{A\,E_2{}^2} \\ &= \frac{(s_x - s_y)\,x\,y}{(1 + 2\,s_y)\,(x^2 + y^2)} \\ &= \frac{(s_x - s_y)}{(1 + 2\,s_y)\left(\dfrac{x}{y} + \dfrac{y}{x}\right)} \quad \ldots\ldots\ldots\ldots\ldots(20) \end{aligned}$$

This will be a maximum when $\frac{x}{y} + \frac{y}{x}$ is a minimum, *i. e.* $\tan\theta + \cot\theta$ is a minimum.

$$\tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\frac{1}{2}\sin 2\theta} = \frac{2}{\sin 2\theta}$$

The minimum value of this $= 2$ when $\sin 2\theta = 1$, *i. e.*, $\theta = 45°$.

∴ Maximum shear strain occurs at 45° to direct strains.

$$\therefore \delta_\theta = \frac{s_x - s_y}{2(1 + 2s_y)}$$

$$= \frac{s_x - s_y}{2}(1 - 2s_y) \text{ approx.}$$

$$= \frac{s_x - s_y}{2} \text{ neglecting products of strains.}$$

There will be an equal angular distortion on the other diagonal.

∴ Total angular distortion = shear strain $= 2\delta\theta = (s_x - s_y)$.

∴ Equivalent maximum shear stress

$$= \text{shear strain} \times G = (s_x - s_y)\,G.$$

Now let the principal stresses be p and q

then
$$s_x = \frac{p}{E} - \frac{\eta q}{E}$$

$$s_y = \frac{q}{E} - \frac{\eta p}{E}$$

$$\therefore (s_x - s_y) = \frac{(p - q)(1 + \eta)}{E}$$

$$\therefore (s_x - s_y)\,G = \frac{(p - q)(1 + \eta)\,G}{E}$$

$$\therefore \text{ Equivalent maximum shear stress} = \frac{(p - q)(1 + \eta)\,G}{E}$$

Resilience.—The work done per unit volume of a material in producing strain is called *resilience.* Consider the case of a body subjected to a simple tensile strain. In going from the point A to the point B, Fig. 12, very near to it, the average stress acting is f. Therefore, if $AB = x$, the work done by the force f in straining the material from the point A to the point B will be equal to $f \times x$. Now, if x is the increase in unital strain and f is the intensity of stress, the volume of material acted upon is unity. Now, AB is assumed to be very small, and $f \times x$ is equal to the area of the shaded portion of the stress-strain curve.

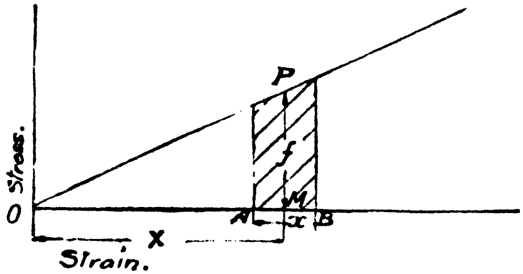

FIG. 12.—Resilience

Therefore, the resilience is equal to the area of the stress-strain curve up to the point M,

$$i.e.\ \text{resilience} = \text{area of } \Delta\, PMO$$

$$= \frac{1}{2} f \times X$$

$$\text{Now, } \frac{f}{X} = \text{Young's modulus} = E$$

$$\therefore\ X = \frac{f}{E}$$

$$\therefore\ \text{resilience in tension} = \frac{f^2}{2\,E}$$

$$\text{similarly in shear the resilience} = \frac{s^2}{2\,G}$$

where s is the shear stress.

Stresses and Strains due to Sudden or Dynamic Loading.—If a load is applied suddenly to a structure, vibration will ensue, and the strain—and thus the stress—will reach twice the value which would occur if the load were gradually applied.

This will be made clear from considering a diagram, Fig. 13 (1), where the force is plotted against the strain. We have seen that, with gradual loading of an elastic body, the curve representing the relation between the strain and the load in direct stress is represented by a straight line A D, the area below the line giving the work done up to a given point.

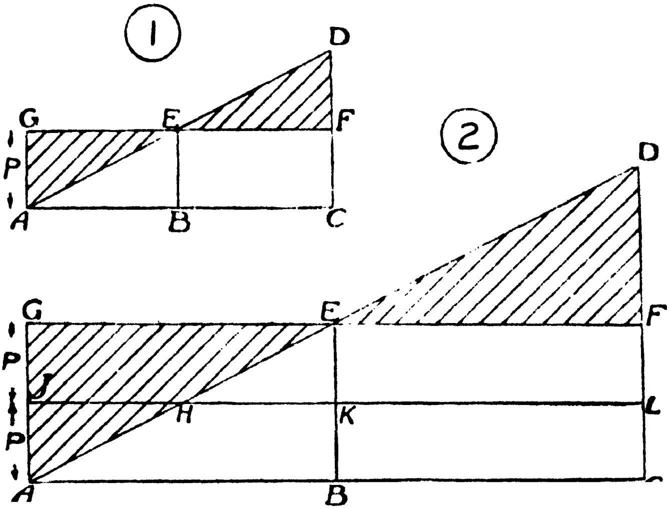

FIG. 13.—Sudden or Dynamic Loading.

Now let A G represent a force P; then when the strain gets to the point B, the work done by the force will be equal to the area of the rectangle A B E G, whereas the work done in straining the material is only equal to the area of the triangle A B E, so that there is an amount of work equal to the area of the triangle A E G still available for causing increased strain. The strain therefore increases until the area of the triangle E F D is equal to that of the triangle A E G. It is clear that A C = 2 A B, or *that the strain—and thus the stress—is twice that in the case of gradual loading.*

If a force is suddenly reversed from − P to + P, then the total strain and stress will be the same as that due to a sudden load of 2 P, and again when the strain reaches the point B, Fig. 13 (2), there will be an amount of work represented by the area of the triangle A E G still available for causing strain, which therefore continues to the point C. Thus the maximum tensile strain will be equal to H L. If the loading were gradual the strain would be H K, and as H L = 3 H K, we see that *a load suddenly reversed causes three times the strain and stress which occur if such reversal takes place slowly.*

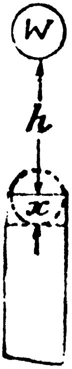

FIG. 14.

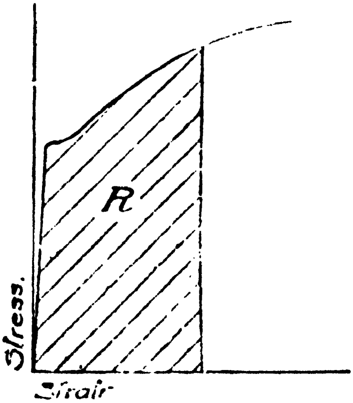

FIG. 15.

In each of these cases the *additional* strain or stress which occurs is equal to the amount of variation. Such additional stress has been called the *dynamic increment*, and we therefore see that *the equivalent gradual stress due to a sudden or dynamic stress f_d which varies by an amount v is given by $f_d + v$.*

Strain and Stress due to Impact.—Suppose a weight W falls from a height h on to a structure and let the deformation or strain in the direction of h be x, Fig. 14. Then the work done by the weight is equal to W $(h + x)$. Now this work is absorbed in straining the structure. Consider first the case in which the resulting strain is within the elastic limit. The work done in such case is equal to the volume multiplied by the resilience. We have shown that in tension or compression the resilience is equal to $\frac{f^2}{2\,E}$ and therefore in this case we get $W(h + x) = \frac{\text{volume} \times f^2}{2\,E} = \frac{V\,f^2}{2\,E}$. Then if x is negligible compared with h

we have $$W \times h = \frac{V\,f^2}{2\,E}$$

or $$f = \sqrt{\frac{2\,E\,W\,h}{V}}$$

If the weight strikes with a velocity v,

$$h = \frac{v^2}{2\,g}$$

or $$f = \sqrt{\frac{2\,E\,.\,W\,v^2}{2\,g\,V}} = v\sqrt{\frac{E\,W}{g\,V}}$$

We will consider resilience in bending and torsion when dealing with beams and shafts.

Strain beyond Elastic Limit.—If the strain is beyond the elastic limit, it follows, from the reasoning given on p. 33, that the work done per unit volume in straining is equal to the area below the stress-strain curve. If this area is R, Fig. 15, then we have $R = W\,h$ or $\frac{W\,v^2}{2\,g}$

From this the stress can be found.

Numerical Example.—*A bar of ½-inch diameter stretches ⅛ inch under a steady load of 1 ton. What stress would be produced in the bar by a weight of 150 lbs. which falls through 3 inches before commencing to stretch the rod—the rod being initially unstressed and the value of E taken as 30×10^6 lbs. per square inch. (B.Sc. Lond.)*

Area of bar ½″ diam. = ·196 sq. in.

$$\therefore \text{Stress under load of one ton} = \frac{1}{\cdot 196} \text{ tons per sq. in.}$$

$$= \frac{2240}{\cdot 196} \text{ lb. per sq. in.}$$

$$\therefore \text{Strain} = \frac{\text{Stress}}{E} = \frac{2240}{\cdot 196 \times 30 \times 10^6}$$

Now $\frac{1}{8}''$ = strain × original length

$$\therefore \text{ Original length} = \frac{\frac{1}{8}}{\text{Strain}} = \frac{\cdot 196 \times 30 \times 10^6}{2240 \times 8}$$

$$\therefore \text{ Volume} = \text{length} \times \text{area of section.}$$

$$= \frac{\cdot 196 \times \cdot 196 \times 30 \times 10^6}{8 \times 2240}$$

$$= 64\cdot 33 \text{ cub. ins.}$$

Work done by 150 lbs. in falling 3 inches

$$= 3 \times 150 = 450 \text{ in. lbs.}$$

$$\therefore \frac{64\cdot 33 \times f^2}{2\,E} = 450$$

$$f = \sqrt{\frac{900\,E}{64\cdot 33}}$$

$$= \sqrt{\frac{900 \times 30 \times 10^6}{64\cdot 33}}$$

$$= \underline{20{,}480 \text{ lbs. per sq. in.} \quad \textit{Ans.}^*}$$

Temperature Stresses.—Suppose a bar of length l is heated $t°$ F. and a is the coefficient of expansion. Then, unless prevented, the length of the bar will become $l\ (1 + a\,t)$, *i. e.* the increase in length will be $a\,t\,l$.

If the bar is rigidly fixed so that this expansion cannot take place, then there will be in the bar a strain equal to $a\,t\,l$, and the unital strain will be $\frac{a\,t\,l}{l} = a\,t$.

This strain will produce a compressive stress of $a\,t \times E$ where E is Young's modulus.

Now for mild steel $a = \cdot 00000657$ per degree Fahrenheit, and E = 13,000 tons per square inch.

$$\therefore \text{ The stress per } °\text{F.} = \cdot 00000657 \times 13{,}000$$

$$= \cdot 0854 \text{ tons per square inch.}$$

Taking a range of temperature of 120° F., the stress due to temperature = 120 × ·0854 = 10·25 tons per square inch. This is more than the safe stress for mild steel, so that the importance of designing so that the expansion may take place becomes quite evident.

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* This problem could be solved if E were not given; it would be found to cancel out.

* **Heterogeneous Bars under Direct Stress.**—If a bar, composed of two different materials—such as steel and concrete, or steel and copper—firmly connected to each other, be subjected to a pull or a thrust, the two materials must be *strained* by equal amounts, and since the values of Young's modulus for the two materials are different the *stresses* in the two materials will be different.

Suppose one material has a cross-sectional area A and Young's modulus E, the resulting stress being f; and let the corresponding quantities for the other material be A_1, E_1, f_1.

Then, if under a pull or thrust P the unital strain is x, we have

$$x = \frac{f}{E} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$x = \frac{f_1}{E_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\text{and } P = Af + A_1 f_1 \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

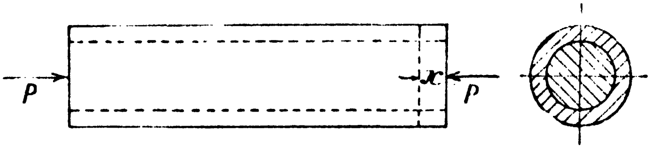

FIG 16.

Af and $A_1 f_1$ being the loads carried by each of the materials.

From (1) and (2) $f_1 = E_1 x = \frac{E_1 f}{E}$

$$\therefore P = f\left(A + \frac{E_1 A_1}{E}\right) \quad \ldots\ldots\ldots\ldots\ldots (4)$$

$$\text{or } f = \frac{P}{A\left(1 + \frac{E_1 A_1}{E A}\right)} \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Now if a new bar is taken wholly of the first material of such area A_2 that the stress under a load P is the same as that in the compound bar, we have

$$f = \frac{P}{A_2}$$

$$\text{or } A_2 = A\left(1 + \frac{E_1 A_1}{E A}\right) \ldots\ldots\ldots\ldots\ldots (6)$$

This quantity A_2 may be called the *equivalent area of homogeneous material*, and the consideration of this problem has become in recent years much more important on account of the progress made in reinforced concrete construction. Returning to the general problem we see that

$$f_1 = \frac{P}{A_1\left(1 + \frac{E\,A}{E_1\,A_1}\right)} \text{..................} (7)$$

The load carried by the first material then comes equal to

$$f\,A = \frac{P}{1 + \frac{E_1\,A_1}{E\,A}} \text{......................} (8)$$

and that carried by the second comes equal to

$$f_1\,A_1 = \frac{P}{1 + \frac{E\,A}{E_1\,A_1}} \text{......................} (9)$$

Since these are not the same there will be an adhesive stress resisting force tending to make one material slide relatively to the other.

This adhesive stress may be computed as follows assuming that the load is applied uniformly.

$$\text{Load per sq. in.} = \frac{P}{(A + A_1)}$$

$\therefore$ Load actually distributed to area A

$$= A\,\frac{P}{(A + A_1)}$$

$$\text{Load carried} = \frac{P}{1 + \frac{E_1\,A_1}{E\,A}} \text{.................(from 8)}$$

$\therefore$ Difference = load carried by adhesion calling $\frac{E}{E_1} = m$

$$= \frac{P}{1 + \frac{A_1}{m\,A}} - \frac{P.A}{A + A_1}$$

$$= P\left\{\frac{m\,A}{m\,A + A_1} - \frac{A}{A_1 + A}\right\}$$

$$= \frac{P\,A\,A_1\,(m - 1)}{(A + A_1)\,(A_1 + m\,A)}$$

$$= \frac{f_1\,A\,A_1\,(m - 1)}{(A + A_1)}$$

NUMERICAL EXAMPLE. — *A reinforced concrete column (Fig. 17) for which* $m = 15$, *is 20 inches square and has 4 1¼" steel rods embedded in it. Find the load on the column when the stress in the concrete is 450 lbs. per sq. in. and the adhesive force.*

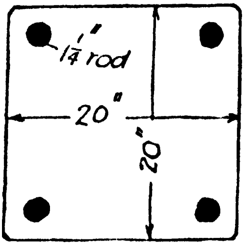

FIG. 17.

$$\text{In this case } A = 4 \times \frac{\pi}{4} \times 1{\cdot}25^2 = 4{\cdot}91 \text{ in.}^2$$

$$A_1 = 20 \times 20 - 4{\cdot}91 = 395 \text{ nearly}$$

$$\therefore P = f_1 \, . \, A_1 \left(1 + \frac{E\,A}{E_1\,A_1}\right) \ldots\ldots\ldots\ldots\ldots \text{(from 7)}$$

$$= 450 \times 395 \left(1 + \frac{15 \times 4{\cdot}91}{395}\right)$$

$$= 210{,}870 \text{ lb. nearly}$$

$$\text{Adhesive force} = \frac{450 \times 4{\cdot}91 \times 395 \times 14}{400} = 30{,}540 \text{ lbs. nearly}$$

CHAPTER II

THE BEHAVIOUR OF VARIOUS MATERIALS UNDER TEST

Properties other than Elastic.—In addition to the elastic properties of materials there are other strength properties which are of very great importance in the practical use of the materials.

Ductility is the property of a material which allows it to be worked without cracking; the strict use of the term refers to the capacity for being drawn out which a ductile metal possesses.

Malleability is the property which allows a material to be hammered out and is very similar to ductility.

Brittleness is lack of ductility or malleability.

Hardness may be defined as the power of a material to resist denting by another material. (For tests for hardness see pp. 396–404.)

The above properties are all relative ones and vary with the same material according to the treatment which it receives; thus by "tempering" a metal we harden it and by "annealing" it we soften it or render it more ductile, and some metals are hardened by plunging them into water when heated, whereas others are annealed by the same process.

With reference to ductility it is important to remember that so long as a metal maintains its elasticity it has no ductility, *i. e.* a metal which possesses ductility cannot exhibit the fact until the yield point has been reached. In our calculations for the strength of various details we shall base nearly all our formulæ on the assumption that our material is elastic, and so we must not expect the formulæ to hold after the elastic limit has been reached. This is a point of very great importance.

Apart from the convenience in the manufacture of articles which ductility gives, it has considerable value from the point of view of safety and strength, because a material does not lose its strength when it first starts drawing out, and the yield may either give us timely warning of excessive loading or, in the case of steam boilers and like fluid-retaining devices, the yielding may actually remove the excessive pressure. As we shall see later, however, the effect of taking a material beyond its yield point is to harden it.

The usual test for ductility is the elongation in fracture by tension.

*** The Cause of Failure of Materials under Stress.**—In recent years a very large amount of attention has been given to the question of the cause of failure of materials under test, and it is doubtful if the vital importance of this problem has been fully realised by practical engineers. As we shall see later, however, the choice of safe working stresses really depends in a large measure upon the view taken as to which of the various theories is correct. If we consider the question carefully we shall see that failure cannot occur by compression only; if a material be prevented from escaping laterally, no amount of compression can rupture it. Even a fluid like water will resist a compression stress of very great magnitude if the vessel containing it is strong enough to resist failure by tension or shear. The late M. Armand Considère showed experimentally that concrete could not be crushed when given adequate lateral support, and he also proved that the very brittle material glass could be bent cold without fracture when placed in a liquid under great hydraulic pressure. Marble has been bent without fracture by Professor F. D. Adams of McGill University when placed in steel cylinders and compressed, and Professor Ira H. Woolson crushed a cylinder of concrete encased in steel into the form shown in Fig. 18, and yet when the encasing cylinder of steel was removed the strength of the concrete was found to be not appreciably different from that of concrete which had not been similarly treated. The question therefore resolves itself whether tension or shear is the cause of failure, and we have reason to believe that in ductile materials such as mild steel failure occurs by shear and in brittle materials such as cement or concrete by tension. We will return to this after considering the various theories of failure; there are four principal theories which we will consider.

1. PRINCIPAL STRESS OR RANKINE THEORY.—According to this theory, which was adopted by the great Glasgow professor, Rankine, the failure occurs when the maximum principal stress exceeds a certain value. We have seen (p. 19) that for a normal or direct stress f and shear stress s the principal stress is given by the relation

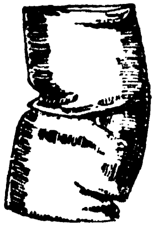

FIG. 18.

$$p = \frac{f}{2}\left(1 + \sqrt{1 + \frac{4s^2}{f^2}}\right) \quad \ldots\ldots\ldots\ldots (1a)$$

$$\text{or } p = \frac{f}{2} + \frac{1}{2}\sqrt{f^2 + 4s^2} \quad \ldots\ldots\ldots\ldots (1b)$$

and the inclination θ of this stress to the normal stress and to the shear stress is given by the relation $\tan 2\theta = \frac{2s}{f}$.

This stress p is the simple normal stress (tension or compression) equivalent in effect to the combined normal and shear stresses.

In the limiting case in which the direct stress f is zero we get $p = s$ and $\tan 2\theta =$ infinite, *i.e.* $\theta = 45°$, i.e. *a shear stress is equivalent to a normal stress of the same intensity and is at 45° to it, or the shear and tensile strengths of the material should be equal.*

2. Principal Strain or St. Venant Theory.—According to this theory, which was favoured by the great French elastician after whom it is named, the failure occurs when the maximum principal strain exceeds a certain value. We have seen (p. 29) that for a normal stress f and a shear stress s the equivalent principal stress is given by the relation

$$p_e = \frac{f}{2}\left\{(1-\eta) + (1+\eta)\sqrt{1+\frac{4s^2}{f^2}}\right\} \ldots\ldots (2)$$

and taking $\eta = \frac{1}{4}$

$$p_e = \frac{f}{2}\left\{\frac{3}{4} + \frac{5}{4}\sqrt{1+\frac{4s^2}{f^2}}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (3a)$$

$$\text{or } p_e = \frac{3f}{8} + \frac{5}{8}\sqrt{f^2+4s^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3b)$$

The inclination of this principal stress is the same as in the previous case.

In the limiting case in which the direct stress f is zero we get $p = \frac{5s}{4}$, i.e. *a shear stress is equivalent to a normal stress of four-fifths of the shear stress, or the shear strength of a material is four-fifths of the tensile strength.*

3. Equivalent Shear Stress or Guest or Tresca Theory.—According to this theory, which is associated with the name of Mr. J. J. Guest, who was one of the first to carry out careful experiments upon the subject, and is usually attributed to Tresca, failure occurs by sliding of the particles over each other, *i. e.* by shear. From p. 20 we get

$$\text{Equivalent shear stress} = \sqrt{\frac{f^2}{4} + s^2} \ldots\ldots\ldots (4)$$

and acts at an angle of 45° to the normal stress.

To compare a simple shear with a simple tension or compression by this formula we put $s = o$ and we get

$$\text{Equivalent shear stress} = \frac{f}{2}$$

i. e. *a shear stress is equivalent to a normal stress of half its magnitude or the shear strength of a material is one-half of its tensile strength.*

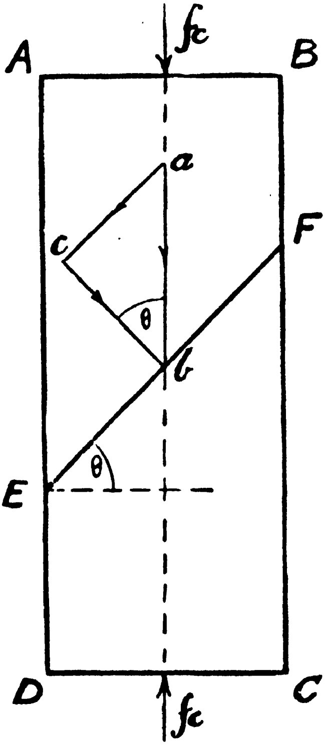

FIG. 19.—Navier's Theory.

It is interesting to note that as shown on p. 32 the equivalent shear stress comes the same whether worked from the point of view of stress or of strain, and so there is logical support for this theory.

4. SLIDING WITH INTERNAL FRICTION OR NAVIER THEORY. —This theory deals with materials subjected to compressive stresses and attempts to explain the fact that short cylinders

of brittle material usually fail by sliding or shearing along a line E F, Fig. 19, inclined at an angle θ from 55° to 65°, on the ground that the particles are capable of exerting frictional resistances.

Consider a short column A B C D of unit sectional area subjected to an ultimate compressive force u_c, which causes failure, and consider the forces across a section E F, the ultimate or breaking shear stress in the material being u_s.

The force u_c acting along the section E F can be resolved into shear and normal components $a\,c$, $c\,b$ respectively, equal to $u_c \sin\theta$ and $u_c \cos\theta$.

If μ is the angle of friction for the material, the normal component $c\,b$ causes a frictional resistance equal to $\mu . c\,b$, i.e. $\mu . u \cos\theta$.

Just before failure the shearing force acting upon E F = u_s × area of section.

$$= \frac{u_s \times \text{normal section}}{\cos\theta} = \frac{u_s}{\cos\theta}$$

(because normal section is of unit area).

When therefore failure is about to take place—

Total force causing failure $= a\,c = u_c \sin\theta$ equals

Total force resisting failure $= \mu u_c \cos\theta + \dfrac{u_s}{\cos\theta}$

$$\text{i.e. } u_c (\sin\theta - \mu\cos\theta) = \frac{u_s}{\cos\theta}$$

$$\text{or } u_c = \frac{u_s}{\cos\theta\,(\sin\theta - \mu\cos\theta)} \quad \ldots\ldots(1)$$

Regarding u_s and μ as constant we now wish to find the value of θ which will make u_c as small as possible; this value of θ will be that at which failure will occur. u_c will be as small as possible when $\cos\theta\,(\sin\theta - \mu\cos\theta)$ is as large as possible.

$$\text{Let } y = \cos\theta\,(\sin\theta - \mu\cos\theta)$$

$$= \frac{\sin 2\theta}{2} - \mu\cos^2\theta$$

$$\text{For a maximum } \frac{dy}{d\theta} = 0$$

$$\text{i.e. } \frac{2\cos 2\theta}{2} - \mu\, 2\cos\theta\,(-\sin\theta) = 0$$

$$\text{i.e. } \cos 2\theta + \mu \sin 2\theta = 0$$

$$\mu = -\frac{\cos 2\theta}{\sin 2\theta} = -\cot 2\theta \ldots (2)$$

If $\mu = \tan\phi$, ϕ being the angle of friction, this gives $\cot 2\theta = -\tan\phi$

$$\text{or } 2\theta = 90^\circ + \phi$$

$$\text{or } \theta = 45^\circ + \frac{\phi}{2} \ldots\ldots\ldots\ldots (3)$$

In support of this theory experiments made by Bouton (Washington University, 1891) may be quoted as follows—

Material.	Number of Tests.	Observed Value of ϕ (degrees).	Observed Value of θ (degrees).	$45^\circ + \frac{\phi}{2}$
Cast iron	24	20·6	54·8	55·3
,, (different kind)	24	16·9	55·0	53·4
Limestone	4	33·4	62·2	61·7
Asphalte paving	3	27·3	59·7	58·6
Milwaukee brick	4	27·0	58·2	58·5

Ratio of shear to compressive strength on Navier theory.—From equation (1) we have

$$u_s = u_c \cos\theta\,(\sin\theta - \mu\cos\theta)$$

Now put in this $\mu = \tan\phi = -\cot 2\theta = -\dfrac{\cos 2\theta}{\sin 2\theta}$

$$\text{This gives } \frac{u_s}{u_c} = \cos\theta\left(\sin\theta + \frac{\cos 2\theta}{\sin 2\theta}\cos\theta\right)$$

$$= \cos\theta\left\{\sin\theta + \frac{(\cos^2\theta - \sin^2\theta)\cos\theta}{2\sin\theta\cos\theta}\right\}$$

$$= \cos\theta\left\{\sin\theta + \frac{(\cos^2\theta - \sin^2\theta)}{2\sin\theta}\right\}$$

$$= \cos\theta\left\{\frac{2\sin^2\theta + \cos^2\theta - \sin^2\theta}{2\sin\theta}\right\}$$

$$= \frac{\cos\theta}{2\sin\theta}\left(\sin^2\theta + \cos^2\theta\right) = \frac{\cos\theta}{2\sin\theta}$$

$$= \tfrac{1}{2}\cot\theta \ldots\ldots\ldots\ldots (4)$$

Taking $\theta = 60$ for masonry, this would give

shear strength = ·289 compressive strength,

but the most recent experiments on shear strength of concrete (see p. 79) indicate that this is too low. Shear tests are very difficult to make without introducing bending, which tends to give the shear strength too low.

An interesting theory which may be regarded as a modification of Navier's theory is outlined by Mr. H. Kempton Dyson in a paper read before the Concrete Institute, December 1914.

The first three theories have been very fully tested experimentally in recent years by Messrs. Guest, Hancock, Scoble, C. A. M. Smith, Mason and Turner,* and the result appears to be that the shear stress theory is most reliable for ductile materials while the strain theory is most reliable for brittle materials.

Professor Ewing and Mr. Rosenhain have found by a microscopic examination of the crystals of a ductile metal under strain that beyond the yield point lines of slip are developed in the crystals, thus proving that the failure or yield is a slipping or shear one. Lüder's lines (p. 54) are also indications that in ductile metals the failure is by shear.

It is possible that ductility is a property of shear strength; if a material is weaker in shear than in tension the shear causes the failure and slippage occurs before the tensile strength is reached, thus giving rise to ductility. If the material is relatively stronger in shear than in tension the material breaks before slippage occurs and thus cannot exhibit ductility.

We will now consider the properties of various materials.

CAST IRON

We will deal with cast iron first because it is a brittle material and behaves differently under test from most other metals.

The strength of cast iron varies considerably with its composition, but like all brittle metals it is relatively weak in tension and strong in compression.

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* These experiments will be found fully described and discussed in various articles and letters in *Engineering* for 1909 and 1910.

Fig. 20 shows the stress-strain diagrams for cast iron in tension and compression, the results of the two tests being plotted on one diagram.

It is clear from this diagram that the stress is never strictly proportioned to the strain in tension; this has an important bearing upon the strength of cast-iron beams (see p. 209).

The compression diagram is not continued to failure as the failure would take place by buckling and injure the instrument for measuring the strain. When a cast-iron bar fails in tension, it breaks off "short," *i. e.* it does not produce a waist as indicated in Fig. 2 for mild steel.

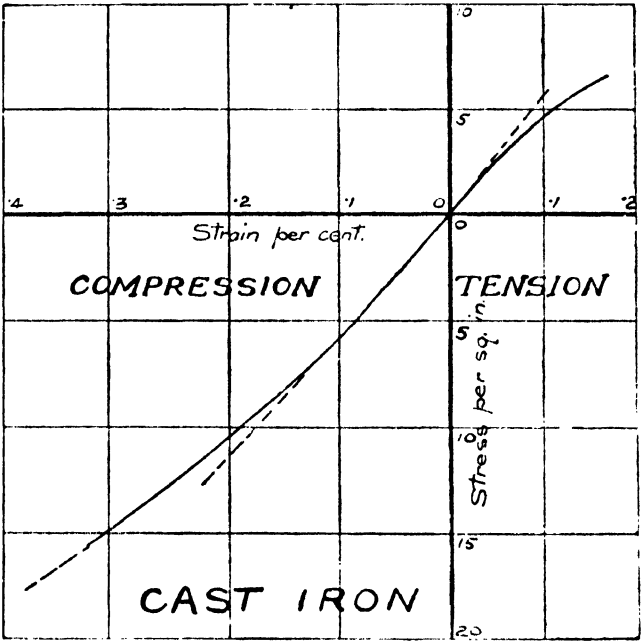

Fig. 20.—Stress-strain Curves for Cast Iron.

When compression tests are made on cylinders which are so short that buckling effects are practically eliminated, the failure takes place by sliding diagonally as indicated in Fig. 21, and for shorter specimens still cracks sometimes develop which split off the outside portion leaving two inverted cones. Some observations on this kind of failure for concrete will be found on p. 69 and apply to cast iron. The difficulty in testing cast iron in compression in very short lengths is that it is so strong that difficulties arise as to the strength of the testing machine.

The tensile strength of cast iron varies from about 7 to 15 tons per sq. in. in extreme cases, but more usually from 8 to 11 tons per sq. in. Figures for the compressive strength show more variation; this is probably due to the fact that the size of the test piece, both as regards its length and breadth, affects the result.

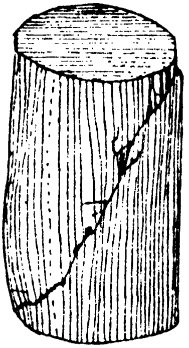

FIG. 21 —Compression Failure of Cast Iron.

The following results of tests made upon ½ in. cubes by the American Foundrymen's Association show that specimens cut from bars of small cross section give much higher results than those from large.

Cross Section of bar from which Cubes were cut.	Crushing strength in tons per sq. in. for cubes cut from				
	Middle half inch.	First half inch.	Second half inch.	Third half inch.	Fourth half inch.
½ × ½	69·0	—	—	—	—
1 × 1	44·5	49·8	—	—	—
1½ × 1½	37·0	39·4	37·0	---	—
2 × 2	32·2	38·9	34·6	---	—
2½ × 2½	31·9	35·4	32·3	31·9	—
3 × 3	28·6	32·5	30·1	28·7	—
3½ × 3½	28·4	30·5	29·6	28·8	28·4
4 × 4	25·4	29·4	27·4	26·6	25·4

STRENGTH OF CAST IRON BEAMS.—Cast-iron when tested in bending shows an apparently greater strength than when tested in pure tension. This is due to the fact, as explained in greater length on p. 209, that the ordinary formula for beams is not strictly applicable for cast iron. The ratio of $\frac{\text{calculated breaking stress from bending}}{\text{tensile breaking stress}}$ is usually about 1·5 for rectangular sections of depth twice the breadth; it increases for round and square sections arranged diagonally and may rise to 3; the ratio becomes nearly 1 for I sections with a thin web.

The following figures give the mean of a large number of tests made by Kirkaldy for the same kind of iron—

Tensile breaking stress = 11 tons per sq. in.
Compression „ = 54 „ „
Calculated bending „ = 17 „ „

The early writers often called the breaking stress calculated from bending tests the "modulus of rupture," but the term is not to be recommended; bending breaking stress is better.

Effect of Temperature on Strength of Cast Iron.—The strength of cast iron increases slightly as the temperature is raised until about 900° F. is reached; it then diminishes rapidly, until at 1100° F. the strength is reduced by nearly 50 per cent. and at 1400° F. by about 75 per cent.

Other properties of cast iron are tabulated on p. 83.

Malleable Cast Iron.—Cast iron is rendered malleable by surrounding the casting with hæmatite or manganese dioxide and exposing it to red heat for many hours, depending upon the size of the casting. The result of the process is to decarbonise the iron and render it similar to mild steel. Some useful information on the subject, especially from the point of view of strength, is given by Mr. C. H. Day in the *American Machinist* of April 21, 1906.

The following results of tests of Mr. Ashcroft are quoted from Vol. CXVII. *Proc. I. C. E.*

	Tons per sq. in.			Elongation. % on 10 in.
	Breaking Stress.	Elastic Limit.	Young's Modulus.	
Tension . .	20·6	8·94	11,620	2·8
Compression	21·6	—	10,240	—
Bending . .	28·8	—	12,330	—
Torsion . .	26·8	—	Rigidity Modulus. 4,120	—

Stanford, in *Trans. Am. Soc. C. E.*, 1895, gives as the mean result of forty-two tests in tension an elongation of 6·61 per cent. and an ultimate stress of 22 tons per sq. in.

Similar results are given by Mr. Day.

STEEL, WROUGHT IRON, AND OTHER DUCTILE METALS

Real and Apparent Maximum Tensile Strength.—We have shown already, on p. 5, a stress-strain diagram for mild steel in tension and pointed out that the last portion D E of the diagram was usually inaccurate and of little commercial importance. The diagram slopes back because the load can be reduced as the area diminishes and the stress still be sufficient to cause fracture. Now this diagram can be corrected if its form is determined very carefully and the areas at the various points are measured.

Taking any point *a*, Fig. 22, on the curve before the specimen began to draw down, we find *bc* by the relation $bc = \frac{ab \times \text{extended length}}{\text{original length}}$. This is based on the assumption that the volume keeps constant; so that

$$\frac{\text{original area} \times \text{original length}}{\text{reduced area} \times \text{extended length}} = 1$$

$$\frac{ab \times \text{extended length}}{\text{original length}} = \frac{ab \times \text{original area}}{\text{reduced area}}$$

The method cannot be used beyond the point at which the waist begins to form.

Taking any point *f* beyond the point of drawing down,

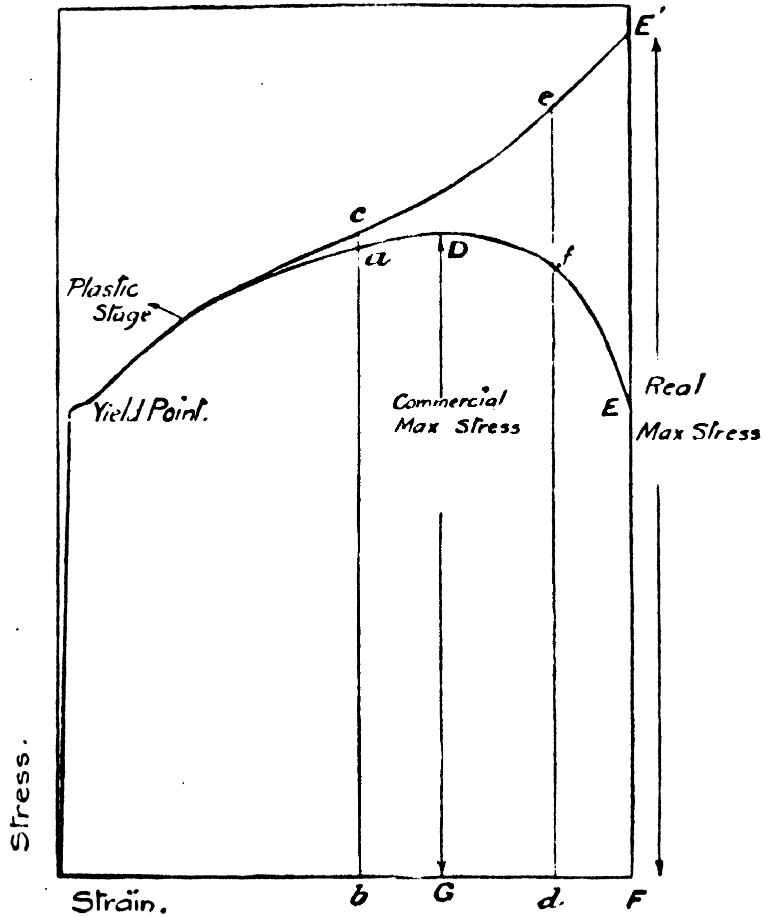

FIG. 22.

Fig. 22, we find $de = \dfrac{df \times \text{original area of bar}}{\text{actual area}}$; and by doing this for a number of points we get the corrected curve *c e* E′, then F E′ gives the real maximum stress as opposed to G D, which is the apparent or commercial maximum stress. It is very difficult to get points *f* accurately.

It thus appears that the actual maximum stress at failure is considerably more than usually measured. Upon the Guest theory that failure occurs by shear, the tensile strength should be twice the shear strength, but ordinary commercial tests indicate that it is about 1¼ times; it is probable that if the true tensile strength were compared instead of the apparent or commercial strength, the agreement would be more in favour of the Guest theory.*

The common form of fracture of a ductile metal is shown in Fig. 23 and consists of a kind of crater, the angle of the sides being approximately 45°. This strengthens the theory that failure is by sliding or shear.

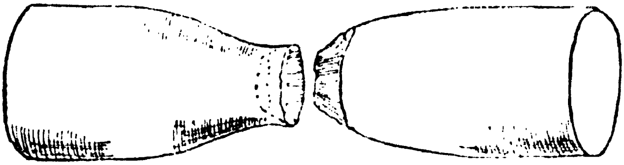

FIG. 23.—Tension Failure of Ductile Metal.

Lüder's Lines.—When a highly polished specimen or one provided with a very thin layer of scale is tested in tension or compression, lines at about 45° to the axis of the specimen and of spiral form in the case of round sections are found to develop directly the yield point is reached. This was first noticed by Lüder and supports the theory that the yield is really due to diagonal shearing.

Fig. 24 shows these lines for thin mild-steel tubes in compression and were given in a paper by Mr. W. Mason, M.Sc., of Liverpool University.†

Percentage Increase in Length and Decrease in Area.—The percentage elongation of the specimen and the decrease in area are usually regarded as reliable tests of ductility of the material, but it is clearly useless to specify the percentage elongation unless the diameter and the

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* See a paper by Professor Carus Wilson, *Proc. Roy. Soc.* 1890.

† *Proc. Inst. M. E.* 1909.

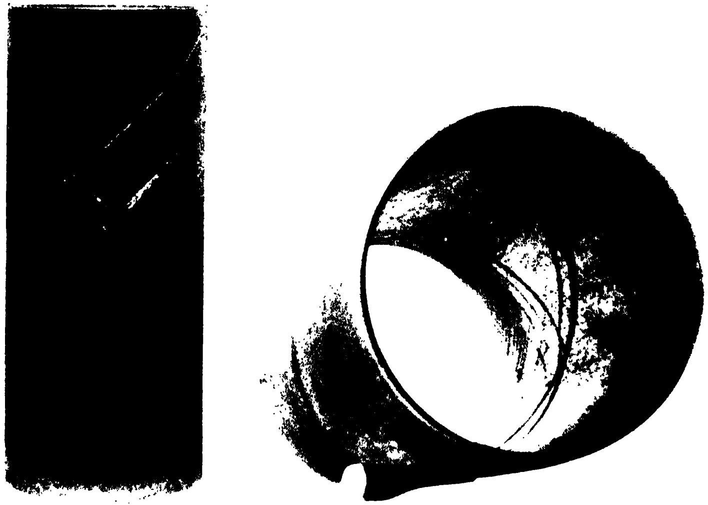

FIG. 24.—Lüder's Lines.

[*To face page* 54.

original length are both specified, because most of the extension occurs in the centre portion where the specimen draws down.

This question has been treated very fully by Professor Unwin in Vol. CLV., *Proc. Inst. C. E.*, and the following figures are taken from his paper as indicative of the general results—

ROTHERHAM STEEL BOILER PLATE (area ·5830 in.^{2})

Extension in inches in each half inch.

Number from Fracture	7	6	5	4	3	2	1	Fracture	1	2	3	4	5	6	7	8	9	10
Extension	·07	·08	·10	·10	·11	·13	·18	·47	·19	·13	·12	·12	·11	·11	·10	·09	·10	·09

Percentage elongation and different gauge lengths.

Gauge Length (inches)	2	4	6	8	10
% Elongation . . .	48·5	36·0	30·9	27·6	25·9

He suggests the formula—

$$\% \text{ elongation} = 100\left(\frac{c\sqrt{A}}{l} + b\right)$$

where A is the original area in sq. in.
l is the gauge length in inches
b and c are constants for a given material.

The following values of c and b are given by Professor Unwin—

Metal.	c	b
Mild steel	70	18
Gun metal (cast) . . .	8·3	10·6
Rolled brass	101·6	9·7
Rolled copper	84	·8
Annealed copper . . .	125	35

By the formula we can obtain the probable percentage extension on any length of any other metal if that on two specified lengths are known.

The percentage contraction in area is not so commonly specified now as formerly, but some authorities consider that the reduction of area is a better criterion of ductility.

TWO WAISTS IN A TENSION SPECIMEN.—It happens very occasionally that two waists form in a tension specimen, failure taking place at the one which draws down most rapidly. This will affect the elongation and the test should be discarded.

FRACTURE NEAR ONE SHOULDER OF SPECIMEN.—If fracture occurs near one shoulder of the specimen (see Fig. 167) the elongation will be less than normal owing to the effect of the shoulder, and such a test should be discarded.

Effect of Abrupt Change of Section upon the Tensile Strength.—The effect of an abrupt change of section such as in a screw thread or a sharp-edge groove does not have a very marked effect upon the ultimate tensile stress of a material when tested in the ordinary way although it does affect the ductility. As we shall show later, however (p. 91), it has considerable effect in tests by repeated loading.

The sharpness of the groove will have a weakening effect, but the presence of the larger area near it will have a strengthening effect.

Fig. 25 shows the results of some tests by Sir Benjamin Baker * which are interesting; the breaking stresses in tons per sq. in. are given below each figure. *a* is an ordinary tension specimen, *b* and *c* have saw cuts, at both and one side respectively, *d* has semicircular notches and *e* has a central hole with saw cuts at each side; after the saw cuts were made the bars were heated and the cuts closed. *d* is the strongest; this would be expected, on the shear theory, as the diagonal line is relatively larger;

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* *Proc. Inst. C. E.*, Vol. LXXXIV.

b is weaker than *d* on account of the abrupt change of section; *c* is weaker still because the load is not central, and in *e* the hole probably gives an accentuation of the effect of the abrupt edges. For other results on the effects of holes *see* Chap. XIV.

Effect of Overstrain.—If a ductile metal is loaded beyond the yield point and the load removed, and the specimen is then loaded up again at once, it is found that the new yield point is higher, but the elastic limit is slightly lower. The overstrain also increases the ultimate or breaking tensile stress. This is shown in Fig. 26 (*a*) in which *b* is the initial or primitive elastic limit, and *a* is the initial yield point; at the point *c* the load is taken off and then the specimen is loaded up almost immediately; the new elastic limit *e* is much lower than previously and the yield point *d* higher. If some hours had been allowed to elapse between taking off the load and reloading, the elastic limit would nearly return to its previous value *b* but the yield point would go higher still to the point *g*.

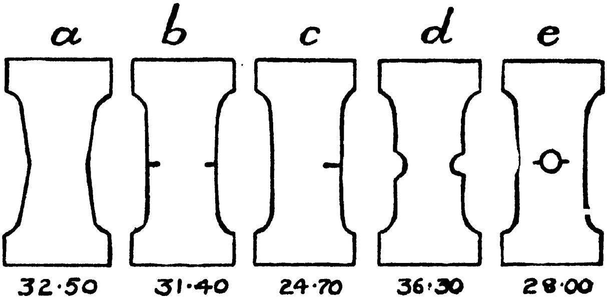

FIG. 25.

In Fig. 26 (*b*) is shown the effect of keeping the load on for some time at the point *c* before increasing it further. The curve in full lines shows the effect of keeping the load fixed for about ten minutes, and in dotted lines *k l* the effect of keeping it for ten days.*

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* For fuller information a paper by Professor Ewing, *Proc. Roy. Soc.* 1880, should be consulted.

This hardening effect of overstrain is well known in practical work. Copper wire becomes very brittle by bending it backwards and forwards, and steel wire in the process of drawing becomes very hard indeed.

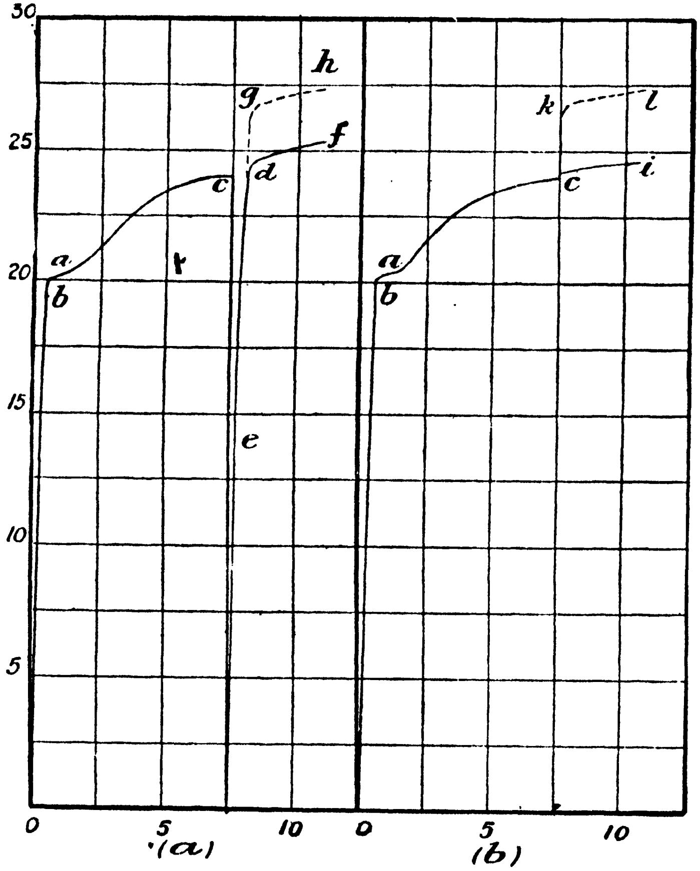

Fig. 26.—Overstrain.

Recovery of Elastic Limit from Overstrain.—As indicated above, the elastic limit slowly recovers its original value after it has been allowed to rest for a few hours; it then will increase as the time of rest is extended and ultimately gets above its final value and gets near to its new yield point.

Mr. Muir * has shown that the temperature of boiling water gives an almost immediate recovery of the elastic limit to near the new yield point which will be as high as if the material had been allowed to rest for several days.

HARDENING BY QUENCHING.—The hardening effect of overstrain is not the same as that effected by heating the metal to a high temperature and quickly cooling by quenching. This has the effect of making the metal very brittle, and there is practically no yield point, the specimen breaking off short with practically no extension.

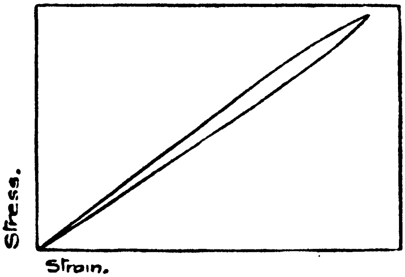

FIG. 27.

MECHANICAL HYSTERESIS.—If a specimen of ductile material is loaded up beyond the elastic limit and the load is taken off slowly and the strains noted for descending loads, the stress-strain diagram for descending loads will be found not to coincide with that for the ascending load, the two curves forming a loop as indicated in Fig. 27; this, by analogy with magnetic hysteresis, is called a mechanical hysteresis loop. In experiments of this kind great care is necessary to eliminate errors of the instrument on the return, but many experimenters have found similar results well within the elastic limit. Very careful experiments by Mr. Bairstow, however, at the National Physical Laboratory,† suggest that this phenomenon does not occur unless

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* *Phil. Trans. Roy. Soc.*, 1899.

† *Ibid.*, vol. 210.

the "natural" elastic limit is passed. These "natural" elastic limits are dealt with on p. 87.

Tensile Strength of Various Steels and Wrought Iron.—Fig. 28 shows typical stress-strain diagrams for various kinds of steels and wrought iron; the strength properties depend to a large extent on the heat treatment and amount of "working" in manufacture.

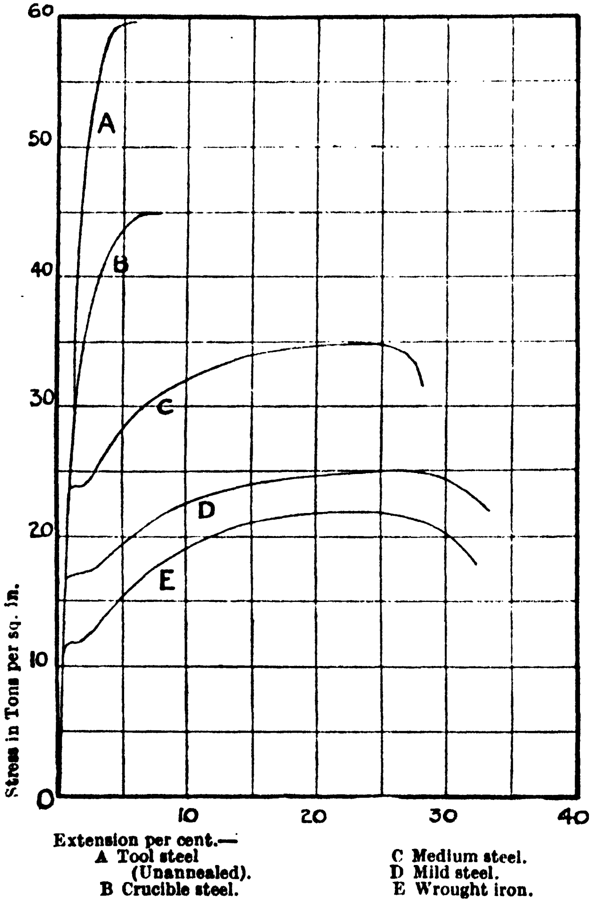


Fig. 28.—Stress-strain Diagrams for various Steels and Wrought Iron.

Effect of Varying Amounts of Carbon on Strength.—The effect of increasing the carbon in the steel is to increase the strength at the expense of the ductility. The following figures are taken from Harbord and Hall's *Metallurgy of Steel* (Griffin) for normal steels.

Stresses in Tons per sq. in.

Percentage of Carbon.	Breaking Stress.	Elastic Limit.	% Reduction of Area.
·13	21·5	13·0	71·2
·18	27·8	15·8	60·2
·25	29·5	17·3	54·8
·47	32·5	18·0	46·3
·72	43·2		25·9
·81	50·4		15·8
·95	51·1		15·4
1·31	45·5		5·6

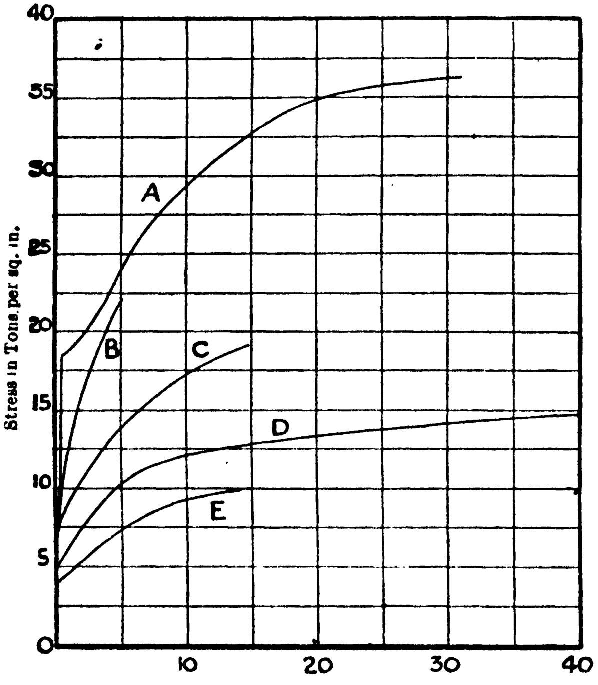
The proportion of carbon does not have an appreciable effect on the value of Young's modulus, nor does tempering or other hardening process.

Alloyed Steels.—The following figures give mean values for some examples of various alloyed steels.

Kind of Steel.	Tons per sq. in.		% Elongation.
	Breaking Stress.	Elastic Limit.	
Nickel [·2 % C, 3·2 % Ni]	42	27	26 on 3 in.
Tungsten			
[·15 % C, ·29 % Mn, ·40 % W.]			
Annealed	25·5	18	39·6 on 2 in.
Unannealed	31·0	24	33
[·46 % C, ·28 % Mn, 8·33 % W.]			
Annealed	42·5	25·5	32·6
Unannealed	64·0	45·0	2·57
Vanadium [·20 % C, ·27 % V, ·48 % Mn] .	47·1	37·4	22·0 on 2 in.
Chromium			
[·4 % C, 5 % Cr.]			
Annealed	24	18	24 on 2 in.
Hardened	55	49	12

"Quality Factor."—This term has been used by some engineers for the result of adding the breaking stress to the percentage elongation.

Stress-strain Diagrams for various Ductile Metals.—Fig. 28*a* shows typical stress-strain diagrams for a number



Extension per cent.—
A Aluminium Bronze.
B Hard Brass.
C Annealed Brass.
D Rolled Annealed Copper.
E Rolled Aluminium.

Fig. 28*a*.—Stress-strain Diagrams for various Metals.

of ductile metals. These must be regarded as only average diagrams, because these metals vary in their elastic properties to a considerable extent, depending on the method of working and upon their constitution in the case of alloys.

In most cases the early portion of the stress-strain diagram is never quite straight, but there is usually a clearly defined yield point.

Effect of Temperature upon the Strength of Steel.—The effect of temperature upon the ultimate strength of steel is to first cause a slight diminution, then an increase up to about 500° F., and finally a progressive diminution in strength for temperatures beyond; the elastic limit, however, falls progressively as the temperature increases. This is shown in Fig. 29, which represents the mean results of

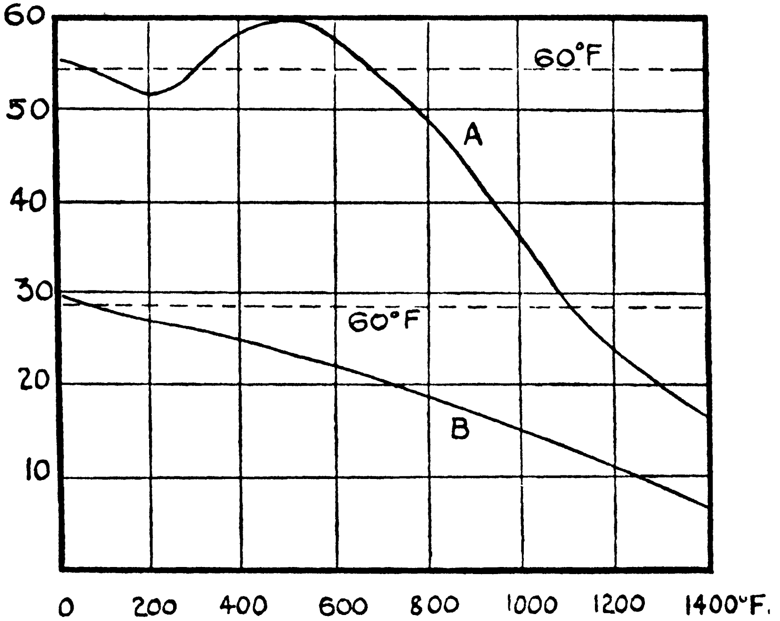


FIG. 29.—Temperature Effect on Strength of Mild Steel.

experiments on steel made at Watertown Arsenal, U.S.A., in 1888. Stresses are in 1000 lbs. per sq. in.

Curve A shows the variation of ultimate strength, and curve B the variation in elastic limit.

For the effect of temperature upon the strength of various other materials the reader is referred to Johnson's *Materials of Construction* (Wiley & Sons).

Compressive Strength of Ductile Metals.—When a ductile metal is tested by compression upon a short cylinder

(Fig. 30 *a*) (in long specimens the failure will always take place by buckling; and it is this which determines the safe stresses in compression members), the cylinder reaches a yield point which usually agrees very fairly well with that in tension and then bulges out almost indefinitely, as indicated in Fig. 30*b*, to a slightly reduced scale, until it fails by cracking transversely. It is not always possible to cause breaking by compression, because the flow becomes so great and the recorded results of ultimate crushing or compression tests therefore show considerable variation and are not of very great value. To obtain anything like a reliable result we should always allow for the changed area in a similar manner to that described for obtaining the true strength in tension.

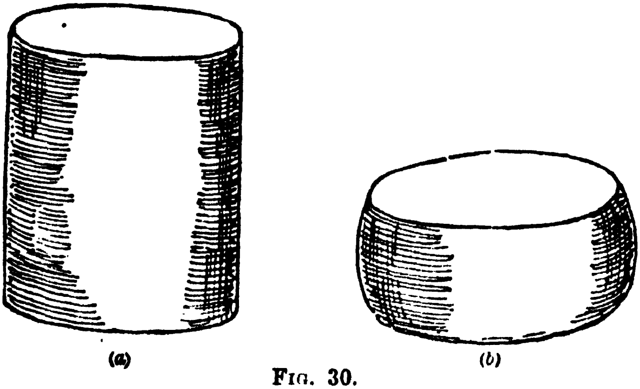


Fig. 30.

Shear Strength of Metals.—It is not easy to obtain a condition of true shear in testing. Fig. 31 (*a*) shows the deformation in shearing on punching a piece out of a bar or plate of ductile material. The lines across the bar indicate lines initially parallel and the deformation is such as to cause tensile and compressive stresses across the rectangles shown in dotted lines. Alongside, in diagram (*b*), is shown a sketch of an actual shear failure of a special phosphor bronze taken from a paper by Mr. E. G. Izod.* The most accurate method of testing for shear is by torsion upon thin tubes.

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* *Proc. I. M. E.*, 1906.

TIMBER

Timber is not an isotropic material, *i.e.* its strength properties are not the same in all directions, and there is considerable variation in the results of tests for the same kind of timber. This is because the strength depends upon the age of the timber, its dryness, the portion of the tree from which it has been cut, and even upon the kind of soil upon which it has been grown.

The strength of timber is greatest when the weight of moisture is about 5 per cent. of the total and decreases to about half this when the timber is green or very wet.

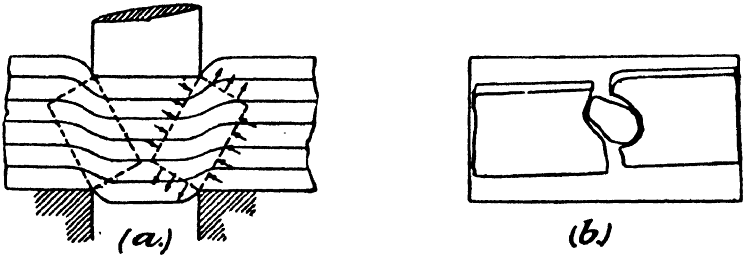

FIG. 31.—Shear Failures.

The moisture is usually determined by taking shavings by boring and weighing them before and after drying in an oven at a temperature of about 212° F.

In scientific tests of timber, such as Bauschinger's tests,* a standard moisture of about 15 per cent. is usually taken. Average values of strengths of various kinds are tabulated on p. 82.

TENSILE STRENGTH.—The tensile strength of timber is very much greater when the pull is parallel to the grain than when it is across it. Considerable trouble is experienced in gripping the specimen satisfactorily in a tension test on account of the tendency to shear or crush the ends.

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* See Unwin's *Testing of Materials* (Longmans).

The stress-strain diagram for straight-grained timber for tension parallel to the grain is practically straight up to fracture.

COMPRESSIVE STRENGTH.—In the compression of short cylinders or cubes of timber in a direction parallel to the grain, the lateral swelling causes the wood to split up into a number of strips or thin tubes which fail by buckling, the line of failure usually following an inclined line as indicated in Fig. 32.

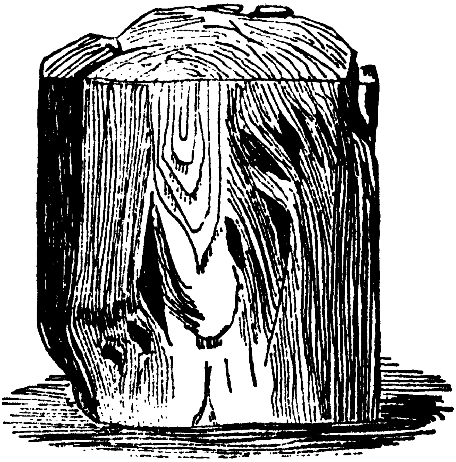

FIG. 32.—Compression Failure of Timber.

Care must be taken that the ends are quite parallel, so that the pressure is uniformly distributed.

When tested by crushing across the grain, the strength of timber is less than when the pressure is parallel to the grain. This strength is more one of hardness, *i. e.* resistance to penetration, and Johnson regarded the ultimate strength in this direction as that which gave 15 per cent. of indentation.

SHEAR STRENGTH.—The shear strength of timber is, as we would expect, very much less along the grain than across it, and in most bending tests of timber the initial failure is really by shear along the grain rather than by tearing of the fibres.

As shown in Chap. XVI. the maximum shear stress upon a rectangular section of breadth b and depth d for a central load W is

$$s = 1{\cdot}5 \text{ mean shear stress} = \frac{1{\cdot}5\,W}{2\,b\,d} = \frac{{\cdot}75\,W}{b\,d}$$

From this the shear strength of timber along the grain can be calculated indirectly by finding the load on a relatively short beam which will split the timber lengthwise as opposed to tearing the fibres.

The results of tests by Mr. Izod* by direct shear are shown by the following table—

SHEAR STRENGTH OF TIMBER (Izod's Experiments).

(Stresses in lbs. per sq. in.)

Kind of Wood.	Ultimate Tensile Strength.	Ultimate Shear Strength.		"Crippling" Stress across Grain.	% Moisture.
		Along Grain.	Across Grain.		
Pine .	9,200	470	4,900	1,900	15·7
Oak .	16,000	890	5,300	—	12·6
Deal .	7,800	440	2,700	1,200	10·6
Teak .	9,800	1,000	4,000	2,800	10·0

The "crippling stress" is the stress at which the timber was found to shear through about three-fourths of its area; considerable increase of load, however, was required before complete shear occurred.

BENDING STRENGTH.—Tests by bending form one of the most satisfactory methods of testing timber. If the section is rectangular, of breadth b and depth d and the span is l, then we have for a breaking central load W as proved in Chap. VII.

$$\text{Breaking stress} = \frac{W\,l}{4} \div \frac{b\,d^2}{6} = \frac{3\,W\,l}{2\,b\,d^2}$$

---

* *Proc. I. M. E.*, 1906 (1).

As we indicated on p. 51, this breaking stress is sometimes called the "modulus of rupture."

Young's modulus may be calculated by finding the deflection δ for a given safe load W by the formula

$$\delta = \frac{W l^3}{48 E I} = \frac{W l^3}{4 b d^3 . E}$$

$$\therefore E = \frac{W l^3.}{4 b d^3 . \delta}$$

In these bending tests it is a good plan to take a standard size of beam, *e.g.* 1″ × 1″ × 12″.

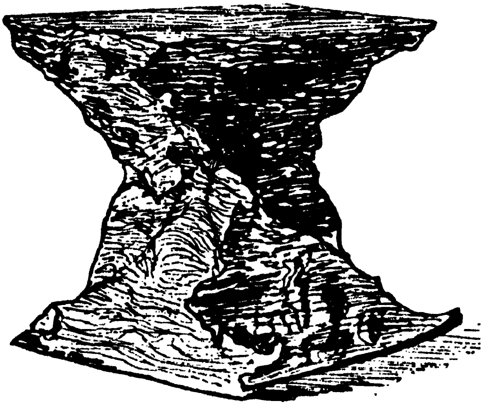

FIG. 33.—Compression Failure of Concrete Cube.

STONE, CONCRETE, CEMENT AND LIKE BRITTLE MATERIALS

Compressive Strength.—When stone, concrete, cement and like materials are tested in compression in the form of cubes or short cylinders, fracture nearly always occurs by splitting in diagonal planes in the manner indicated in Fig. 33. This is commonly referred to as a "shear failure," the failure being attributed to the shear stresses on the diagonal planes at 45° to the axis. We have seen already (p. 10) that on such planes there is a shear stress of equal intensity to the compressive stress. There are, however, strong reasons for supposing that the fracture of brittle materials is due to tension, and the most careful experiments on cement and concrete show that the shear strength is greater than the tensile strength (cf. p. 79).

Tension Theory of Failure.—When a block of material is compressed longitudinally it swells laterally, as shown in dotted lines in Fig. 34*a*, the ratio between lateral swelling y and the longitudinal compression x being Poisson's ratio (η), and one theory is that the limit of compressive strength of

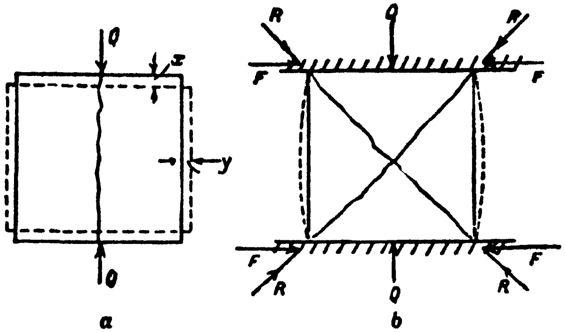

FIG. 34.

the material is reached when the tensile strain y reaches the limit of tensile strain for the material; in the case of a block in a testing machine, there is nearly always a very large friction force F (Fig. 34*b*) induced, which prevents the lateral expansion and causes the block to bulge as indicated in dotted lines, thus causing resultant stresses R in a diagonal direction, which cause the apparent shear fracture and make the specimen to appear stronger than it really is. It has been known for many years that the measured compressive strength of blocks depends upon the material placed between the press head and the block. The following figures quoted from Unwin's *Testing of Materials* are of interest in this connection—

Material.	Crushing Load in tons on 4 in. cubes.	Material between cube and press head of Testing Machine
Portland stone	57·7	Two millboards
	52·6	One lead plate
	45·6	One lead plate ⅛ in. smaller all round
	33·5	Three lead plates
Yorkshire grit	79·7	Two millboards
	80·0	Cemented between two strong iron plates with plaster of Paris
	56·2	One lead plate
	35·9	Three lead plates

The lead plates were ·085 in. thick in each case, and the fracture was longitudinal, as in Fig. 34*a*, in each case with lead plates, and diagonal with the millboards. Professor Unwin, in commenting upon this, says that the "lead falsifies the result of the experiment," but we do not see why he should not consider the lead as giving the more correct result and the millboard figures as being false.

Professor Perry apparently takes the latter view, for he says in *Applied Mechanics* (Cassell): "There is much published information on the fracture by compression of blocks of stone, cement and bricks. In almost every case care is taken in loading the usually short specimens that friction at the ends shall prevent the material swelling laterally. When sheet lead is inserted at the ends, it gives a small amount of lateral freedom, and in every case the breaking load is lessened by its use, and therefore it is said to be wrong to use lead. I consider all this published information to be nearly valueless, except that there is some probability that half the usually published ultimate compressive strength for a cube is the true resistance to compression in the material."

There is, of course, in the case of a pure compression, a shear stress across a diagonal plane, and for materials like mild steel, in which the shear strength is less than the compressive strength, this shear stress probably causes the ultimate failure, and thus determines the compressive strength.

Ratio between Tensile and Compressive Strength for Concrete.—The consideration of the transverse strain enables us to calculate the ratio between the tensile and compressive strength of the material upon this theory.

Let u_t = ultimate tensile strength of the material.
u_c = ultimate compressive strength of the material.
E_t = Young's modulus in tension at failure.
E_c = Young's modulus in compression at failure.
η = Poisson's ratio.

Then, compressive strain $= x = \dfrac{u_c}{E_c}$

$$\therefore \text{ lateral strain} = y = \eta\, x = \frac{\eta\, u_c}{E_c}$$

$$\therefore \text{ tensile stress} = u_t = \frac{\eta \,.\, u_c \,.\, E_t}{E_c} \quad \ldots\ldots\ldots\ldots(1)$$

Ratio of tensile to compressive strength

$$= \frac{u_t}{u_c} = \frac{\eta\, E_t}{E_c} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now this ratio, according to different authorities, varies from one-eighth to one-twelfth, according to the usual method of determining compressive stress, and depends on the age of the concrete, the higher value occurring usually at ages of three months and more. η for concrete is not fully known, and in the absence of further information we will assign to it the value $\frac{1}{4}$, which is the theoretical value for a perfectly elastic solid.

E_t also has not been very fully determined, but Hatt, for a 1 : 2 : 4 mixture gives $E_t = 2{\cdot}1 \times 10^6$ lb. per sq. in., which is approximately equal to the value usually accepted for E_c. For a rough consideration, therefore, we will take $\dfrac{E_t}{E_c} = 1$. This would give $\dfrac{u_t}{u_c} = \dfrac{1}{4}$.

If, as Professor Perry suggests, the actual compressive strength is about one-half that usually published, the above figure compared with published figures gives $\frac{u_t}{u_c} = \frac{1}{8}$, which bears comparison with the figures usually given, agreeing with the lower limit.

All of these points are of very great importance, and it would be of great value if very careful experiments were made to determine η, E_t and E_c for the same mixture at the same age, and to see how nearly true is the suggestion that the compressive strength is given in terms of the tensile strength by the above formula.

Another consideration which enters into the problem is that tensile strengths as determined by the usual briquettes are somewhat less than the actual tensile strengths, the discrepancy being due to the variation in the distribution of the tensile stress across the specimen (see p. 78). According to various authorities the actual tensile strength is 1·5 to 1·75 the mean strength, and allowance for this would bring the value of $\frac{u_t}{u_c}$ from $\frac{1}{12}$ to $\frac{1}{14}$, which agrees very well with experimental values.

We have already dealt with Navier's theory for this problem (p. 45).

Effect of Relative Height and Breadth of Compression Specimens.—Very careful experiments in 1876 by Professor Bauschinger upon sandstone prisms have shown that the compressive strength of sandstone prisms decreases slightly when the relative height to breadth is greater than for a cube and increases when the relative height is less.

Fig. 35 shows a curve expressing the results of Bauschinger's tests and is a modified form of a similar curve given by Professor Johnson,* and tends to support both the Navier theory and the transverse tension theory that we have just given, because in one case we should have that the cube and more dumpy specimens prevent the rupture along the line given

* *The Materials of Construction* (Wiley & Sons).

by the theory, and in the other case the effect of the friction in preventing the natural transverse swelling is greater with a dumpy section than with a comparatively tall one.

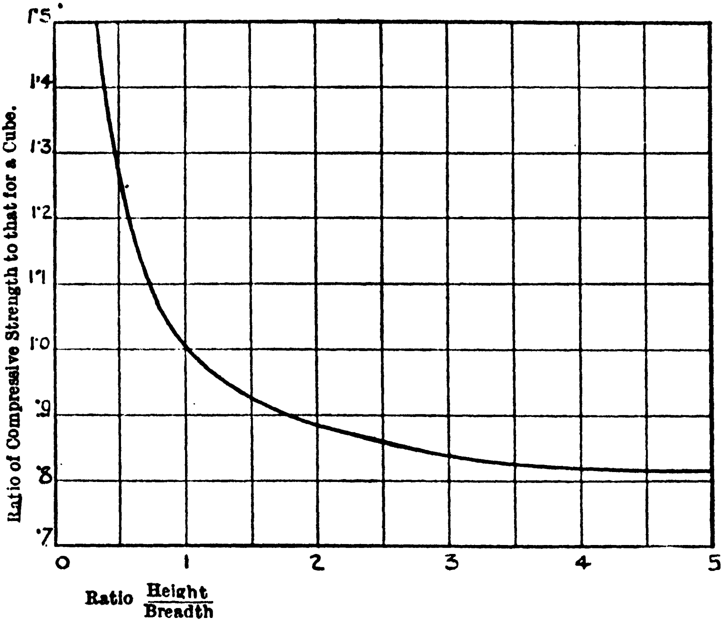

FIG. 35.—Effect of Height upon Strength of Sandstone Blocks in Compression.

Bauschinger recommended the following formula to represent the results of these tests—

$$u_c = \sqrt{\frac{4\,A}{p}}\left(a + b\,\frac{\sqrt{A}}{h}\right)$$

where u_c = ultimate crushing stress
A = area of cross-section
p = perimeter of cross-section
h = height of specimen
a, b = constants.

Strength of Cube with Chamfered Edges.—Fig. 36 shows the results of Bauschinger's tests upon chamfered specimens, the part of the curves in full lines representing the range over which the actual experiments were carried.

The curve marked A is for comparison of the strength of the chamfered block with a cube of the same size as the large area, and the curve marked B is for comparison of the strength of the chamfered block with that of a cube of the same size as the small area.

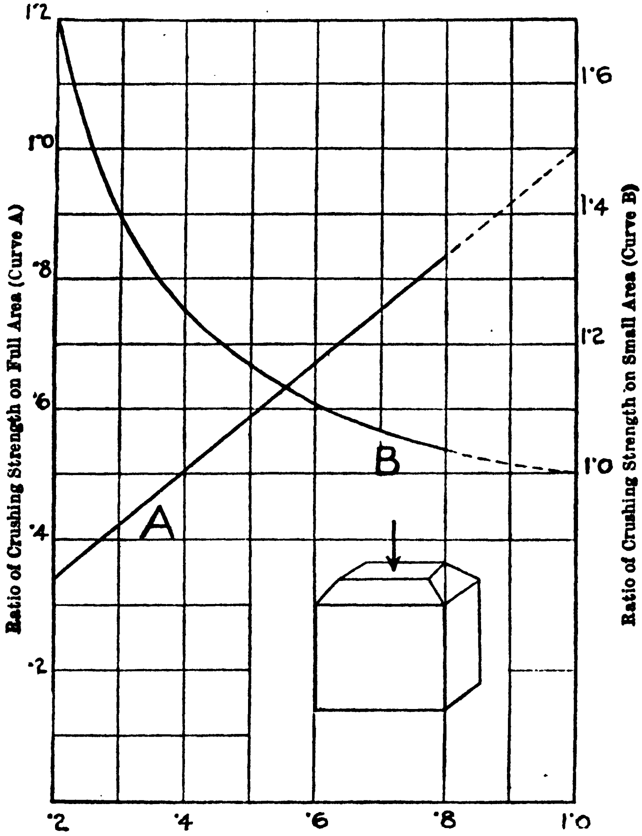

FIG. 36.—Compression Strength of Cube with Chamfered Edges.

Stress-Strain Diagram for Portland Cement and Concrete in Compression.—The kind of concrete which we will consider is composed of a mixture of Portland cement, sand and broken stone or brick, gravel or like material, which is called "aggregate."

The composition is usually referred to in the ratio of volumes of cement : sand : aggregate, *i.e.* a 1 : 2 : 4; concrete is one composed of 1 part cement, 2 parts sand and 4 parts aggregate.

The stress-strain diagram for concrete in compression is never quite straight so that there is no elastic limit, the exact curve depending on the composition and on the time after setting.

The curve shown in Fig. 37 is almost exactly a parabola.

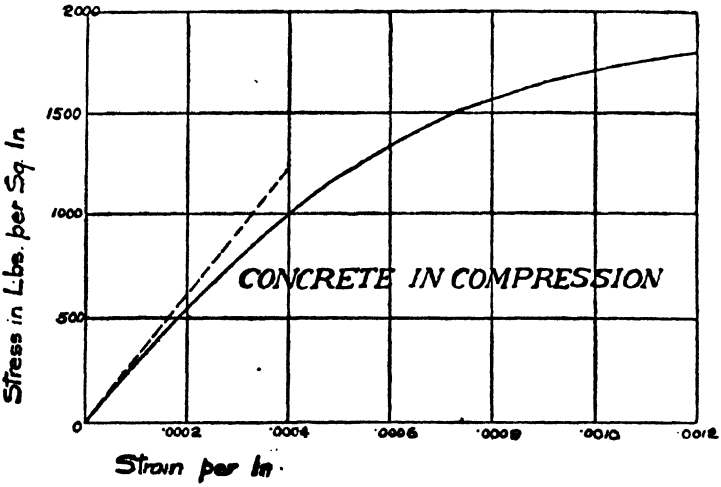

FIG. 37.—Stress strain Diagram for Concrete in Compression.

This curve is for a 1 : 3 : 6 concrete, 90 days old, which was tested by Mr. R. H. Slocum, of the University of Illinois Some authorities assume that the curve is a parabola, but in practice it is seldom that the curve comes so near to a parabola as the above. The stress-strain curve is, however, nearly always of a similar shape, the strains increasing more quickly than the stresses. It is extremely important to remember that with cement and concrete the relations between stress and strain vary largely with the quality and proportions of ingredients, and cannot be taken as almost constant as in the case of steel. In tension a somewhat similar curve is obtained, but as cement and concrete are practically never used in tension, much less work has been done on its tensile properties.

Young's Modulus for Concrete.—In a material like concrete Young's modulus E is not constant, so that we must give the stress at which the ratio is taken if it is to have any real value.

The initial value of E is obtained by drawing a tangent to the curve at the origin as indicated in the figure.

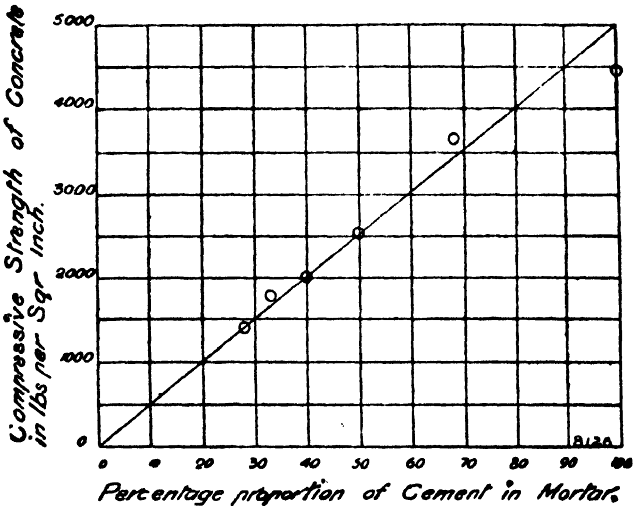

Fig. 38.

$$\text{We then have initial } E = \frac{1200}{\cdot 0004} = 3 \times 10^6 \text{ lbs. per sq. in.}$$

$$\text{Similarly final } E = \frac{1800}{\cdot 0012} = 1\cdot 5 \times 10^6 \text{ lbs. per sq. in.}$$

The usual value of E taken in reinforced concrete calculations is 2×10^6 lbs. per sq. in.

Effect of Composition and Age upon the Compressive Strength of Concrete.—The compressive strength of concrete is roughly proportional to the proportion of the cement in the mortar. Fig. 38 shows a diagram plotted from the results of experiments by Mr. G. W. Rafter, of New York.

Clean, pure silica sand and Portland cement were used, and the aggregate consisted of sandstone broken so as to pass through a 2-inch ring, containing 37 per cent. of voids when rammed.

The compressive strength increases with age, and Fig. 39

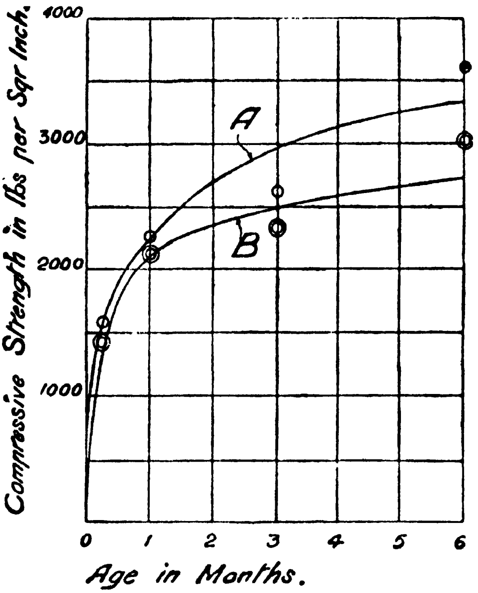

FIG. 39.

shows on a diagram the results of experiments made at the Watertown Arsenal, U.S.A., in 1899.

Curve A is for a mixture of one part of cement, two parts of sand, four parts of aggregate; and curve B is for a mixture of 1 : 3 : 6. The figures given are for the same brand of cement.

Tensile Strength of Portland Cement and Concrete.—The tensile strength of concrete is about $\frac{1}{10}$ of its compressive strength, but it is not usual to allow for any tension in the concrete in practice.

The standard method of testing the strength of neat cement is, however, by tension, so that the tensile strength is of considerable importance. (For method and apparatus for testing see Chap. XIV.)

The British standard specification requires the following strength of Portland cement.

Briquettes 1 sq. in. in section (see Fig. 190) must develop at least the following strength—

Neat cement.

After 7 days (1 in moist air, 6 in water) . .	450 lbs.
„ 28 „ „ „ 27 „ . .	500 „

$$\text{Breaking strength} = \text{strength at 7 days} + \frac{40000}{\text{strength at 7 days}}.$$

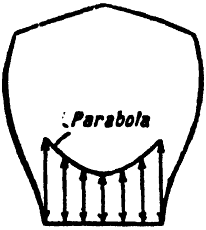

FIG. 40.

One part cement and three parts sand.

After 7 days (1 in moist air, 6 in water) . .	200 lbs.
„ 28 „ „ „ 27 „ . .	250 „

$$\text{Breaking strength} = \text{strength at 7 days} + \frac{10000}{\text{strength at 7 days}}.$$

VARIATION OF STRESS IN BRIQUETTE.—It can be shown that the stress is not quite constant over the briquette, but varies somewhat as indicated in Fig. 40; this means that the test strengths are always a little less than their actual values.*

Strength of Concrete in Shear.—Early experimenters found the shear strength of concrete to be from ·12 to ·2 of its compressive strength, but recent experiments suggest that the shear strength is considerably greater than this and depends upon the kind of aggregate (gravel, broken stone, broken brick, etc.) used. The most exhaustive investigation is that by Professor A. N. Talbot, who has also carried out many valuable experiments on the strength of reinforced concrete beams. The results of these experiments were published in a Bulletin of the Engineering Department of the University of Illinois.

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* See also Chap. XIV.

Two different methods of testing were used; in the first the shear strength was obtained by punching a hole in a concrete plate, and in the second by means of concrete beam with the ends fixed. There was considerable variation in the results, as will be seen from the following summary taken from *Engineering* of June 6, 1907—

Summary of Shear Tests. (Professor Talbot.)

Form of Specimen.	Kind of Concrete.	Method of Storing.	Number of Tests.	Strength.			Ratio of Shear to Compression.	
				Shear.	Compression.		Cube.	Cylinder.
					Cube.	Cylinder.		
				lbs. per sq. in.	lbs. per sq. in.	lbs. per sq. in.		
Plain plate	1·3·6	Air	9	679	1230	—	0·55	—
	1·3·6	Water	7	729	1230	—	0·59	—
	1·3·6	Damp sand	4	905	2428	1322	0·37	0·68
	1·3·6	Do.	1	968	1721	1160	0·56	0·83
	1·2·4	Do.	5	1193	3210	2430	0·37	0·49
Recessed block	1·3·6	Air	17	796	1230	—	0·65	—
	1·3·6	Water	6	692*	1230	—	0·56	—
	1·3·6	Do.	5	879	1230	—	0·71	—
	1·3·6	Damp sand	4	1141	2428	1322	0·47	0·86
	1·3·6	Do.	1	910	1721	1160	0·53	0·78
	1·2·4	Do.	5	1257	3210	2430	0·39	0·52
Reinforced recessed block	1·3·6	Air	4	1051	1230	—	0·86	—
	1·3·6	Damp sand	4	1821	2428	1322	0·75	1·38
	1·3·6	Do.	1	1555	1721	1160	0·90	1·39
	1·2·4	Do.	5	2145	3210	2430	0·67	0·88
Restrained beam	1·3·6	Do.	4	1313	2428	1322	0·54	1·00
	1·3·6	Do.	1	1020	1721	1160	0·59	0·88
	1·2·4	Do.	6	1418	3210	2430	0·44	0·58

* Specimens injured in removing the forms.

The conclusions to which these Illinois experiments lead are that the resistance of concrete to shear is dependent on the strength of the stone used, as well as on the strength of the mortar; and in the richer mixtures the stone appears to exercise the greater influence. With hard limestone and 1-3-6 concrete sixty days old the shearing strength may be expected to reach 1100 lbs. per sq. in.; and with 1-2-4-mixture 1300 lbs. per sq. in. There is reason to believe that if tests can be made with the load applied evenly over the shearing section, so as to obtain the true resistance to simple shear, the results will be found to be higher than those already obtained.

An important point brought out by Professor Talbot's investigations was the influence which variations in the constitution of the concrete have on the shearing strength. The compressive strength of concrete is largely affected by the strength of the cement, but the shearing strength is influenced more by the strength of the aggregate. For this reason it does not seem well to express the shearing strength in terms of the compressive strength. The method has the advantage, however, that an idea is gained of their relative action.

If, as indicated by these experiments, the shear strength of concrete is greater than the adhesion between concrete and steel, then there is an advantage over plain bars for reinforcement in those bars such as the twisted or indented bars which cannot be withdrawn from the concrete without shearing it.

Adhesion between Concrete and Steel.—It is absolutely necessary in a reinforced concrete structure that there shall be a good bond between the concrete and the steel, for the latter will bear its share of the stress only so long as there is no relative movement between the steel and the concrete. If a concrete beam were cast with holes throughout its length on the lower side and steel rods were inserted loosely into these holes, the strength of the beam would be practically no greater with the rods than without, because relative movement between the steel and concrete would be possible. A concrete beam with the reinforcing bars loose is like a plate girder without any rivets. The adhesive strength for plain bars can be found as follows: Let O be the perimeter of the bar and l its length; then if f is the safe adhesive stress, the adhesive force F that can be carried is given by

$$F = O\,l\,f$$

f can be found experimentally by embedding a rod in concrete and finding the force necessary to pull it out and dividing the resulting stress by the factor of safety (usually taken as about 6). Most authorities take a safe adhesive stress of 60 lbs. per sq. in.

NUMERICAL EXAMPLE.—*Find the length in relation to the diameter of a round bar that must be embedded in concrete in order that the tensile stress of* 16,000 *lbs. per sq. in. will be reached as soon as the safe adhesion stress of* 60 *lbs. per sq. in.*

Let d = diameter of rod in inches.

Let l = length of rod in inches.

Then load to reach safe tensile stress

$$= P = \text{stress} \times \text{area}$$
$$= 16000 \times \frac{\pi d^2}{4}$$

Load to reach safe adhesive stress

$$= P = \text{stress} \times \text{length} \times \text{perimeter}$$
$$= 60 \times l \times \pi d$$

If these are equal, $60\,l \times \pi d = 16000 \times \frac{\pi d^2}{4}$

$$l = \frac{16000}{60 \times 4} \times \frac{\pi d^2}{\pi d}$$
$$= 67\,d \text{ nearly}$$

∴ the bar must be embedded for a length equal to 67 diameters.

STRENGTH OF TIMBER (Normal Values)

(Stresses in thousands of pounds per sq. in.)

Kind of Timber.	Ultimate Strength.					Young's Modulus (millions of lbs. per sq. in.).	Weight in lbs. per cu. ft.
	Tensions along Grain.	Compression along Grain.	Shear across Grain.	Shear along Grain.	Bending.		
Ash	8–14	6–8	2–4	·4–·7	10–12	1·6	50
Beech . . .	9–18	4–6	—	—	8–10	1·3	44
Dantzig Fir .	6–9	3–5	2·7	·4	4500	·5	33
Elm . . .	6–12	5–8	3–5	·6–·9	6–8	1·6	34
Oak	9–15	4·5–9	3–5·5	·6–·9	7–10	1·7	58
Pitch Pine . .	8–12	4–6	4·9	·4	7·5	1·5	42
Red Pine . .	6–9	4–6	3	·4	4–6	1·2	27
Teak . . .	12–14	12–14	4	1	12–16	2·4	49
Yellow Pine .	6–9	4–6	3·5	·4	4–8	1·7	32

NORMAL CRUSHING STRENGTH OF CEMENT, STONES, ETC.

Material.	Weight in lbs. per cu. ft.	Ultimate Crushing Strength. (Thousands of lbs. per sq. in.)
Brick (London stock)	115	2·5
,, (Staffordshire blue) . .	140	7
Brickwork in Cement . . .	100–150	1·25–2·5
Cinder Concrete (1 : 2 : 4) . .	97	1·8 after 28 days
Granite	170	12–20
Gravel Concrete 1 : 2 : 4 . .	120	2·4 after 28 days
,, ,, 1 : 3 : 6 . .	130	1·8 ,, ,,
Portland Cement	90	7
Portland Stone	145	5
Sandstone	135–145	5–10
Slate	175	10

ELASTIC PROPERTIES OF VARIOUS MATERIALS (Normal Values).

Metals and Miscellaneous Materials. (Values in Tons per sq. in.)

Poisson's Ratio.	Material.	Weight in lbs. per cu. in.	Ultimate Stress.			Elastic Moduli.		Tensile Yield. Point.
			Tensile.	Crushing.	Shear.	E	G	
·27	Mild steel *	·288	26–32	—	20–25	13000	5200	16
·27	Wrought-iron	·277	20–25	25–30	16–20	12500	5000	12
·25	Cast-iron	·26	8–11	40–60	12–14	6000–9000	2400–3600	—
—	Aluminium castings . .	·092	4–6	—	—	4000	1700	—
—	„ bars . . .	·096	6–10	—	4–7	4000	1700	2–6
·33	Brass, cast	·301	10–12	5–6	—	5000	2000	—
—	„ annealed wire . .	·307	15–20	—	—	6500	—	—
—	„ hard wire . . .	·307	20–25	—	—	7500	—	—
—	Bronze, aluminium . .	—	35	45	20	7500	—	—
—	„ phosphor . .	—	14–16	—	16–18	6000	2300	—
·33	Copper, cast	·310	8–10	20	11–12	5000–6000	1900–2300	—
—	„ annealed . . .	·316	12–14	—	—	5000	—	—
—	„ hard drawn wire	·32	26–30	—	—	5500	2200	—
—	Delta metal, cast . . .	·310	20	—	—	5500	2200	8
—	„ „ rolled bar .	·310	34	—	18	6000	2500	23
—	Gun-metal	·306	16–18	—	16–18	5000	2000	—
—	Tin, cast	·262	1·5–2·5	5–6	—	3000	—	—
—	Zinc, cast	·252	2–3	—	—	3000	—	—
—	„ rolled	·252	7–10	—	—	5500	—	—
·27	Glass (flint)	·109	1·2	10	—	3500	1300	—
—	Leather belt	·035	2	—	—	—	—	11

* See also tables on p. 61.

CHAPTER III

REPETITION OF STRESSES: WORKING STRESSES

Repetition or Variation of Stresses.—In the design of machines and structures, we very often have to deal with cases in which the stresses vary in amount from one time to another; such cases occur in nearly every machine part subjected to rotary and reciprocatory movements and in structures which have to resist wind-pressures and rolling loads. In recent years, a large amount of investigation has been carried out on the strength of materials which are subjected to alternating stresses. The stress required to cause rupture in a material which is gradually increasingly stressed is called the *static breaking stress*, and is the stress obtained in the ordinary testing machines.

Fairbairn discovered in connection with some tests on wrought-iron girders, that a girder can be ruptured by repeatedly applying a load equal to about one-half of the static breaking load.

The first exhaustive investigation on the subject was conducted by Wöhler on behalf of the Prussian Ministry of Commerce, and was published in 1870. Wöhler's experiments extended over a period of twelve years, and had results which at the time were very startling, and the importance of which has only in comparatively recent years been appreciated by engineers.

The general result of these and subsequent experiments is to show that the stress necessary to rupture a material when such stress is repeated a very large number of times is considerably less than the static stress.

In Wöhler's experiments, which were carried out in tension, bending and torsion, some of the variations were from zero to a maximum in tension or compression and some were for a complete reversal of stress.

In one form of Wöhler's apparatus for testing by reversal of stress in bending, the specimen was in the form of a projecting beam or cantilever A (Fig. 41) clamped at the end of a shaft E, mounted between bearings B. The shaft was rotated by means of a belt surrounding a pulley C, and the specimen was of circular section and loaded at the end by a spring D, and in the rotation the compression and tension sides changed places gradually, thus giving a gradual reversal of stress. To balance the forces on the machine, a specimen was mounted at each end of the shaft.

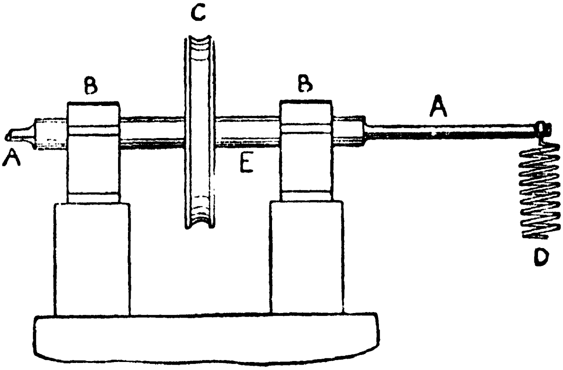

Fig. 41.—Wöhler's Experiments.

In another form of apparatus a beam was mounted upon knife edges to one of which a spring was connected by levers. The load was applied in the centre by a spring rod which was lifted periodically by a crank upon a rotating shaft, thus gradually applying the load and taking it off again.

Full accounts of the experiments will be found in Unwin's *Testing of the Materials of Construction.* We will take some examples of his results :—

For Krupp's Axle Steel—

Statical breaking stress = 52 tons per sq. in.
Breaking stress from zero to maximum = 26·5 „ „ „
„ „ for reversed stresses = 14·05 „ „ „

For Wrought-Iron—

Statical breaking stress = 22·8 tons per. sq. in.
Breaking stress from zero to maximum = 15·25 „ „ „
„ „ for reversed stresses = 8·6 „ „ „

In the first case the *range* of stress is in one case 26·5 and in the case of reversal is – 14·05 to + 14·05, *i. e.* 28·1, whereas the corresponding figures in the second case are 15·25 and 17·2.

Sir Benjamin Baker carried out similar experiments in this country and obtained similar results.

For mild steel of static strength from 26·8 to 28·6 tons per square inch, he obtained a breaking stress of 11·6 tons per square inch for reversal of stress.

Bauschinger carried out a large number of experiments on the same lines as those of Wöhler, and extended them to a larger number of materials.

For Bessemer Steel his results were—

Static breaking stress = 28·6 tons per sq. in.
Breaking stress from zero to maximum = 15·7 „ „ „
„ „ for reversal stresses = 8·55 „ „ „

With regard to these breaking stresses for variations of stress, it should be remembered that these are the least stresses for which the specimen would break after a very large number of repetitions.

In carrying out tests of this kind a number of specimens are taken, and the range or amount of variation of stress is altered for different specimens or sets of specimens, and when the range comes below a certain value the specimen will not break within the time over which the experiment lasts. The results are expressed on a diagram in which the range of stress is plotted against the number of repetitions required to cause fracture; or, in the case of variations from zero to a maximum or of complete reversal of stress, the limit of stress is plotted against the number of repetitions. Such a curve is shown in Fig. 42. From such curves the *apparent* stress, at which an infinite number of repetitions could be made without fracture, is obtained, and this is taken as the least breaking stress. The word "apparent" is used because no record appears to exist of a number of repetitions more than about fifty millions, and it has been suggested that perhaps lower stresses still would be obtained if the repetitions were extended still more.

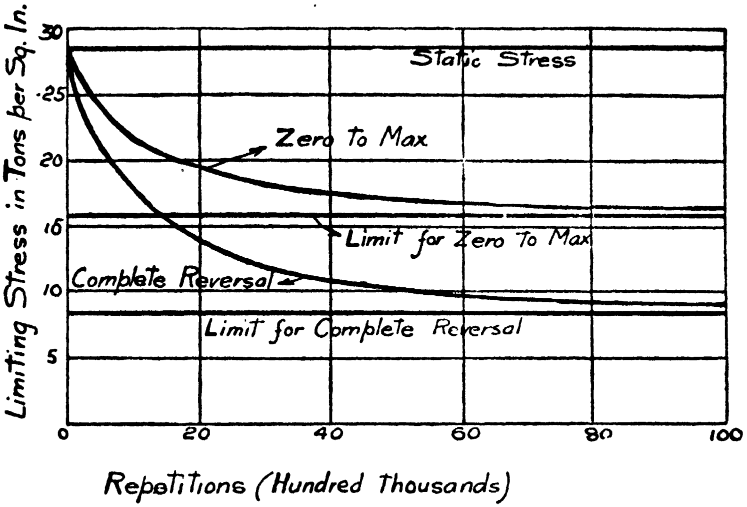

FIG. 42.—Repetitions of Stress.

Bauschinger suggested that there was some relation between the range of stress which a material would stand and the elastic limit. This elastic limit was what he called the "natural elastic limit," *i.e.* that obtained after the material has been subjected to a few variations of stress. We thus get the theory of natural elastic limits, which states that the range of repetition of stress which a material can resist indefinitely without failure is the range between the natural elastic limits in tension and compression.

Dr. Stanton and Mr. Bairstow published in Vol. CLXVI. of *Proc. Inst. C.E.* an important paper on the subject, giving the results of experiments conducted at the National Physical Laboratory.

They used a machine in which the specimen formed part of the piston-rod in a steam-engine mechanism; the specimen thus was subjected to reversals of direct stress, and a variation in the limiting stresses was obtained by varying the relative dimensions of the mechanism.

This research had some important results, the principal ones of which are—

(*a*) An alteration of the rate of repetition from 60 to 800 per minute has no marked effect on the results obtained.

(*b*) The range of stress which moderately high-carbon steels can stand is comparatively greater than that for low-carbon steel and wrought iron. This confirms Wöhler's opinion, and is contrary to the common idea that a comparatively brittle material can withstand less variation than a ductile one.

(*c*) The limiting stress which iron and steel can bear depends on the range of stress, and is almost independent of the actual values between the limits $\frac{7}{5}$ and $\frac{5}{7}$. This means that a stress variation from, say, 7 tons per square inch in tension to one of 5 tons per square inch in compression has the same effect as one from 6 tons per square inch in tension to 6 tons per square inch in compression.

Although the authors agree that more work must be done before a definite statement can be made, their experiments go to support Bauschinger's theory as to the elastic limits.

Effect of Rate of Repetition upon Results.—There seems to be some unexplained difference in the results of experiments upon the effect of speed upon the results. Wöhler's experiments were at 60 repetitions per minute, as indicated above. Stanton and Bairstow found no appreciable effect of increasing to 800 per minute; Reynolds and Smith,* however, employing the machine described in Chap. XIV., found that there is a progressive diminution in the resistance against repetition for repetitions of 1300 per minute and upwards; Eden, Rose and Cunningham,† however, experimenting with short rotating beams with a uniform bending moment over an appreciable length, found no such effect for speeds of 1300 per minute.

The following summary of results, pp. 94, 95, taken from the last-mentioned paper, gives a clear idea of the results of the various experiments on the subject. The only explanation of these contrary results appears to be in the difference in the design of the machines; at the high speeds it is possible that some secondary influence had a marked effect upon the results.

The Fatigue of Metals.—The phenomena described above are often referred to as the "fatigue of metals"; the suggestion being that the stress causes a change in the molecular structure of the metal and that the metal gets fatigued after a time and so breaks down under a smaller load. The bulk of the evidence, however, appears to be against that view and in favour of the theory that ultimate failure will occur only if the plastic limit is exceeded and thus the effects of overstrain become accumulative.

Specimens cut out of pieces that have been fractured by repetition of stress do not exhibit any weakening that the fatigue idea suggests.

The subject is still full of difficulties from the point of view of a satisfactory explanation of the results. For instance, the effect of overstrain is to cause the material to become brittle, and yet the more brittle kinds of steel (the high carbon steels) show less effect than the mild steel.

One explanation, called Foster's Theory, is that the mechanical hysteresis (p. 59) causes a very small permanent strain at each repetition and that the effect of these permanent strains is cumulative so that ultimately the permanent strain becomes sufficient to cause failure.

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* *Phil. Trans. Roy. Soc.*, 1902.

† *Proc. I. M. E.*, 1911.

The appearance of the fracture in these experiments is always different from that for ordinary static tensile tests; that for mild steel being more like a hard steel. This is probably due to the effect of overstrain upon the properties of the metal.

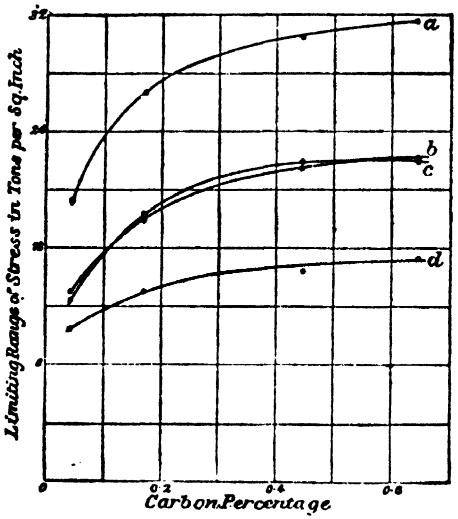

FIG. 43.—Repetitions of Stress—Abrupt Change of Section.

Effect of Sudden Change of Section upon Results.—The effects of sudden change of section have been investigated by Dr. Stanton and Mr. Bairstow,* who obtained the results shown in Fig. 43, in which the limiting stresses for failure by repeated loading are plotted against the carbon percentage. The results may be summarised as follows.

1. The resistance of a screw-cut specimen varied from 67 to 70 per cent. of the maximum resistance of the corresponding material, the fracture always taking place at the end of the thread.

2. The resistance of a specimen having a moderately

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* See *Engineering*, April 19, 1907.

rapid change of section varied from 65 to 72 per cent. of the maximum resistance of the corresponding material.

3. The resistance of a specimen having a sudden change of section varied from 47 to 52 per cent. of the maximum resistance of the corresponding material. In this form of specimen the low carbon material appears to realise a larger percentage of the maximum resistance than the higher carbon materials; but it is worthy of notice that even under conditions which are commonly supposed to be the most fatal to high carbon steels—*i. e.* a sudden change of section—the actual resistance of the 0·4 and 0·6 carbon steels is approximately 40 per cent. greater than that of the iron.

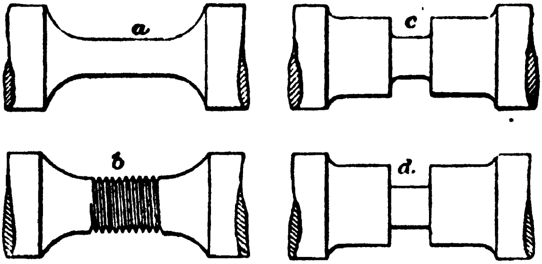

FIG. 43*a*.—Repetitions of Stress—Abrupt Change of Section.

The above resistances are, of course, estimated per unit of area, so that in calculating the strength of a screwed rod under alternating stress it will be further necessary to take into account the area at the bottom of the threads, so that the total reduction in resistance may well be more than 50 per cent. of its maximum value.

In the case of screw-threads there is a further possible source of weakness due to faulty machining in the cutting of the screw. If the bottoms of the threads are not properly curved, but left with a sharp angle, there can be no doubt that risks of the development of a crack are very considerably increased. It seems quite probable that failures of steam-engine crosshead bolts, which have broken under very low ranges of stress, may be due to this cause.

Cast Iron.—Very little work appears to have been done on repetitions of stress for cast iron, but from a small number of experiments by the author in reversal by bending the same general result was obtained, the limiting breaking stress in this case being nearly one-quarter of the static stress.

Equivalent Stress Formulæ.—STRAIGHT LINE FORMULA.—If f_r is the greatest stress that can be applied for an indefinite period for a range of stress r, and f_s is the static breaking stress of the material, the results of repeated load experiments can be expressed approximately by the relation

$$f_r = f_s - r$$

For a variation from zero to f_r, i.e. $r = f_r$; this gives $f_r = \frac{f_s}{2}$

For a reversal f_r to $-f_r$, i.e. $r = 2f_r$; this gives $f_r = \frac{f_s}{3}$

UNWIN'S FORMULA.—Unwin has given a formula from which the equivalent static stress for a given range of stress can be found. This formula gives, when plotted, a curve sometimes known as *Gerber's parabola*.

The formula is

$$f_r = \frac{r}{2} + \sqrt{(f_s^2 - n\,r\,f_s)}$$

where n is a constant depending on the nature of the material.

For mild steel we may take $n = 1{\cdot}5$.

Now if the variation is from zero to f_r, then $r = f_r$

$$\therefore f_r = \frac{f_r}{2} + \sqrt{f_s^2 - 1{\cdot}5\,f_r f_s}$$

Solving this equation we get $f_r = {\cdot}6\,f_s$. For complete reversal $r = f_r - (-f_r) = 2f_r$

$$\therefore f_r = f_r - \sqrt{f_s^2 - 3\,f_r f_s}$$

$$\text{or, } f_r = \tfrac{1}{3}f_s.$$

Relation between Repetition of Stress and Sudden Loading.—The similarity between the results of experiments on the variation of stresses and the reasoning given on p. 34

with regard to sudden loading has led many authorities to think that Wöhler's experiments were really experiments on sudden loading. The alternative point of view is that the two questions are distinct, and that therefore separate allowance should be made for each in the design of machines and structures.

One of the first difficulties to overcome in reconciling the questions is that strain is not proportional to stress beyond the elastic limit, and that, therefore, beyond this point twice the strain would not cause twice the stress (see Fig. 2). There is, however, the fact that if a material is strained beyond the yield point, the yield point will be found to have been raised on a subsequent testing; therefore, if this action goes on indefinitely with each repetition of stress, the yield point will ultimately become so high that the dynamic argument will apply up to the breaking point.

Although there are still many points which require to be decided in this controversy, for practical reasons we prefer to allow for one or the other, but not both, in design. The reason for this is as follows: Suppose that the safe working stress for mild steel for a constant and gradual load is 7·5 tons per square inch. Then, on the dynamic theory the safe stress for a reversing and sudden load is one-third of this, *i.e.* 2·5 tons per square inch. If we now make a separate allowance for the repetition of stresses, our working stress would be $\frac{1}{3} \times 2{\cdot}5$, or ·8 ton per square inch. As there is no question of *impact* in this, this seems an absurdly low working stress, and experience shows that it is not necessary to make the allowance for both points of view.

WORKING STRESSES

The Conflict between Theory and Practice.—An engineer has been tersely described by a somewhat characteristic American as "a man who can do for one dollar what a fool can do for two." Although from an æsthetic

SUMMARY OF RESULTS OF EXPERIMENTS

Quoted from Eden, Rose and Cunningham

NOTE.—The values for "Range of Stress" causing fracture after 10^8 alternations of stress in this Table are copied from the published accounts of the experiments of Turner, Reynolds and Smith, and of Stanton and Bairstow. The corresponding figures for the tests of Wöhler, Baker, and

			Tensile Test Figures.	
Experimenter.	Type of Endurance Test.	Material.	Tenacity. Tons per square inch.	Limit of Elasticity. Tons per square inch.
Wöhler*	Rotating Cantilever	Phœnix Iron	21·3	—
		Homogeneous Iron	28·1	—
		Vickers' Steel Axles	27·7	—
		Firth's Tool Steel	55	—
Baker*	Rotating Cantilever	Soft Steel	27·7	—
		Fine Drift Steel	54	—
Rogers†	Rotating Cantilever	Steel C (0·32% C.) as rolled	29·3	16·7
		Steel C annealed	26·5	7·1
Eden, Rose and Cunningham‡	Rotating Beam Uniform Bending Moment	½-in. bright-drawn Wrought-Iron bar	33·8	26·5
		½-in. bright-drawn Mild Steel rod A	35·7	25·0
Turner§	Fixed cantilever rotating deflection	Mild Steel	27·3	18·8
		Nickel Steel	48·0	36·1
Stanton and Bairstow‖	Reciprocating Weight direct tension and compression ratio $\frac{\text{tension}}{\text{compression}} = 1{\cdot}4$	Wrought Iron No. 2	25·6	13·4
		Piston-rod Steel	43·8	19·6
Reynolds and Smith¶	Reciprocating Weight direct tension and compression ratio $\frac{\text{tension}}{\text{compression}} = 1{\cdot}15$	Mild Steel annealed	25·8	—
		Cast Steel annealed	48·0	—

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* *The Testing of Materials of Construction.* Unwin.

† "Heat Treatment and Fatigue of Iron and Steel." Rogers. *Journal of Iron and Steel Institute,* No. 1, 1905.

‡ For other metals see *Proc. Univ. of Durham Phil. Soc.,* vol. iii. p. 251.

§ "The Strength of Steels in Compound Stress, and Endurance under

UPON THE REPETITION OF STRESS.

on the endurance of metals (*Proc. I. M. E.*, 1911).

Rogers have been estimated from stress-revolutions diagrams plotted from the various published results.

The tenacity figures for the Reynolds and Smith tests are for short test-pieces two diameters long.

Endurance Test Figures.			
Speed. Alternations of Stress per minute.	Range of Stress causing fracture after 10^6 alternations. Tons per square inch.	Ratio $\frac{\text{Range of Stress}}{\text{Tenacity}}$	Ratio $\frac{\text{Range of Stress}}{\text{Limit of Elasticity}}$
60–80	22·8	1·07	—
60–80	24·5	0·87	—
60–80	24	0·87	—
60–80	30·9	0·56	—
50–60	25	0·90	—
50–60	27·5	0·51	—
400	32	1·09	1·92
400	27·7	1·05	3·9
250			
620	34·6	1·02	1·3
1,300			
300			
600	39	1·09	1·56
1,300			
250	35·6	1·3	1·9
250	52·5	1·1	1·46
800	19·2	0·75	1·43
800	28·3	0·65	1·44
1,337	20·9	0·81	—
1,428	20·1	0·78	—
1,516	19·2	0·75	—
1,656	18·1	0·70	—
1,744	15·2	0·59	—
1,917	12·4	0·48	—
1,320	20·1	0·42	—
1,660	18·3	0·38	—
1,820	16·8	0·35	—
1,990	13·1	0·27	—

Repetitions of Stress." L. B. Turner, *Engineering*, 11th and 25th Aug. 1911.

‖ "On the Resistance of Iron and Steel to Reversals of Direct Stress." Stanton and Bairstow. *Proc. Inst. Civil Engineers*, 1906, vol. clxvi.

¶ "On a Throw Testing Machine for Reversals of Mean Stress." Professor Osborne Reynolds and J. H. Smith. *Phil. Trans. Royal Society*, 1902.

standpoint this seems to be a somewhat too mundane description of the engineer's vocation, we must not forget that the most scientific construction is the one which best fulfils the conditions for the least cost.

There is in reality no conflict between theory and practice in designing; each has its own place, and each is dependent on the other. The theory will tell us what is the best design as far as the economical arrangement of material goes. The best-designed structure is one which would be about to collapse at all sections at the same time; or, in other words, the various parts are so designed that the stresses in them are equal. This is all that the theory sets out to do. Practice, on the other hand, determines whether the theoretical design is in reality the cheapest in the end. Questions of workmanship, cost of erection and upkeep have to be considered, and it is only by balancing these with the theory that the really scientific design is obtained.

In dealing with the theoretical side of design we must never forget that, if we are to be guided by theory at all, we should see that we use the best theory. The disdain for theory that ultra-practical men often possess is largely due to the fact that their theoretical knowledge is not sufficiently comprehensive; they have not realised the conditions which have to be fulfilled before a certain theory is applicable, and so they probably use some formula for a case for which it was never intended.

Another point to be remembered is that practical rules for use in design are not necessarily sound because the machines or structures resulting therefrom satisfactorily fulfil their function. Such rules may make the design much heavier, and therefore much more costly, than necessary. Our aim in the theoretical investigations should be to eliminate as many uncertainties as possible, and not to be merely content with erecting something which will stand.

Commercial Aspect of Design.—If the word "scientific" is used in its best sense, the commercial aspect differs very slightly from the scientific aspect. There are certain points, however, that we would like to deal with which point to the necessity of considering the merely commercial aspects. First, there is the question of the sizes of sections adopted. Care should be taken that as much as possible is used of the same section, and that such section should be easily obtainable. The cost of a given structure may be increased largely because a section is specified which has to be rolled specially —although sections figure in makers' catalogues they are not always readily obtainable. In riveted work, too, much additional cost is often involved by an unnecessarily irregular pitch of the rivets, and fancy forms of cleated connections are often shown which have no advantage over the simple forms.

The designer should avoid curved lines in structural steelwork wherever possible in his design. It costs a lot to cut plates to a curve, and there is generally no reason for them. Some might urge that curved forms are more pleasing to the eye, and some go as far as to put cast-iron rosettes on the plates of plate-girders. But it is better to agree that no steel structure is artistically beautiful, and that to attempt to decorate it by curved gusset plates and rosettes is to make it really more ugly, because it has cost more and is still an eyesore to the artist. There is, also, a theoretical objection to curved members, viz. that the loading on such bars is eccentric, and stresses are therefore much increased.

Where practice necessitates our putting theory aside somewhat, we should always keep this in mind in our calculations. For instance, theoretically the centre-line of the rivets in a T section should coincide with the centroid line of the section. In practice this is impossible, as the head of the rivet could not then be closed. But we must remember in designing that the load is eccentric and that due allowance must be made for this.

We shall in this book be concerned only with the strength of structural details and machine parts, but it may be pointed out that in designing castings the pattern-maker must be considered and that ease in machining must be kept in mind; in fact, everything must be done which will save needless expense.

Working Stresses and Factor of Safety.—The question of the working stresses to adopt in practice is of the utmost importance, and if our design is to be of any real value we must have clear ideas as to such working stresses.

In dealing with working stresses we often speak of the **factor of safety.** This may be defined as the factor by which the working stresses may be multiplied to give stresses which will result in failure. This phrase is one which is often used glibly without any real meaning; and it has been suggested that in many cases it would be better called the factor of ignorance. If we design a structure with a factor of safety of four, say, we certainly do not as a rule mean that the structure could bear four times the load without failure. This is because there are certain contingencies that we do not allow for in our design. Our aim should be, however, to make our calculations so that the factor of safety has as exact a meaning as possible. This can be done only by choosing our working stresses skilfully and by making allowance for as many points as possible. For steel-work it is common to adopt as a working stress in tension one-quarter of the breaking stress in tension and to say therefore that the factor of safety is 4. Many designers forget, however, to make the due allowance for live or variable loads. The basing of the factor of safety on the breaking stress is also open to a very serious objection, viz. that the *elastic limit* of the material is the point which really determines the safety of the structure. If the stresses are above the elastic limit, failure is almost certain to ensue, especially in the case of compression members or struts. It would, therefore, be better to base the working stresses on the elastic limit or the yield point, since in pure tension the two points are close together, and the yield point is much easier to measure and specify for a definite minimum value of such limit in the steel. The point commonly urged against this method of procedure, viz. that the elastic limit is a much more variable quantity than the breaking stress, seems to us to be one in favour of its adoption. It is certain that stresses beyond the elastic limit are very dangerous, and if this quantity is a variable one we ought to know it for the material that we are using, and base our working stresses on it accordingly. We would suggest that the dead-load or static working stress should be taken as one-half of the natural elastic limit.

The following tables of stresses may be used for obtaining the usual working stresses adopted for *dead loads* in design :—

Material.	Working Stress.			Dimensions of Stresses.
	Tension.	Compression.	Shear.	
Mild steel* . . .	7	7 (bending) 6 (direct)	5	tons per sq. in.
Wrought iron . .	5	5 (bending) 4 (direct)	4	,, ,,
Cast iron . . .	½	4	½	,, ,,
Oak	16	13	5 (across grain)	cwt. per sq. in.
Pine, yellow . .	3	6	3 (across grain)	,, ,,
Cement concrete 1 : 2 : 4 . .	60	600 (bending) 500 (direct)	60	lbs. per sq. in.
Granite	—	35	—	tons per sq. ft.
Sandstone . . Yorkstone . .	—	20	—	,, ,,
Limestone . . .	—	15	—	,, ,,
Brickwork in cement mortar	·5 (adhesion)	8	—	,, ,,
,, in lime mortar	·4 (adhesion)	6	—	,, ,,

---

* Many authorities allow ½ ton more for each of the stresses in mild steel. We have already seen that according to one theory the working shear stress for ductile metals should be half the tensile strength, viz. 3·75 tons per sq. in. for mild steel. This figure is not, however, in common use.

Allowance for "Live" Loads or Variable Loads.—There are two principal methods of allowing for live loads which are in effect the same.

(*a*) EQUIVALENT DEAD-LOAD METHOD.—According to this method the static stresses are used and the loads are increased to give the equivalent dead load. The ways for allowing this, are—

(1) equivalent dead load = dead + 2 live load.

This may be called the dynamic formula.

(2) equivalent dead load

$$= w_e = \frac{n r + \sqrt{n^2 r^2 + 4\left(w - \frac{r}{2}\right)^2}}{2}$$

where r is the variation of load, and w is maximum load, n being a constant which may be taken as 1·5 for steel. This formula is deduced from Unwin's formula for Wöhler's experiments.

For steel we get

$$w_e = \frac{1{\cdot}5\, r + \sqrt{2{\cdot}25\, r^2 + 4\left(w - \frac{r}{2}\right)^2}}{2}$$

When the variation is from zero to a maximum, we have $r = w$.

Then $w_e = 2{\cdot}1\, w$.

(3) equivalent dead load = maximum load + variation.

(*b*) VARIABLE WORKING STRESS METHOD.—According to this method the working stress is varied according to the relative amounts of live and dead loads.

The common ways of allowing for this are—

(1) Launhardt-Weyrauch method.

$$\text{Working stress} = \frac{f}{1{\cdot}5}\left(1 + \frac{\text{minimum load}}{2 \times \text{maximum load}}\right)$$

f being the static or dead-load working stress.

(2) Dynamic method.

$$\text{Working stress} = \frac{f}{1 + \dfrac{\text{live load}}{\text{total load}}}, \ f \text{ being as before.}$$

Take as a simple numerical example the case of a member of a roof truss in which the dead load is a tension of 5 tons, and the wind on one side causes a tension of 2 tons and on the other side a compression of 1 ton. The various methods give the following results :—

(*a*) (1) Equivalent dead load $= 5 + 2 \times 2 = 9$ tons.

(2) „ „ $$= \frac{1{\cdot}5 \times 3 + \sqrt{2{\cdot}25 \times 9 + 4\left(7 - \frac{3}{2}\right)^2}}{2}$$

8·2 tons.

(3) „ „ $= 7 + 3 = 10$ tons.

(*b*) (1) Working stress $$= \frac{f}{1{\cdot}5}\left(1 + \frac{4}{14}\right)$$
$$= \frac{6f}{7}$$

(2) „ „ $$= \frac{f}{1 + \frac{2}{7}} = \frac{7f}{9}$$

Assuming the material to be mild steel.

(*b*) (1) gives working stress = 6 tons per square inch.

(2) „ „ = 5·4 „ „ „

Taking the material as mild steel, the requisite number of square inches in the sectional area of the tie are—

(*a*) (1) $\frac{9}{7} = 1{\cdot}28$ square inch.

(2) $\frac{8{\cdot}2}{7} = 1{\cdot}17$ „ „

(3) $\frac{10}{7} = 1{\cdot}43$ „ „

(*b*) (1) $\frac{7}{6} = 1{\cdot}17$ „ „

(2) $\frac{7}{5{\cdot}4} = 1{\cdot}30$ „ „

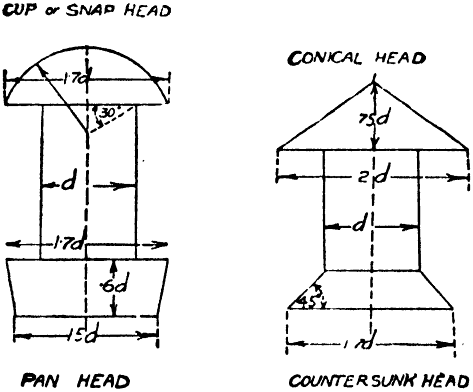
If consideration of variation of stresses be neglected altogether, we should have—area $= \frac{5 + 2}{7} = 1$ square inch.

CHAPTER IV

RIVETED JOINTS; THIN PIPES

Forms of Rivet Heads. — The most common forms of rivet heads and their usual proportions are shown in Figs. 44, 45.

For structural work the snap-headed rivets are most usual, but countersunk rivets are used where necessary to prevent projections from the surface of the plate. Snap-heads take a length of rivet equal to about 1¼ times the diameter.



FIGS. 44, 45.—Forms of Rivet Heads.

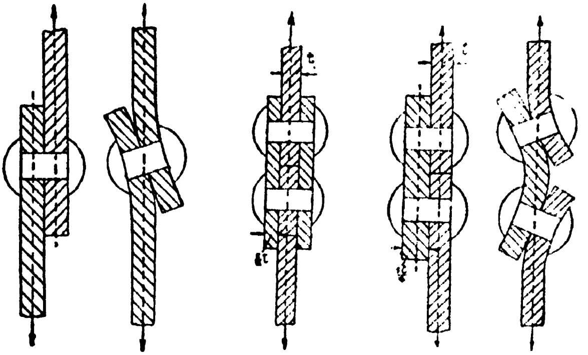
It is usual in practice to adopt a diameter of rivet when cold equal to one-sixteenth of an inch less than the diameter of the hole, but in all calculations the diameter of the rivet is taken as being equal to that of the hole.

Forms of Joints.—(*a*) LAP JOINTS AND BUTT JOINTS.—

In the *lap joint* the plates overlap as shown in Fig. 46. This form of joint has the disadvantage that the line of pull is such as to cause bending stresses, tending to distort the joint as shown.

In the *butt joint* the edges of the plate come flush, and cover plates are placed on each side as shown, the thickness of each cover plate being usually five-eighths that of the main plates. In this form of joint the pull is central, so that there are no bending stresses.

In the *single cover joint,* which is a cross between the lap joint and the butt joint, there are bending stresses developed, tending to distort the joint as shown.



LAP JOINT. BUTT JOINT. SINGLE COVER JOINT.

Thickness of Cover $\frac{5t}{8}$ Thickness of Cover $\frac{5t}{4}$

FIG. 46.—Forms of Riveted Joints.

It is clear from the above that the butt joint should be adopted wherever possible.

(*b*) CHAIN RIVETING AND ZIG-ZAG OR STAGGERED RIVETING.—The different rows of rivets in a joint may be arranged in chain form or zig-zag form, as shown in Figs. 47, 48. As we shall see later, the zig-zag form is more economical, and should be used whenever possible.

The essential feature of zig-zag riveting is that the rivets in alternate rows are displaced laterally by half the distance between the rivets, *i. e.* by half the pitch of the rivets. In the form shown the joint is in a tie bar of a bridge and the rivets form a triangle; it is common in boiler and like riveting to make the pitch of the outermost of three rows twice that of the innermost row; the result is a special kind of zig-zag riveting which we may call triangular riveting.

Methods by which a Riveted Joint may Fail.—A riveted joint may fail in any of the following ways:—

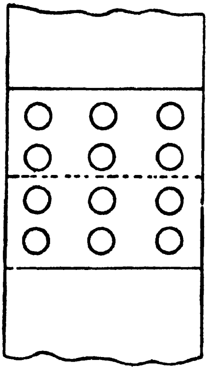


Fig. 47.—Chain Riveting.

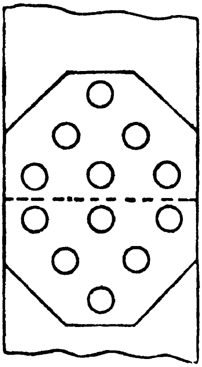


Fig. 48.—Zig-zag Riveting.

(1) By tearing of the plate.
(2) By shearing of the rivets.
(3) By crushing of the rivets.
(4) By bursting through the edge of the plate.
(5) By shearing of the plate.

Fig. 49 shows these methods of failure.

(4) and (5) are allowed for by the following rule: The minimum distance between the centre of a rivet and the edge of the plate is 1½ *d*, where *d* is the diameter of the rivet.

If this rule is adhered to the joint will always fail first in one of the ways (1), (2), (3).

The aim in designing a joint should be to make the force necessary to cause failure in the various ways equal.

We will now consider the various ways of failure in detail, taking in each case a strip of plate equal to the pitch of the rivets.

(1) Tearing of the Plate.—In this case the width along

which fracture will occur is $(p - d)$, and as the thickness of the plate is t, the area of fracture $= (p - d)\,t$.

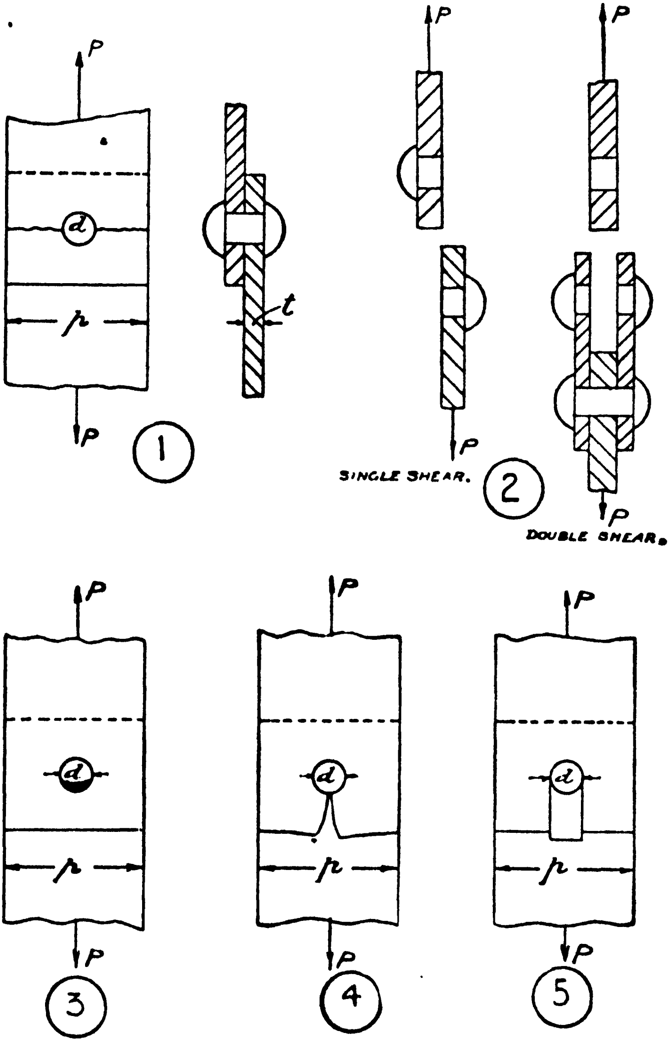


FIG. 49.

Therefore, if f_t is the *safe* tensile stress in the material, the safe load which the joint can carry is equal to

$$P = f_t\,(p - d)\,t \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

(2) SHEARING OF THE RIVETS.

In the case of single shear, the area sheared $= \frac{\pi d^2}{4}$

„ double „ „ $= \frac{2 \pi d^2}{4}$ *

Therefore if s is the safe shear stress on the rivet, the safe forces on the joint as regards shear are respectively

$$\left.\begin{array}{l} P = s \frac{\pi d^2}{4} \\ P = s \frac{2 \pi d^2}{4} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

(3) CRUSHING OR BEARING OF RIVETS.—In this case the crushing or bearing area is taken as the diameter of rivet multiplied by the thickness of the plate, i. e. $d \times t$. Therefore, if f_B is the safe bearing stress on the rivet, the safe force on the joint as regards bearing is equal to

$$P = f_B . d . t \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The values of f_t and s may be taken as given in Chap. III. For f_B, 10 tons per square inch may be taken for mild steel, and 8 tons per square inch for wrought iron. These figures are higher than for ordinary compression, and are obtained from the results of experiments.

For structural work the strength of the joint as regards bearing will often be less than as regards shear, because the plates are often thin compared with the diameter of the rivet.

Efficiency of Joint.—The efficiency of a joint is the percentage ratio of the least strength of a joint to that of a solid plate, *i. e.*

$$\text{Efficiency} = \eta = \frac{\text{Least strength of joint}}{\text{Strength of solid plate}}$$

Diameter of Rivets.—For the most economical joint the diameter of the rivet should be such that the shear and bearing strength are equal.

---

* A Board of Trade rule states that this should be taken as $\frac{1 \cdot 75 \pi d^2}{4}$, and this rule is often though not universally adopted.

Unwin's Formula, which is in common use, gives

$$d = 1{\cdot}2\sqrt{t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

For single shear we have

$$\text{shear strength} = \frac{\pi d^2}{4}\,.\,s$$

$$\text{bearing strength} = d\,t\,.\,f_B.$$

If these are equal

$$d = \frac{4\,t\,f_B}{\pi\,s} = 2{\cdot}54\,t \;(\text{if } f_B = 2\,s) \quad \ldots\ldots\ldots\ldots\text{(D)}$$

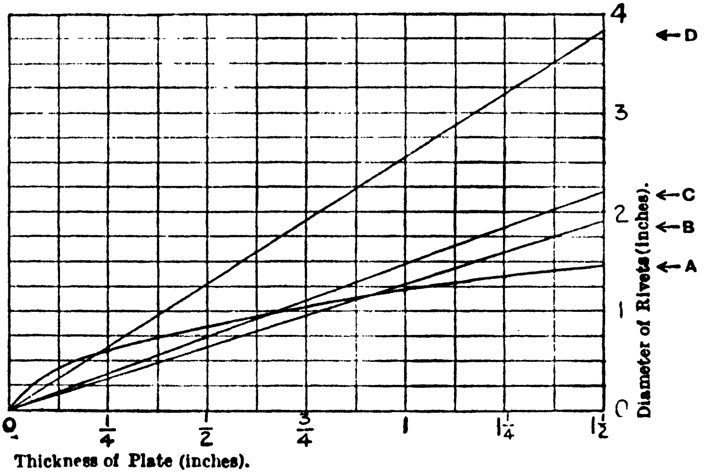


FIG. 50.—Diameter of Rivets.

For double shear we shall have

$$\frac{2\,\pi\,d^2}{4}\,.\,s = d\,t\,f_B$$

$$d = \frac{2\,f_B\,.\,t}{\pi\,s} = \frac{4\,t}{\pi} = 1{\cdot}27\,t \ldots\ldots\ldots\ldots\text{(B)}$$

or on the Board of Trade rule

$$\frac{1{\cdot}75\,\pi\,d^2}{4}\,.\,s = d\,t\,f_B$$

$$d = \frac{4\,f_B\,.\,t}{\pi\,.\,1{\cdot}75\,s} = \frac{32\,t}{7\,\pi} = 1{\cdot}45\,t \ldots\ldots\ldots\text{(C)}$$

These values are plotted in Fig. 50.

It is clear from this figure that for large thicknesses of plate for single shear the theoretical value gives diameters of rivet which are impossible in practice.

NUMERICAL EXAMPLES.—(1) *A tie bar in a bridge consists of a flat bar of steel* 9 *inches wide by* 1¼ *inches thick. It is to be spliced by a double butt joint. Determine the diameter of the rivets and their number, and give sketches showing the proper pitch and arrangement of the rivets.* (*B.Sc. Lond.*)

According to Unwin's formula $d = 1{\cdot}2\sqrt{t} = 1{\cdot}34$ inches. This is, however, rather high for practice, and so we will adopt $d = 1$ inch.

Assuming that the rivets are arranged in zig-zag fashion, the strength of the joints against tearing through the outside rivet is equal to $7\,(9 - 1) \,.\, 1\frac{1}{4} = 70$ tons. ?

Shear strength of each rivet $= 5 \,.\, \frac{2\pi}{4} \,.\, (1)^2 = 7{\cdot}85$ tons.

∴ Number of rivets required for shear

$$= \frac{70}{7{\cdot}85} = 8{\cdot}93 = \text{say } 9.$$

Bearing strength of each rivet $= 10 \times 1 \times 1\frac{1}{4} = 12{\cdot}5$ tons.

∴ Number of rivets required for bearing $= \frac{70}{12{\cdot}5} =$ say 6.

9 rivets would thus be ample as regards bearing.

The joint would then be arranged as shown in Fig. 51, the centre two rows being chain-riveted.

We will now consider the strength of this joint under various ways of failure.

If the plate tears along the line A A, the force necessary to reach the safe limit of stress is, as we have shown above, 70 tons.

Now suppose that the plate tore along B B, shearing off the rivet in A A.

Then strength of line B B $= 7\,(9 - 2)\frac{5}{4} = 61{\cdot}25$ tons.

Strength of one rivet $= 7{\cdot}85$ tons.

$\therefore$ Total strength against failure along B B

$$= 61{\cdot}25 + 7{\cdot}85 = 69{\cdot}1 \text{ tons.}$$

Now suppose plate tore along C C, shearing off the three rivets.

Then strength of line C C $= 7\,(9 - 3)\,.\,\dfrac{5}{4} = 52{\cdot}5$ tons.

Strength of three rivets $= 23{\cdot}55$.

$\therefore$ Total strength against failure along C C

$$= 52{\cdot}5 + 23{\cdot}55 = 76{\cdot}05 \text{ tons.}$$

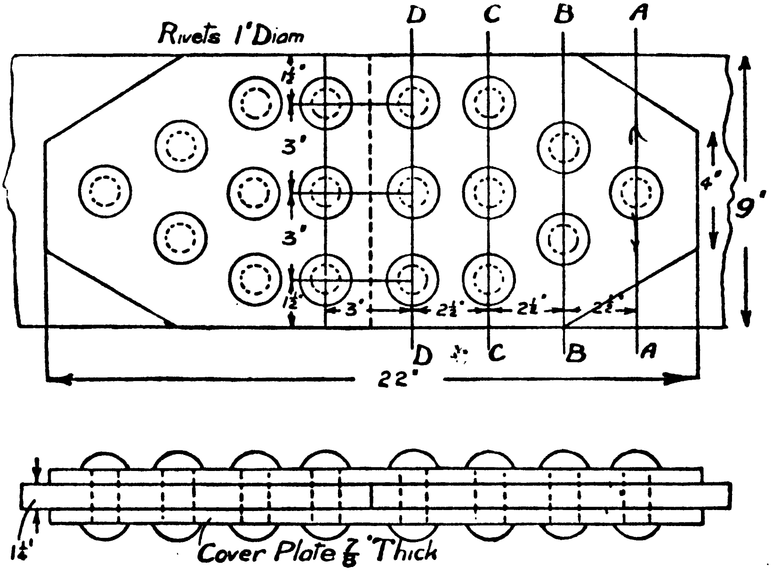


FIG. 51.

Finally, suppose cover plates tore along D D, then strength

$$= 7\,(9 - 3)\,.\,2\,.\,\frac{7}{8} = 73{\cdot}5 \text{ tons.}$$

From the above we see that the weakest section is along B B.

$$\text{Then efficiency of joint} = \frac{\text{Least strength of joint}}{\text{Strength of solid plate}}$$

$$= \frac{69{\cdot}1}{9 \times 1\tfrac{1}{4} \times 7} = \frac{69{\cdot}1}{78{\cdot}8} = 87{\cdot}8\,\%.$$

If instead of zig-zag riveting we had adopted chain riveting with three rows of three rivets (9 in all) the least strength would be $(9-3)\,1\frac{1}{4}\times 7 = 52{\cdot}5$ tons.

$$\therefore \text{ efficiency of joint} = \frac{52{\cdot}5}{78{\cdot}8} = 66{\cdot}7\ \%.$$

If we had four rows of chain riveting with two rivets in each row (8 in all), the least strength would be $(9-2)1\frac{1}{4}\times 7 = 61{\cdot}25$ tons.

$$\therefore \text{ efficiency of joint} = \frac{61{\cdot}25}{78{\cdot}8} = 77{\cdot}7\ \%.$$

The above shows that the zig-zag riveting is considerably more efficient than the chain riveting, and is therefore more economical.

(2) *Design a double-riveted lap joint to connect two steel plates ½ in. thick with steel rivets, the tensile strength of the plates before drilling being* 30 *tons per sq. in.; the shearing strength of the rivets* 24 *tons per sq. in.; and the compressive strength of the steel* 43 *tons per sq. in. Find the efficiency of the joint.* (*A.M.I.C.E.*)

For ½ in. plates Unwin's formula would give

$$d = 1{\cdot}2\sqrt{{\cdot}5} = {\cdot}85 \text{ in., say } \tfrac{7}{8} \text{ in.}$$

The joint is a double-riveted lap, therefore there will be two rivets in single shear in a width of plate equal to the pitch.

∴ Strength against tearing per pitch

$$= f_t(p-d)\,t$$
$$= 30\,(p-d)\,\frac{1}{2} = 15(p-d) \quad \text{..(1)}$$

∴ Strength against shearing per pitch

$$= s\,.\,\frac{2\,\pi\,d^2}{4}$$
$$= \frac{24\,.\,2\,\pi}{4}\,.\left(\frac{7}{8}\right)^2 \quad \text{..........(2)}$$
$$= 28{\cdot}9 \text{ tons.}$$

If these are equal $15\left(p-\frac{7}{8}\right) = 28{\cdot}9$

$$\therefore p = \frac{28{\cdot}9}{15} + \frac{7}{8}$$
$$= 1{\cdot}93 + {\cdot}87 = 2{\cdot}80, \text{ say } 3 \text{ ins.}$$

The bearing stress for a force of 28·9 tons would be equal to

$$\frac{28{\cdot}9}{\frac{7}{8} \times \frac{1}{2} \times 2} = 33 \text{ tons per sq. in.}$$

the bearing area of each rivet being $\frac{7}{8} \times \frac{1}{2} = {\cdot}437$ sq. in.

This is less than the allowable value of 43 tons per sq. in., showing that a larger diameter of rivet might be used with greater economy, but $\frac{7}{8}$ in. diameter is in most cases more suitable in practice.

The efficiency of joint in this case is equal to

$$\frac{28{\cdot}9}{30 \times 3 \times \frac{1}{2}} = \frac{28{\cdot}9}{45} = 64{\cdot}2\ \%$$

The joint then comes as shown in Fig. 52.

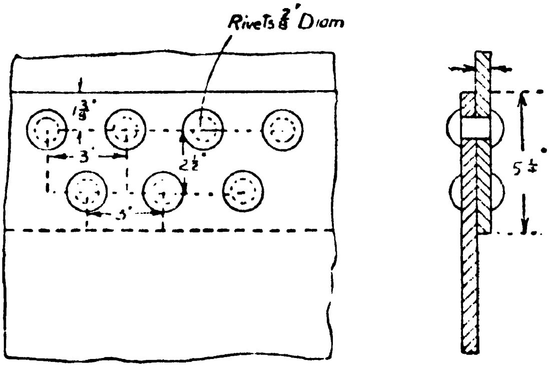


FIG. 52.

(3) *A steel-plate tie bar in a bridge is subject to a tension due to dead load only of* 16 *tons. The stress due to live load only varies from* 36 *tons tension to* 10 *tons compression. The tie bar is* $\frac{3}{4}$ *in. thick and is to be joined to the side plate of a girder by means of a* $\frac{1}{2}$ *in. gusset plate and double-cover butt joint. Select suitable working stresses and design the joint, arranging the rivets so that the tie bar is weakened by only one rivet section. (B.Sc. Lond.)*

The maximum load in this case is 36 + 16 = 52 tons, and the minimum load 16 − 10 = 6 tons.

Using the Launhardt-Weyrauch formula, we have

$$\text{Working Stress} = \frac{f}{1{\cdot}5}\left(1 + \frac{\text{min. stress}}{2\ \text{max. stress}}\right)$$

$$= \frac{f}{1{\cdot}5}\left(1 + \frac{6}{104}\right) = {\cdot}705\,f$$

This gives a tensile stress of 4·93, say 5 tons per sq. in.; a shear stress of 3·52, say 3·5 tons per sq. in.; and a bearing stress of 7 tons per sq. in.

According to Unwin's formula $d = 1{\cdot}2\sqrt{{\cdot}75} = 1{\cdot}04$ in., but for practical reasons $\frac{7}{8}$ in. would usually be adopted.

We now require to find the necessary width of the tie bar. Let this be w.

Then $\left(w - \frac{7}{8}\right)\frac{3}{4}$ is the equivalent cross-sectional area.

$\therefore \left(w - \frac{7}{8}\right) . \frac{3}{4} . 5$ must be equal to the maximum pull of 52 tons.

$$\therefore \left(w - \frac{7}{8}\right) = \frac{52 \times 4}{3 \times 5} = 13{\cdot}89$$

$$\therefore w = 13{\cdot}89 + {\cdot}875 = \text{say } 15 \text{ inches.}$$

The strength of each rivet in double shear is equal to

$$\frac{2\pi}{4} . \left(\frac{7}{8}\right)^2 . 3{\cdot}5 = 4{\cdot}22 \text{ tons.}$$

$$\therefore \text{Number of rivets required for shear} = \frac{52}{4{\cdot}22} = 12{\cdot}3.$$

We will use 14, as they give the best arrangement.

The strength of each rivet in bearing is equal to

$$\frac{3}{4} . \frac{7}{8} . 7 = 4{\cdot}58 \text{ tons.}$$

$\therefore$ 14 rivets will be ample for bearing.

The joint is then arranged as shown in Fig. 53. It is very important in such joints that the centre line of the rivets should coincide with the centre line of the tie bar, or else the pull in the bar would be eccentric. In such joints, therefore, the rivets should always be arranged symmetrically with regard to the centre line of the tie bar.

(4) *Find the number of rivets necessary to the gusset plates, etc., at the base of a steel stanchion to the stanchion proper, the load carried being* 150 *tons. The diameter of the rivets is* $\frac{7}{8}$ *in. and the thickness of the plate* $\frac{1}{2}$ *in.*

The rivets usually have to be designed in such cases so that they will carry the whole load, so that if the stanchion itself

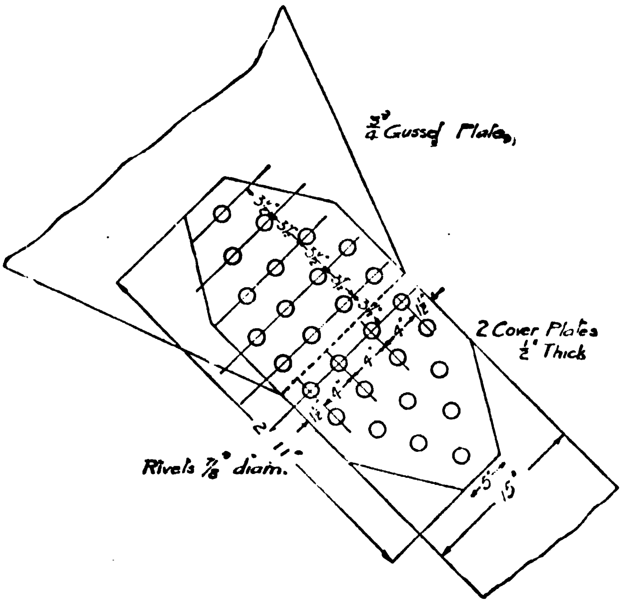


FIG. 53.

does not bear on the base plate the rivets will distribute the load satisfactorily.

$$\text{The strength of each rivet in single shear} = \frac{\pi}{4} \cdot \left(\frac{7}{8}\right)^2 \cdot 5 = 3{\cdot}01 \text{ tons.}$$

$$\text{The strength of each rivet in bearing} = \frac{7}{8} \cdot \frac{1}{2} \cdot 10 = 4{\cdot}37 \text{ tons}$$

$$\therefore \text{Number of rivets necessary} = \frac{150}{3{\cdot}01} = 50 \text{ nearly.}$$

Some Practical Considerations in Riveted Joints.—PUNCHING AND DRILLING OF RIVET HOLES.—It is quite common in this country for specifications to state that rivet holes must be drilled out of the solid. Punching is known to injure to some extent the material in the neighbourhood of the hole, and is thus often objected to. The extent to which punched holes weaken a structure such as a plate girder compared with drilled holes does not appear to have been satisfactorily determined, although such determination from a practical point of view would seem to be absolutely necessary, since there is an increase in cost entailed in drilling the holes. In recent years punching machines and means for obtaining an accurate pitch of the holes have been improved considerably, and when we consider the increased cost of the drilling, punching is preferable in many cases. In recent years "gang" or multiple drilling machines have been introduced which lessen the cost of drilling; one great advantage that drilling possesses is that the plates to be joined can be clamped together and drilled right through, thus ensuring accurate registering of the holes. A good compromise is to punch the hole $\frac{1}{4}$ to $\frac{1}{8}$ inch less than required, and to reamer out to size, the damaged metal being thus removed; but this is considerably more expensive than plain punching. A method of allowing for the damage of metal due to punching which has been suggested, and which we consider preferable, is to add $\frac{1}{8}$ inch to the diameter of the hole in calculating the tearing or tensile strength. This adds very little to the size of the plate and saves in cost of production. The point that should be very carefully seen to is that the holes are accurately pitched, so that the holes will register well when the parts are assembled, and will not require excessive drifting as is the case when the spacing of the holes is inaccurate. It is probable that many more joints are unsatisfactory because the rivets do not fill the holes, owing to the latter not registering accurately, than because the metal has been injured owing to punching the holes.

There is considerable friction between the plates in a riveted joint, but this is not allowed for in calculations of the strength.

Pitch and Spacing of Rivets.—In order to prevent moisture getting between the plates and causing bulging due to rusting, or to prevent local buckling in the case of compression members, it is common to stipulate that the pitch of rivets shall not be greater than 6 ins., or sixteen times the thickness of the thinnest plate. The designer should remember that pitches from 3 ins. upwards, increasing by half-inches, should be used, and odd fractional pitches avoided, except where absolutely necessary. As far as economically possible, the same pitch should be used throughout, and in many cases, for girder work, etc., 4 ins. is used unless special conditions require a different pitch.

Working Strength of Steel Rivets

Diam. of Rivets in ins.	Area in sq. ins.	Strength in single shear at 5 tons per sq. in.	Bearing Strength at 10 tons per sq. in. Thickness in ins. of plate.							
			$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{1}{2}$	$\frac{9}{16}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{3}{4}$
$\frac{3}{8}$	·1104	·55	1·17	1·41	1·64	1·87	2·11	2·34	2·59	2·81
$\frac{1}{2}$	·1963	·98	1·56	1·87	2·18	2·50	2·81	3·12	3·43	3·75
$\frac{5}{8}$	·3068	1·53	1·95	2·34	2·72	3·12	3·51	3·90	4·30	4·68
$\frac{3}{4}$	·4418	2·21	2·34	2·81	3·27	3·75	4·21	4·69	5·16	5·63
$\frac{7}{8}$	·6013	3·01	2·72	3·27	3·82	4·37	4·91	5·46	6·02	6·56
1	·7854	3·93	3·12	3·75	4·37	5·00	5·62	6·25	6·87	7·50

Thin Pipes and Cylinders.—Suppose that a thin cylinder of diameter d, Fig. 54, and thickness t, is subjected to a pressure of intensity p. This pressure will tend to burst the pipe along a longitudinal section, and the pressure on the ends will tend to cause failure across the circumferential section $x\,x$. In thick pipes the stress will vary across the section, and is dealt with in Chap. XVII.

Longitudinal Section.—Consider a length l of the pipe. Then the radial pressure p at any point can be resolved into a component $o\,b$ parallel to any diameter under consideration and a component $a\,b$ normal to it. The resultant normal force will be the same as the pressure acting over the diameter.

If f_t is the tensile stress across the section, we have

$\therefore$ Force tending to burst pipe $= p \times \text{area} = p\,d\,l.$

Force resisting bursting $= f_t \times \text{area}.$

$= f_t \times 2\,l\,t$ ($l\,t$ on each side).

$$\therefore f_t = \frac{p\,d\,l}{2\,l\,t}$$

$$= \frac{p\,d}{2\,t} \qquad\qquad (1)$$

This stress f_t is often called the *hoop stress.*

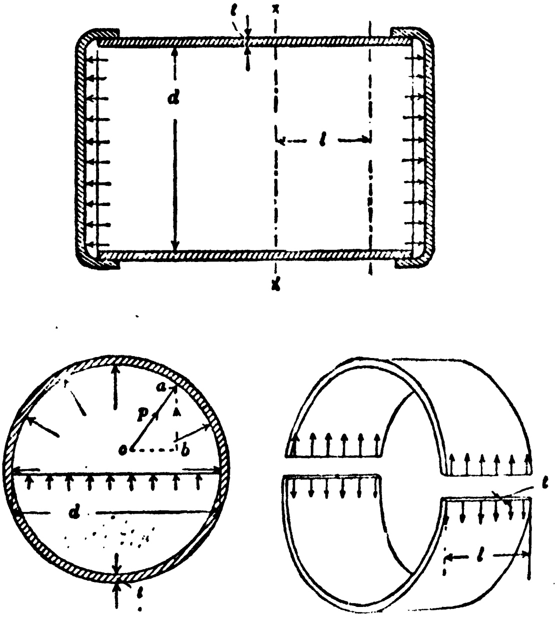


Fig. 54.—Stresses in Thin Pipes.

Circumferential Section.—If f_t' is the tensile stress across the section $x\ x$, we have

Force tending to cause failure $= p \times$ area of end.

$$= p \times \frac{\pi\,d^2}{4}$$

Force resisting failure $= f_t' \times$ area

$= f_t' \times \pi\,d\,t$

(because the pipe is thin).

$$\therefore f_t' = p\,\frac{\pi d^2}{4} \div \pi d t$$

$$= \frac{p d}{4 t} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$= \tfrac{1}{2} f_t$$

Therefore the stress across a longitudinal section is twice that across a circumferential section; for this reason longitudinal joints of boilers are made stronger than circumferential ones.

* EQUIVALENT STRESSES ON STRAIN THEORY.—On any small cube of the material with sides parallel and normal to x x, there will be a hoop stress f_t, a longitudinal stress $\frac{1}{2} f_t$, and a radial stress which is p on the inside and o on the outside of the tube and may generally be neglected.

$$\therefore \text{ Strain in longitudinal direction } = \frac{f_t}{E} - \frac{\eta f_t}{2E}$$

$$= \frac{f_t}{E}\left(1 - \frac{\eta}{2}\right)$$

$$= \frac{7}{8}\frac{f_t}{E} \text{ for } \eta = \frac{1}{4}$$

$$\therefore \text{ Equivalent hoop stress } = \frac{7}{8} f_t$$

$$\text{Similarly stress on circumferential section } = \frac{f_t}{2} - \eta f_t$$

$$= \frac{f_t}{4}$$

On the equivalent strain theory, therefore, the pressure on the ends strengthens the pipe.

NUMERICAL EXAMPLE.—*A boiler* 7 *ft.* 6 *ins. in diameter has to sustain a pressure of* 80 *lbs. per sq. in. If the efficiency of the joints is* 70 *per cent. and the safe stress is* 4 *tons per sq. in., find the thickness that the boiler should have and the necessary pitch of rivets on the longitudinal butt joint.*

$$t = \frac{p d}{2 f} = \frac{80 \times 90}{2 \times 4 \times 2240} = \frac{90}{224}$$

Efficiency of joint = 70 per cent.

$$\therefore \text{ Thickness must be } \frac{t \times 100}{70} = \frac{90 \times 100}{224 \times 70} = \cdot 573 \text{ in.}$$

say ⅝ in.

$$\text{Diameter of rivets} = 1{\cdot}2\sqrt{t} = 1 \text{ in. nearly.}$$

$$\text{Efficiency} = 70 \text{ per cent.}$$

$$\therefore \text{ In tension } \frac{p-d}{p} = \cdot 7$$

$$\text{i.e. } p - 1 = \cdot 7\,p$$

$$\therefore \cdot 3\,p = 1$$

$$p = 3{\cdot}3, \text{ say } 3\tfrac{1}{4} \text{ ins.}$$

$$\text{Strength of each rivet in shear} = \frac{\pi}{4} \times 1{\cdot}75\,d^2 \times 5$$

$$= 6{\cdot}9 \text{ tons.}$$

$$\text{Strength of plate per pitch} = 4 \times 2{\cdot}25 \times \frac{5}{8}$$

$$= 5{\cdot}6 \text{ tons.}$$

$\therefore$ Number of rivets required per pitch $= \frac{5{\cdot}6}{6{\cdot}9} = 1$ (as whole numbers only are possible).

$\therefore$ A single row of rivets are required on each side of the joint with a pitch of $3\tfrac{1}{4}$ ins.

Collapse of Thin Pipes under External Pressure.—If thin pipes are subjected to external pressure there will be hoop compression stresses which may be calculated by the same formulæ as we have obtained for the hoop tension due to internal pressure. If there is any inequality in the pipe bending stresses will be induced which will cause failure by collapse before the crushing strength of the material is reached, this collapse being similar to the failure of columns by buckling.

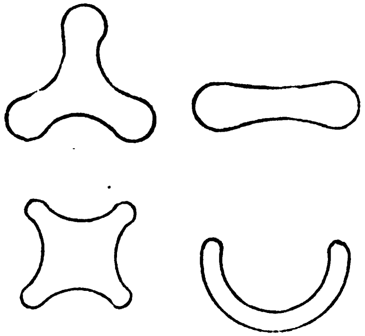


Fig. 55.

Fig. 55 shows some of the forms of collapse of such tubes.

Fairbairn's Formula.—The first well-known experiments on the subject were made about 1860 by Fairbairn. His formula is

$$p = \frac{806,300\, t^{2 \cdot 19}}{\mathrm{L}\, d}$$

p = collapsing pressure in lbs. per sq. in.
d = diameter of tube in inches.
t = thickness „ „
L = length of tube in feet.

Board of Trade Rule.—

Safe pressure $= \frac{60000\, t^2}{(\mathrm{L}+1)d}$ for single-riveted lap-welded tubes

$= \frac{90000\, t^2}{(\mathrm{L}+1)d}$ for welded or double-riveted butt-jointed tubes.

Stewart's and Illinois Experiments.*—These recent experiments were made with great care and proved that except for tubes of length less than about 5 diameters the collapsing pressure is practically independent of the length, so that Fairbairn's and the Board of Trade Rules are not applicable.

Fig. 56 shows the results of Professor Stewart's experiments for 4 and 7 in. tubes; it is clear from this that beyond a certain thickness the collapsing pressure is practically proportional to the thickness. The Illinois experiments confirmed this.

The following formulæ are given :—

Very thin tubes $\left(\frac{t}{d} < \cdot 03\right)$—

Brass: $p = 25,150,000 \left(\frac{t}{d}\right)^3$ (Illinois)

Cold-drawn seamless steel: $p = 50,200,000 \left(\frac{t}{d}\right)^3$ (Illinois)

Stewart finds that the same formula holds for lap-welded Bessemer steel tubes.

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* *Am. Soc. Mech. E.* 1906 (Reid T. Stewart); *Illinois University Bulletin,* 1906 (A. P. Carman and M. L. Carr).

Tubes in which $\frac{t}{d} > \cdot 03$—

$$\text{Brass: } p = 93365\,\frac{t}{d} - 2474 \text{ (Illinois)}$$

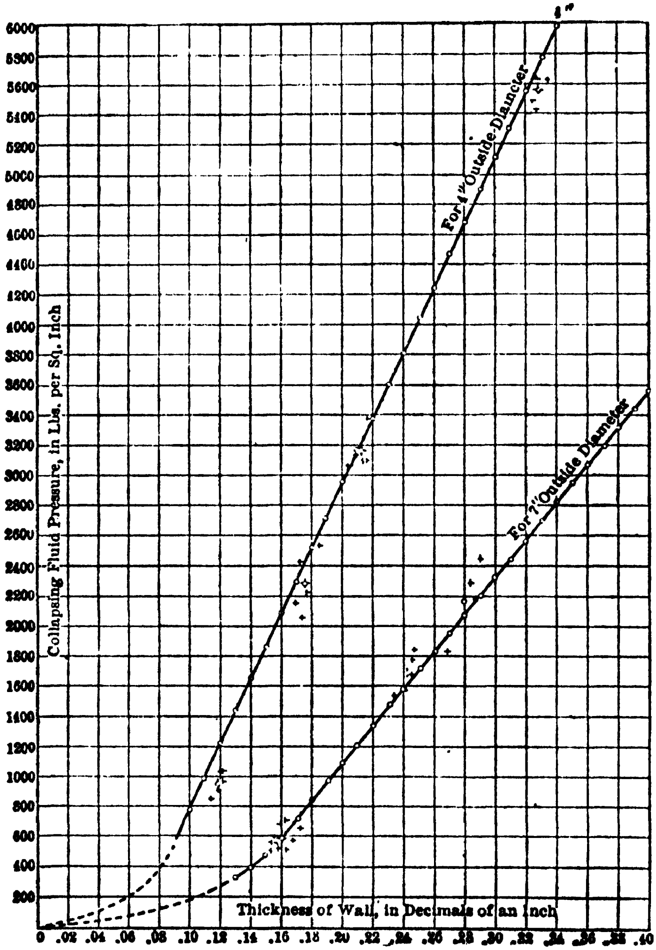

FIG. 56.—Collapse of Tubes. Stewart's Experiments.

$$\text{Seamless cold-drawn steel: } p = 95520\,\frac{t}{d} - 2090 \text{ (Illinois)}$$

$$\text{Lap-welded Bessemer steel: } p = 83270\,\frac{t}{d} - 1025 \text{ (Illinois)}$$

$$= 86670\,\frac{t}{d} - 1386 \text{ (Stewart)}$$

CHAPTER V

BENDING MOMENTS AND SHEARING FORCES ON BEAMS

Definitions.—The *shearing force* at any point along the span of a beam is the algebraic sum of all the perpendicular components of the forces acting on the portion of the beam to the right or to the left of that point.

The *bending moment* at any point along the span of a beam is the algebraic sum of the moments about that point of all the forces acting on the portion of the beam to the right or to the left of that point.

As the beam is in equilibrium under the forces acting on it, the algebraic sum of the forces at any point, and of the moments of the forces about the point, acting *on both* sides must be nothing; so that we shall get the same numerical values for the shearing force and bending moment from whichever side we consider them, but they will be opposite in sign. We will, wherever convenient, consider the shearing force and bending moment of the forces to the right of the section, and we will take an *upward* shearing force and an *anti-clockwise* bending moment as *positive*, the downward and clockwise being taken as negative.

Bending Moment and Shearing Force Diagrams.—If the bending moment and shearing force at every point of the span be plotted against the span and the points thus obtained be joined up, we shall get two diagrams called the Bending Moment (B.M.) and Shear diagrams, and from these diagrams the values of these quantities can be read off at any point of the span. We will consider the forms of these diagrams for various kinds of loading and for various ways of supporting the beam, but will only consider beams with fixed loads. We will use M_P and S_P to represent respectively the bending moment and shearing force at a point P.

B.M. AND SHEAR DIAGRAMS WITH FIXED LOADS

A. Cantilevers,* *i.e.* beams fixed at one end and free at the other, the loads being all at right angles to the length of the beam.

CASE 1. CANTILEVER WITH ONE ISOLATED LOAD.—Let a cantilever, fixed at the end B, Fig. 57, carry an isolated load W at the point A, at distance l from B. Consider any point P at distance x from A.

Then we have $S_P = W$.

This is constant throughout the span.

∴ Shear diagram is a rectangle of height W

Again $M_P = W \times x$

This is proportional to x.

∴ B.M. diagram is a triangle whose maximum ordinate is $W l$, this being the bending moment at the point B.

CASE 2. CANTILEVER WITH TWO ISOLATED LOADS.—Since the B.M. and shear at any point are defined as the sum of the moments and the forces to the left of that point, it follows that the B.M. and shear diagrams for a number of loads can be obtained by adding together the diagrams for the separate loads. In the present case, in which we have loads W_1 and W_2 at distances l_1 and l_2 from the fixed end, the diagrams are obtained by adding together the separate diagrams as shown in Fig. 57 (2).

CASE 3. CANTILEVER WITH UNIFORM LOAD.—Let a uniformly distributed load of w tons per foot run be carried by a cantilever A B of span l. Consider a point P at distance x from the free end A. Then

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* According to our convention the shears and B.Ms. for all the cases of cantilevers that we consider are negative. There is, however, no need to give the sign, unless both positive and negative values occur in the same beam.

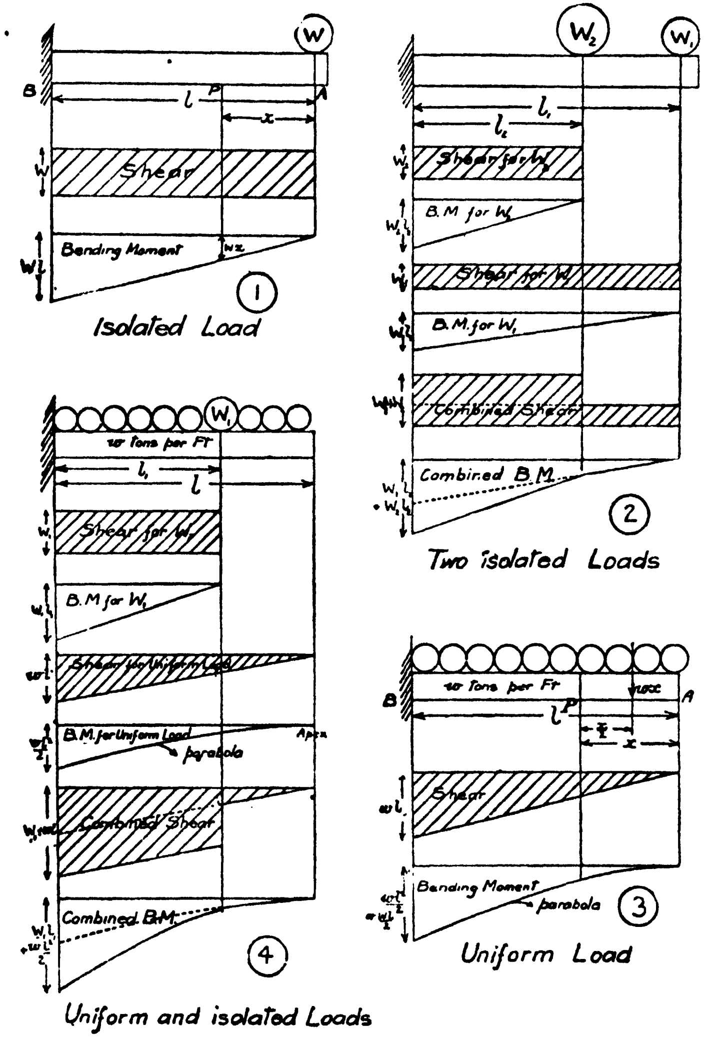

Fig. 57.—B.M. and Shear Diagrams for Cantilevers.

$$S_P = \text{load on A P}$$
$$= w\,x$$

This is proportional to x, and therefore the shear diagram is a triangle, the maximum shear occurring at the end B, and being equal to $w\,l$ or W, if W is the total load on the cantilever.

$$\begin{aligned} M_P &= \text{moment of load } w\,x \text{ about P} \\ &= w\,x \times \frac{x}{2} \\ &= \frac{w\,x^2}{2} \end{aligned}$$

This is proportional to x^2, and therefore the B.M. diagram will be a parabola with vertex at A. The maximum B.M. will be equal to $\frac{w\,l^2}{2}$ or $\frac{W\,l}{2}$ and occurs at B.

CASE 4. CANTILEVER WITH ISOLATED LOAD AND UNIFORM LOAD.—In this case, as in Case 2, the shear and B.M. diagrams are obtained by drawing the separate diagrams in accordance with Cases 1 and 3, and then adding them together as shown in the figure.

CASE 5. CANTILEVER WITH UNIFORMLY INCREASING LOAD.—Suppose a cantilever A B carries a load which increases in intensity uniformly from the free end A to the fixed end B, Fig. 58. This occurs in practice in the case of a vertical wall or side of a tank subjected to water pressure.

Let the intensity of load at unit distance from A be w tons per foot run, then the intensity at any point P at distance x from A will be equal to $w\,x$. The intensity of load at B will be $w\,l$, and the total load equal

$$\frac{w\,l}{2} \times l = \frac{w\,l^2}{2} = \text{W}$$

$$\begin{aligned} S_P &= \text{total load to left of P} \\ &= w\,x \times \frac{x}{2} = \frac{w\,x^2}{2} \end{aligned}$$

∴ Shear diagram is a parabola with vertex at A_1, the maximum shear at B being equal to W.

$$\begin{aligned} M_P &= \text{moment of load to left of P} \\ &= \frac{w\,x^2}{2} \times \frac{x}{3} = \frac{w\,x^3}{6} \end{aligned}$$

∴ B.M. diagram is a curve whose ordinates vary as x^3, such curve being called a parabola of the third order.

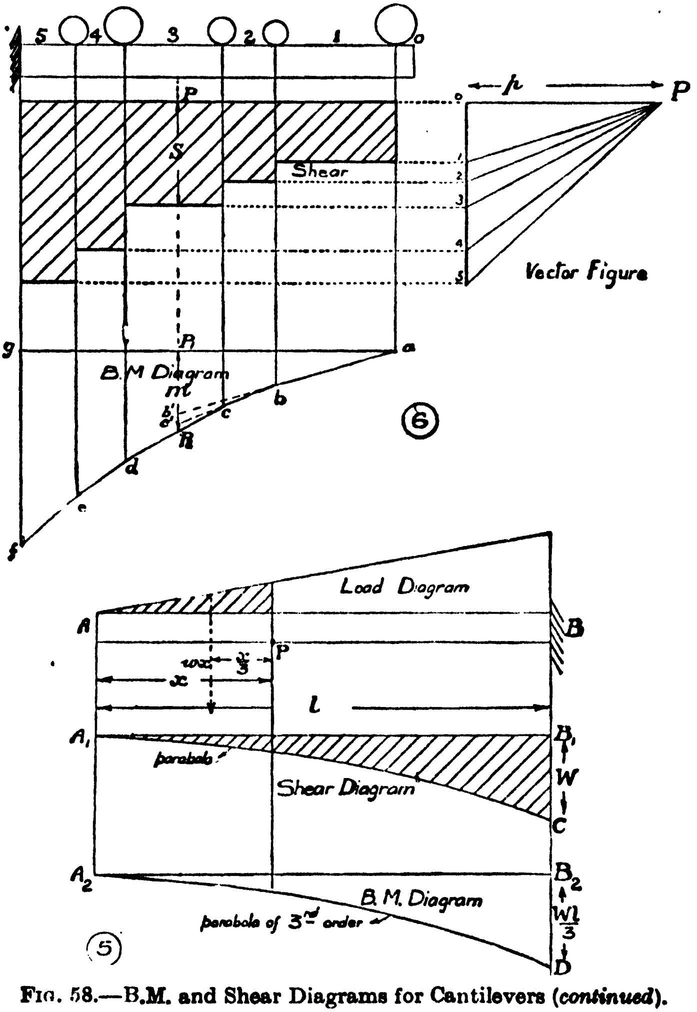

Fig. 58.—B.M. and Shear Diagrams for Cantilevers (*continued*).

The maximum B.M. at B is equal to $\frac{w\,l^3}{6} = \frac{W\,l}{3}$

The diagrams then come as shown in Fig. 58.

CASE 6. CANTILEVER WITH IRREGULAR LOAD SYSTEM.—GRAPHICAL METHOD.—Suppose a number of loads 0,1, 1,2, and so on, Fig. 58, act on a cantilever. To obtain the shear and B.M. diagrams set down 0, 1; 1, 2; 2, 3, &c., down a vector line 0,5 to represent the forces to some convenient scale, and take a pole P at some convenient distance p from the vector line 0,5 and join P to each of the points 0 to 5 on the vector line.

Now across the lines of the forces draw $a\,g$ parallel to P O; across space 1 draw $a\,b$ parallel to P 1; across space 2 draw $b\,c$ parallel to P 2, and so on until the point f is reached.

Then $a\,b\,c\,d\,e\,f\,g$ is the B.M. diagram.

To obtain the shear diagram, project the points 0–5 on the vector line across their corresponding spaces, the line through the point 0 being drawn right across the span, the stepped figure thus obtained being the shear diagram.

PROOF.—Consider any point P along the span, and produce $a\,b$ and $b\,c$ to cut the corresponding ordinate $P_1\,P_2$ of the link polygon at b' and c' respectively.

Now consider the Δs $a\,P_1\,b'$ and P 01.

They are similar, and as the bases of similar triangles are proportional to their heights, we have

$$\frac{P_1\,b'}{0,1} = \frac{a\,P_1}{p}$$

$$\therefore\ p \times P_1\,b' = 0,1 \times a\,P_1$$

But $0, 1 \times a\,P_1$ = moment of force 0, 1 about P.

$\therefore\ p \times P_1\,b'$ = moment of force 0, 1 about P.

Similarly it follows that

$p \times b'\,c'$ = moment of force 1, 2 about P,

and $p \times c'\,P_2$ = moment of force 2, 3 about P.

$\therefore$ We see that $p \times P_1\,P_2 = p\,(P_1\,b' + b'\,c' + c'\,P_2)$
= moment of all forces to right of P about P.
$= M_P$.

$\therefore$ Since p is a constant quantity, it follows that the ordinates of the link polygon represent the bending moments at the corresponding points of the beam.

Now consider the shear S at P. The total force to the right of P is 0, 1 + 1, 2 + 2, 3 = 0, 3, and this is obviously the value given on the shear diagram.

SCALES.—In all graphical constructions it is extremely important to state clearly the scales to which the various quantities are plotted, and to see that such scales are convenient for reading off.

Let the space scale be 1 in. = x feet
and the load scale on the vector line 1 in. = y tons
and let the polar distance be p actual inches.

Then the scale to which the bending moments can be read off is 1 in. = $p \times x \times y$ ft. tons.

p should thus be chosen so as to make this a convenient round number.

To take a numerical example, suppose the space scale is 1 in. = 4 ft. and the load scale is 1 in. = 2 tons, then if p is taken as $2\frac{1}{2}$ ins. the B.M. scale will be 1 in. = $4 \times 2 \times 2\frac{1}{2}$ = 20 ft. tons.

If p has been taken as 2 ins. the B.M. scale would have come 1 in. = 16 ft. tons, which would not be nearly such a convenient scale.

B. Simply Supported Beams.—*i. e.* beams simply resting on two supports, the loading all being at right angles to the length of the beam. Unless it is definitely stated to the contrary, we will always take it that the supports are at the ends of the beam.

In simply supported beams the forces acting are the loads and the reactions at the supports, the sum of the reactions being equal to the total load, and their values being obtained by means of moments. As the ends are freely supported, there can be no bending moment at either end.

We will now consider the following standard cases :—

CASE 1. ISOLATED LOAD IN ANY POSITION.—Let a load W be supported at a point C on a beam A B (Fig. 59) of span l, the distances of the point C from B and A being b and a respectively.

Then to get the reaction R_B at B take moments round A.

$$\text{Then } R_B \times l = W \times a$$

$$R_B = \frac{W \times a}{l}$$

$$\text{Similarly } R_A = \frac{W \times b}{l}$$

Now consider a point p between B and C.

$$S_p = R_B = \frac{+ W a}{l}$$

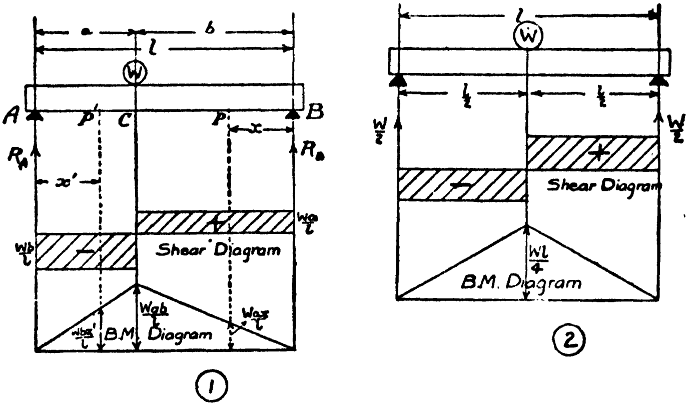

FIG. 59.—Simply Supported Beams. Isolated Load.

∴ between B and C the shear diagram is a rectangle of height $= \frac{W a}{l}$

Now take a point P′ between C and A.

$$S_{p'} = R_B - W$$

$$= \frac{W a}{l} - W = W\left(\frac{a - l}{l}\right) = \frac{- W b}{l} = - R_A$$

∴ Shear between C and A is a rectangle of height

$$= \frac{- W b}{l}$$

In the case of the cantilever there was no need to distinguish between positive and negative shear because there was no change in direction of the shear; but in the present case there is a change in direction, and so we will use the rule given on p. 121.

Now considering the bending moment,

$$M_P = R_B \times x = \frac{W.a.x}{l}$$

This is proportional to x, and therefore the B.M. diagram between B and C will be a triangle, the B.M. at C being equal to $\frac{W\,a\,b}{l}$. If P were between C and A and at distance x' from A we should have

$$\begin{aligned} M_P &= R_B(l - x') - W(l - x' - b) \\ &= R_B l - R_B.x' - Wl + Wx' + Wb \\ &= x'(W - R_B) + Wb - l(W - R_B) \\ &= R_A.x' + Wb - lR_A \\ &= \frac{W\,b\,x'}{l} + Wb - Wb \\ &= \frac{W\,b\,x'}{l} \end{aligned}$$

This is proportional to x', and therefore the B.M. diagram between A and C is also a triangle, the whole diagram then coming as shown in the figure.

CASE 2. ISOLATED LOAD AT CENTRE.—This is a special case of the preceding one, in which $a = b = \frac{l}{2}$

Each reaction is now equal to $\frac{W}{2}$ and the maximum

$$\text{B.M.} = \frac{W \times \frac{l}{2} \times \frac{l}{2}}{l} = \frac{Wl}{4}$$

CASE 3. UNIFORM LOAD OVER WHOLE SPAN.—Let a uniform load of w tons per ft. run cover the whole span A B, and consider a point C at distance x from B.

In this case the two reactions will, from symmetry, be equal, and each have the value $\frac{w\,l}{2}$ or $\frac{W}{2}$

$$\text{Then } S_C = R_B - w\,x = w\left(\frac{l}{2} - x\right)$$

This is a linear relation, therefore the shear diagram will be a triangle as shown, having values ± $\frac{w\,l}{2}$ at the ends and changing sign at the centre.

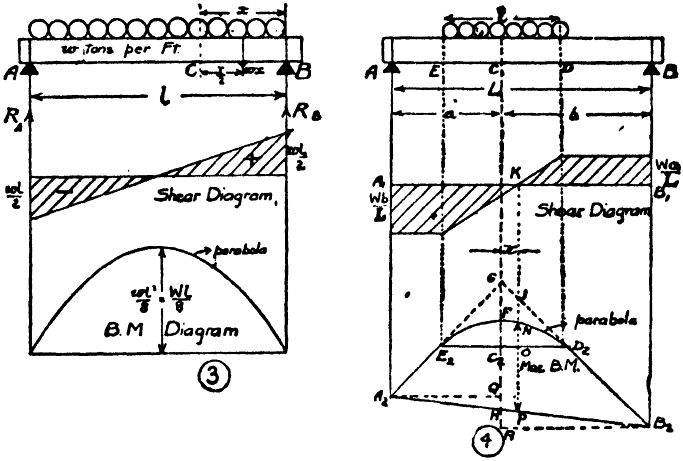

FIG. 59a.—Simply Supported Beams. Uniform Load.

Now consider the bending moment.

$$M_C = R_B \times x - w\,x \times \frac{x}{2}$$

$$= \frac{w\,l\,x}{2} - \frac{w\,x^2}{2} = \frac{w}{2}(l\,x - x^2)$$

This depends on x^2, and therefore the B.M. diagram will be a parabola.

The maximum B.M. will occur at the centre—*i.e.* when $x = \frac{l}{2}$.

$$\text{Then maximum B.M.} = \frac{w}{2}\left\{\left(\frac{l\,.\,l}{2}\right) - \left(\frac{l}{2}\right)^2\right\} = \frac{w}{2}\left(\frac{l^2}{2} - \frac{l^2}{4}\right)$$

$$\frac{w}{2} \times \frac{l^2}{4} = \frac{w\,l^2}{8} \text{ or } \frac{W\,l}{8}$$

CASE 4. UNIFORM LOAD OVER PORTION OF SPAN.—Let a uniform load of w tons per foot run and of length E D equal to l be placed on a beam A B of span L, and let the centre C of the load be at distance a and b respectively from A and B.

Then, if total load $w\,l = W$,

$$R_B = \frac{W\,a}{L} \text{ and } R_A = \frac{W\,b}{L}$$

The shear between B and D will be constant, and will be equal to $\frac{W\,a}{L}$; between D and E the shear will decrease uniformly until at E the shear will be equal to

$$R_B - W = \frac{W\,a}{L} - W = \frac{-W\,b}{L} = -R_A;$$

between E and A the shear will be constant and equal to $\frac{-W\,b}{L}$, the shear diagram then coming as shown on the figure.

The point K at which the shear is zero can be found as follows. Let it be at distance x from the centre C of the load.

$$\text{Then } S_K = R_B - w\left(\frac{l}{2} - x\right) = 0$$

$$i.e.\ \frac{W\,a}{L} - \frac{w\,l}{2} + w\,x = 0$$

$$w\,x = \frac{w\,l}{2} - \frac{W\,a}{L} = \frac{w\,l}{2} - \frac{w\,l\,a}{L}$$

$$\therefore x = \frac{l}{2} - \frac{a\,l}{L} = l\left(\frac{1}{2} - \frac{a}{L}\right)$$

The B.M. diagram can be drawn by setting up a length $C_2F = \frac{W\,l}{8}$, *i.e.* the bending moment at the centre of the short span E D, then produce C_2F to G, making F G equal to

C_2F and join G to E_2 and D_2, and produce to meet the reaction verticals in A_2 and B_2. Join A_2B_2, and we then have the B.M. diagram as shown.

To prove that this gives the correct diagram, consider the bending moment at a point at distance x from the centre of the load.

$$\text{Then } M_x = R_B(b - x) - \frac{w}{2}\left(\frac{l}{2} - x\right)^2$$

$$= \frac{Wa}{L}(b - x) - \frac{W}{2l}\left(\frac{l}{2} - x\right)^2 \ldots\ldots\ldots\ldots(1)$$

$$\text{Now, } \frac{GQ}{GC_2} = \frac{A_2Q}{E_2C_2}$$

$$\therefore GQ = \frac{GC_2 \times A_2Q}{E_2C_2} = \frac{Wl}{4} \times \frac{a}{\frac{l}{2}} = \frac{Wa}{2}$$

$$\text{Similarly } GR = \frac{Wb}{2}$$

$$\therefore QR = \frac{W}{2}(b - a)$$

$$\therefore QH = \frac{QR \times a}{L} = \frac{Wa}{2L}(b - a)$$

$$\therefore GH = GQ + QH = \frac{Wa}{2} + \frac{Wa}{2L}(b - a)$$

$$= \frac{Wa}{2}\left(1 + \frac{b - a}{L}\right)$$

$$= \frac{Wa}{2}\left(\frac{L + b - a}{L}\right) = \frac{Wa(b + b)}{2L} = \frac{Wab}{L}$$

$$\text{Again } \frac{JP}{GH} = \frac{b - x}{b}$$

$$\therefore JP = \frac{b - x}{b} \times GH = \frac{Wa(b - x)}{L}$$

$$\text{Again } \frac{JO}{GC_2} = \frac{OD_2}{C_2D_2}$$

$$\therefore JO = \frac{GC_2 \times OD_2}{C_2D_2} = \frac{Wl}{4} \times \frac{\frac{l}{2} - x}{\frac{l}{2}}$$

$$= \frac{W}{2}\left(\frac{l}{2} - x\right)$$

Then since curve is a parabola—

$$\frac{\mathrm{FC_2 - ON}}{\mathrm{FC_2}} = \left(\frac{\mathrm{OC_2}}{\mathrm{C_2D_2}}\right)^2$$

$$\therefore \frac{\frac{Wl}{8} - \mathrm{ON}}{\frac{Wl}{8}} = \frac{x^2}{\frac{l^2}{4}}$$

$$\therefore \frac{Wl}{8} - \mathrm{ON} = \frac{Wl}{8} \times \frac{4x^2}{l^2} = \frac{Wx^2}{2l}$$

$$\text{or } \mathrm{ON} = \frac{Wl}{8} - \frac{Wx^2}{2l}$$

$$\therefore \mathrm{NP} = \mathrm{PJ} - \mathrm{JO} + \mathrm{ON}$$

$$= \frac{Wa(b-x)}{\mathrm{L}} - \frac{W}{2}\left(\frac{l}{2} - x\right) + \frac{Wl}{8} - \frac{Wx^2}{2l}$$

$$= \frac{Wa}{\mathrm{L}}(b-x) - \frac{W}{l}\left(\frac{x^2}{2} - \frac{l^2}{8} + \frac{l^2}{4} - \frac{lx}{2}\right)$$

$$= \frac{Wa}{\mathrm{L}}(b-x) - \frac{W}{l}\left(\frac{x^2}{2} - \frac{lx}{2} + \frac{l^2}{8}\right)$$

$$= \frac{Wa}{\mathrm{L}}(b-x) - \frac{W}{2l}\left(\frac{l^2}{4} - lx + x^2\right)$$

$$= \frac{Wa}{\mathrm{L}}(b-x) - \frac{W}{2l}\left(\frac{l}{2} - x\right)^2$$

Comparing this with (1) we see that NP gives the B.M. at the given point.

We shall prove later (p. 149) that the B.M. is a maximum at that point of the span where the shear is zero, and so the vertical through K will give the maximum ordinate of the B.M. diagram.

Alternative Construction.—The following alternative construction is usually more accurate in practice. On a horizontal base $\mathrm{A_2B_2}$, Fig. 60, set up $\mathrm{C_2M}$ equal to $\frac{Wab}{\mathrm{L}}$, *i.e.* the B.M. due to an isolated load W placed at C, the centre of the load. Join $\mathrm{MA_2}$, $\mathrm{MB_2}$, cutting the verticals through E and D in N and D respectively, and join ND. On $\mathrm{E_2D_2}$ draw a parabola $\mathrm{E_2QD_2}$ of height equal to $\frac{Wl}{8}$ (the B.M. due to a uniform load on a span E D), then the B.M. diagram comes as shown shaded. If it is desired to have the diagram on a straight base, the parabola may be drawn on the inclined base N D as indicated in dotted lines. The proof of this construction is left as an exercise to the student.

CASE 5. IRREGULAR LOAD.—GRAPHICAL CONSTRUCTION.— Let a number of loads W_1, W_2, W_3, and W_4, be placed anywhere along a span A B. Number the spaces between the loads and set down, 0, 1; 1, 2; 2, 3; 3, 4, as a vertical vector line to represent the loads to some convenient scale, and in *any position* take a point P at convenient polar distance p from the vector line, and join P 0, P 1, P 2, etc.

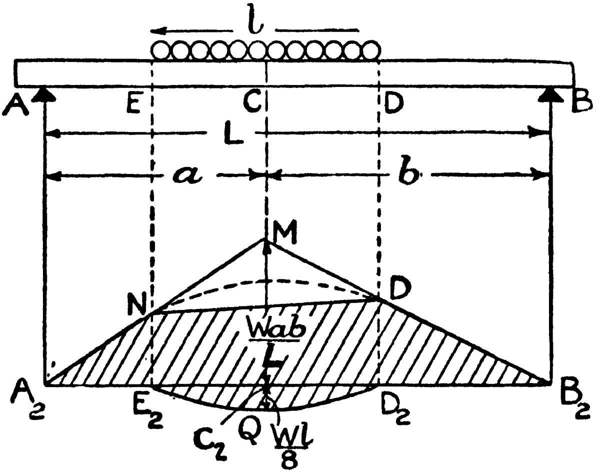

FIG. 60.—Alternative Construction for Uniform Load over Portion of Span.

Across space 0 then draw $a\ b$ parallel to P 0; across space 1 draw $b\ c$ parallel to P 1 and so on until $e\ f$ is reached, this being parallel to P 4.

Join $a\ f$, then the figure a, b, c, d, e, f, a, will give the B.M. diagram for the given load system.

Now draw P x parallel to $a\ f$, the closing link of the link polygon then on the vector line, $4\ x = R_B$ and $x\ 0 = R_A$.

To draw the shear diagram, draw a horizontal line through

x right across the span: this gives the base line for shear. Now project the point 0 horizontally across space 0; project point 1 across space 1 and so on, the stepped diagram thus obtained being the shear diagram.

PROOF.—Produce the links $c\,b$, $d\,c$, $e\,d$, $f\,e$ back to meet the vertical through A in b', c', d', e', and let the first link $a\,b$

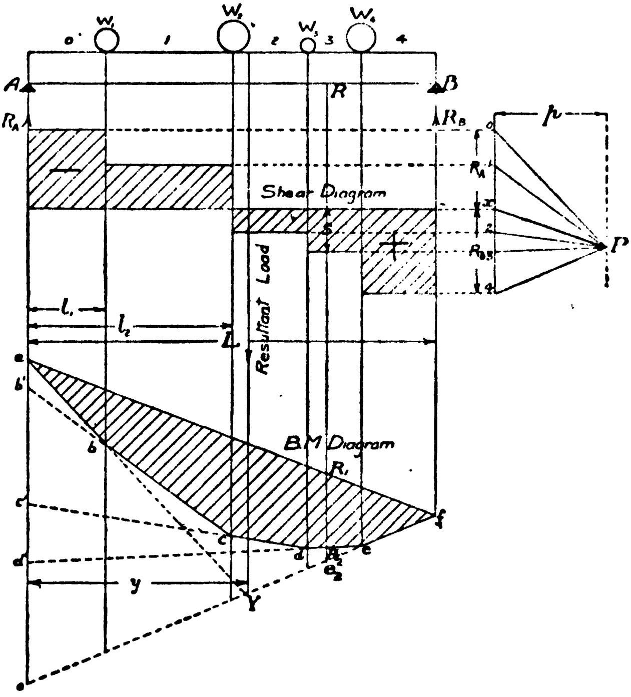

FIG. 61.—Graphical Construction for Shear and B.M. Diagrams.

produced meet the last link $e\,f$ in Y. Then the point Y is the point through which the resultant of the loads acts.

Now the triangles $a\,b\,b'$ and 0 P 1 are similar.

$$\therefore \frac{a\,b'}{l_1} = \frac{0,1}{p}$$

$$\therefore a\,b' = \frac{0,1 \times l_1}{p} = \frac{W_1 \times l_1}{p}$$

$$= \frac{\text{moment of first load about A}}{p}$$

similarly $b'c' = \dfrac{\text{moment of second load about A}}{p}$

and so on

$$\therefore\ a e' = a b' + b' c' + c' d' + d' e'$$

$$= \frac{\text{sum of moments of loads about A}}{p}$$

but $R_B \times L$ = sum of moments of loads about A

$$\therefore\ a e' = \frac{R_B \times L}{p}$$

Now consider Δs $a\, e'\, f$ and $x\, 4\, P$; they are similar:

$$\therefore\ \frac{a e'}{L} = \frac{4 x}{p}$$

$$\therefore\ 4 x = \frac{p \times a e'}{L} = R_B$$

Similarly $x\, 0 = R_A$

Now consider any point R along the span.

$$S_R = R_B - W_4$$
$$= 4 x - 3, 4 = 3 x$$

but the ordinate S of the shear diagram is equal to $3\, x$, and therefore the stepped figure gives the correct shearing force at any point.

Let the vertical through R cut the B.M. diagram in $R_1\, R_2$ and $f\, e$ produced in e_2.

Then by exactly similar reasoning as before

$$R_1 e_2 = \frac{\text{moment of } R_B \text{ about R}}{p}$$

$$R_2 e_2 = \frac{\text{moment of W 4 about R}}{p}$$

$$\therefore\ R_1 R_2 = R_1 e_2 - R_2 e_2$$

$$= \frac{\text{moment of } R_B - \text{moment of W 4 about R}}{p}$$

an $\underline{M_R}$
$= p$

$$\therefore\ M_R = p \times R_1 R_2$$

∴ The ordinate of the B.M. diagram represents the B.M. at any point.

SCALES.—As in the case of the cantilever (p. 126), if 1″ = x feet is the space scale and 1″ = y tons is the force scale, and if the polar distance is p actual inches, then the vertical ordinates of the B.M. diagram represent the bending to a scale 1″ = $p \times x \times y$ ft. tons.

NOTE.—In this construction the bending moment R_1 R_2 is measured *vertically* and not at right angles to the closing line *a f*.

CASE 6. IRREGULAR LOAD—OVERHANGING ENDS.—The construction just described is equally applicable to the case where the ends are overhanging. Fig. 62 shows such a case. Set out the loads down a vector line as before and take any pole P. Now draw *a b* parallel to P 0 across space 0, *i. e.* between the support vertical and the first force line. Then draw *b c* parallel to P 1 across space 1 and so on, the last link *e f* being drawn between the last force line and the reaction vertical. Joining *a f* we get the B.M. diagram as shown.

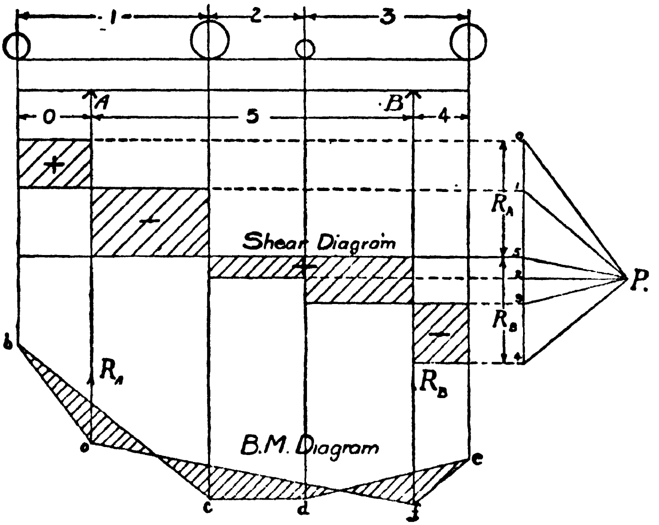

FIG. 62.—Beam with Overhanging Ends—Graphical Construction.

To get the shear diagram draw P 5 parallel to *a f*, then the horizontal through gives the base line for the shear between

A and B. The shears in the end spaces will be equal to the end forces 0, 1 and 3, 4 respectively, as shown on the figure.

This graphical loading is applicable to *all kinds of loading*, and any of the previous standard cases can be worked by its means. In the case of a continuous load the latter should be divided up into a number of small portions, and the load in each portion treated as an isolated load acting down the centre of such portion.

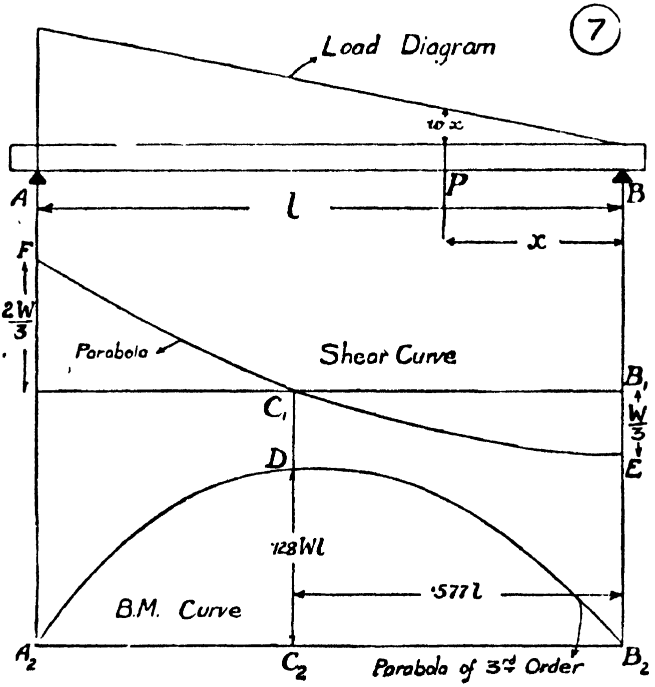

FIG. 63.—Simply supported Beam with Uniformly Increasing Load.

CASE 7. UNIFORMLY INCREASING LOAD.—Suppose a beam A B carries a load which increases in intensity uniformly from the end B to the end A. Let the intensity of the load at unit distance from B be w tons per ft. run; then the intensity at any point P at distance x from B will be equal to $w\,x$ (Fig. 63).

The intensity of the load at A will be equal to $w\,l$, and the total load W will be equal to $w\,l \times \frac{l}{2} = \frac{w\,l^2}{2}$

The resultant load W acts through the centroid of the load curve, *i.e.* at distance $\frac{l}{3}$ from A.

$$\therefore R_B = \frac{W}{3}$$

$$R_A = \frac{2W}{3}$$

$$\text{Then } S_P = \text{total load to right}$$

$$= \frac{W}{3} - \frac{w x^2}{2}$$

This depends on x^2 and therefore the shear curve is a parabola

The point C_1 is obtained as follows—

$$S_c' = 0 = \frac{W}{3} - \frac{w x^2}{2}$$

$$\therefore \frac{w x^2}{2} = \frac{W}{3} = \frac{w l^2}{2 \times 3}$$

$$\therefore x^2 = \frac{l^2}{3}$$

$$x = \frac{l}{\sqrt{3}} = \cdot 577\, l$$

$$M_P = R_B \times x - \frac{w x^2}{2} \cdot \frac{x}{3}$$

$$= \frac{W x}{3} - \frac{w x^3}{6}$$

This depends on x^3, and so the B.M. curve is a parabola of the third order.

The maximum B.M. occurs at the point of zero shear (see p. 149), *i.e.* when $x = \frac{l}{\sqrt{3}}$

$$\therefore \text{Maximum B.M.} = \frac{W l}{3\sqrt{3}} - \frac{w l^3}{18\sqrt{3}}$$

$$= W l \left(\frac{1}{3\sqrt{3}} - \frac{1}{9\sqrt{3}}\right)$$

$$= \frac{2 W l}{9\sqrt{3}} = \frac{2 W l \sqrt{3}}{27}$$

$$= \cdot 128\, W l.$$

The B.M. and shear curves then come as shown in the figure.

Case 8. Uniformly Loaded Beam with both Ends Overhanging.—Let a beam of span L be loaded with a uniform load of w tons per foot run, and let it overhang a distance x at each end, the distance between the supports being l.

The overhanging portions act as cantilevers, and the shear and B.M. diagrams for such portions will be as shown. The B.M. for the centre portion will be a parabola drawn on the base shown dotted, the resulting curve being as shown cross-hatched.

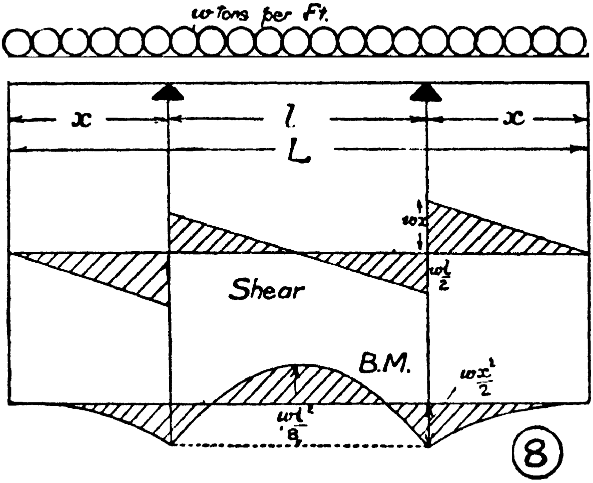

Fig. 63a.—Uniformly Loaded Beam with Overhanging Ends.

If the load on the centre portion of the span were removed, the B.M. diagram would consist of the two end parabolas and the dotted line. This B.M. is opposite in direction to that due to the centre portion, and therefore on replacing the centre load and drawing the parabola, the resulting curve is the difference between the two as shown.

To find the value of x to get the least resultant B.M. we proceed as follows.

As x increases, the B.M. at the supports increases and the resulting B.M. at the centre decreases, so that the least B.M. will occur when the support B.M. is equal to the centre B.M.

$$\text{The support B.M.} = \frac{w\,x^2}{2}$$

$$\text{The centre B.M.} = \frac{w\,l^2}{8} - \frac{w\,x^2}{2}$$

$$\text{If these are equal } \frac{w\,x^2}{2} = \frac{w\,l^2}{8} - \frac{w\,x^2}{2}$$

$$\therefore w\,x^2 = \frac{w\,l^2}{8}$$

$$x = \frac{l}{2\sqrt{2}}$$

$$\therefore \frac{l}{\text{L}} = \frac{l}{l + 2\,x} = \frac{l}{l + \frac{l}{\sqrt{2}}}$$

$$= \frac{1}{1 + \frac{\sqrt{2}}{2}} = \frac{2}{2 + \sqrt{2}}$$

$$= \frac{2\,(2 - \sqrt{2})}{2} = 2 - \sqrt{2} = \cdot 586$$

This gives the position at which the legs of a trestle table should be placed to give the maximum strength to the latter.

CASE 9. UNIFORMLY LOADED BEAM WITH ONE END OVERHANGING.—A beam A B, Fig. 64, of span L is supported at one end A and overhangs the support C at the other end; we wish to find, with a uniformly distributed load, the position of the support C which will be most economical—*i. e.* give the least bending moment.

If the length of the overhanging portion B C is l_2 and the distance A C is l_1, the B.M. diagram will be as shown shaded in the figure; the portion B_1 D is a parabola tangential at B_1 and is the familiar diagram for a cantilever with a uniformly distributed load; the portion A G C_1 is a parabola of height $\frac{w\,l_1^2}{8}$, the usual one for a freely supported span A C; and A_1 D is a straight line.

The maximum positive B.M. will be given by K J, which will be equal to $\frac{w\,(l_1 - a)^2}{8}$ since the B.M. between the point

A_1 and the point E of contraflexure will be the same as for a freely supported beam of span A_1 E, and the maximum negative value is given by C_1 D. Our problem resolves itself into find the position of C to make J K or C_1 D the least possible.

Now, if you move the point C to the left, C_1 D will increase

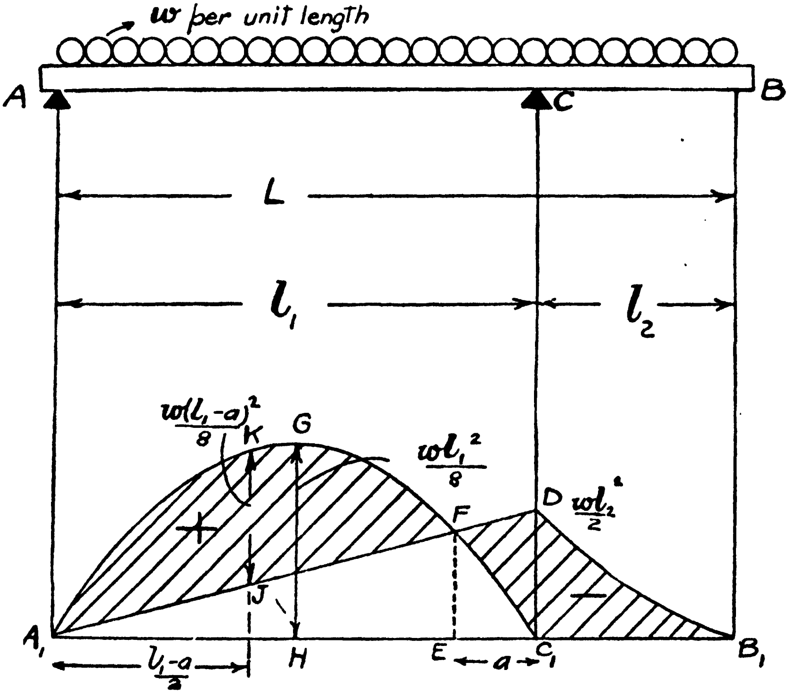

FIG. 64.

and K J will decrease, whereas if C moves to the right the converse happens. If, therefore, C_1 D = K J, movement of C will increase one or the other, so that the least value of either occurs when they are equal.

This gives
$$\frac{w\,(l_1 - a)^2}{8} = \frac{w\,l_2^2}{2}$$
$$i.e.\quad (l_1 - a)^2 = 4\,l_2^2$$
$$\text{or}\quad (l_1 - a) = 2\,l_2 \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Again, by the property of the parabola

$$\text{EF} = w\left(\frac{a\,l_1}{2} - \frac{a^2}{2}\right) \quad \ldots\ldots\ldots\ldots(2)$$

This can be found by taking the B.M. at E for the span $A_1\,C_1$; also by similar Δs.

$$\frac{\text{EF}}{C_1\,\text{D}} = \frac{A_1\,\text{E}}{A_1\,C_1}$$

$$i.e. \quad \text{EF} = \frac{w\,l_2^2}{2} \times \frac{l_1 - a}{l^1} \quad \ldots\ldots\ldots\ldots(3)$$

Combining (2) and (3) we get

$$\frac{w\,a}{2}\left(l_1 - a\right) = \frac{w\,l_2^2}{2}, \frac{l_1 - a}{l_1}$$

$$\text{or,} \quad a = \frac{l_2^2}{l_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Putting this result in (1) we get

$$l_1 - \frac{l_2^2}{l_1} = 2\,l_2$$

$$\text{i.e.} \quad l_1^2 - 2\,l_1\,l_2 - l_2^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

The solution of this quadratic equation gives, taking the positive root—

$$\frac{l_1}{l_2} = \frac{2 + \sqrt{4 + 4}}{2} = 1 + \sqrt{2} = 2{\cdot}414$$

$$\therefore\ 1 + \frac{l_1}{l_2} = \frac{l_1 + l_2}{l_2} = \frac{\text{L}}{l_2} = 1 + 2{\cdot}414 = 3{\cdot}414$$

$$\text{or,} \quad l_2 = \frac{\text{L}}{3{\cdot}414} \qquad i.e. \quad \underline{l_2 = {\cdot}293\,\text{L}}$$

In this case the maximum B.M. will be equal to

$$\frac{w\,l_2^2}{2} = \frac{w \times ({\cdot}293\,\text{L})^2}{2} = \frac{w\,\text{L}^2}{23{\cdot}3}$$

It will be of some interest to compare this result with that which would occur if each end were overhung and the supports were placed so as to give the least B.M. for this condition.

In this case the best condition is given when the overhang is ·207 L. This gives a maximum B.M. equal to

$$\frac{w \times ({\cdot}207\,\text{L})^2}{2} = \frac{w\,\text{L}^2}{46{\cdot}6} \text{ approx.,}$$

which is half that for the previous case.

Steps in Shear Curves.—In practice it is impossible to get absolutely sharp steps in shear diagrams, because the load cannot be transmitted at a mathematical point, but must be distributed over a short length. This has the effect of slightly rounding off the corners of the shear diagram as shown exaggerated in dotted lines on Fig. 65*a*.

NUMERICAL EXAMPLES.—(1) *A freely supported beam of* 20 *ft. span carries a uniformly distributed load of* 5 *tons, and isolated loads of* 3 *and* 2 *tons, at distances respectively of* 4 *and* 5 *ft.* moment *from the ends* (*see Fig.* 65).

We have first to get the reactions R_A and R_B.

Take moments round B.

$$R_A \times 20 = 5 \times 10 + 3 \times 16 + 2 \times 5$$
$$= 50 + 48 + 10 = 108$$
$$\therefore R_A = \frac{108}{20} = 5{\cdot}4 \text{ tons}$$
$$\therefore R_B = 10 - 5{\cdot}4 = 4{\cdot}6 \text{ tons.}$$

The shear diagram then comes as shown in the figure, the amounts of the steps being equal to the isolated loads. The point at which the shear is nothing is found as follows—

Let it be at distance x from B. Then

$$S_x = 0 = R_B - 2 - w.x$$
$$= 4{\cdot}6 - 2 - \frac{5x}{20}$$
$$= 2{\cdot}6 - \frac{x}{4}$$
$$\therefore \frac{x}{4} = 2{\cdot}6$$
$$x = 10{\cdot}4 \text{ feet.}$$

The B.M. at this point will be a maximum, and will be equal to

$$Mx = R_B \times 10{\cdot}4 - 2\,(10{\cdot}4 - 5) - \frac{1}{4}\cdot\frac{10{\cdot}4^2}{2}$$
$$= 47{\cdot}84 - 10{\cdot}8 - 13{\cdot}52$$
$$= 23{\cdot}52 \text{ ft. tons.}$$

The B.M. diagram will consist of a parabola for the uniformly distributed load, the max. ordinate of which is equal to $\frac{5 \times 20}{8} = 12{\cdot}5$ ft. tons. The B.M. diagram for each of the isolated loads will be a triangle, the respective heights being $\frac{3 \times 4 \times 16}{20} = 9{\cdot}6$ ft. tons, and $\frac{2 \times 5 \times 15}{20} = 7{\cdot}5$ ft. tons. Combining these three figures we get the B.M. diagram

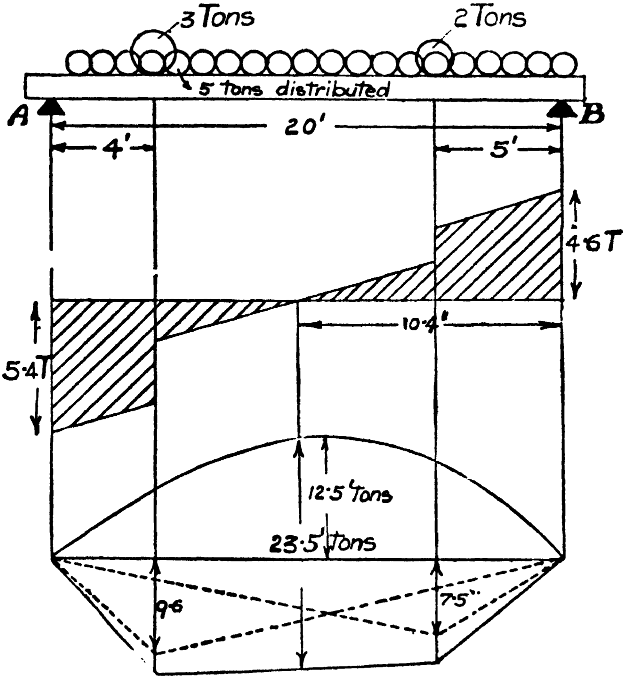

FIG. 65.

shown on the figure, and on scaling off the maximum ordinate it will be found to be 23·5 tons.

NOTE.—In all constructions where diagrams are going to be added together, such diagrams must of course be drawn to the same scale.

(2) *A girder of 24 ft. span is supported at one end, and rests on a column at a point 6 ft. from the other end. The girder carries a uniformly distributed load of 6 tons and an isolated load of 2 tons at the free end. Draw the shear and B.M. curves.*

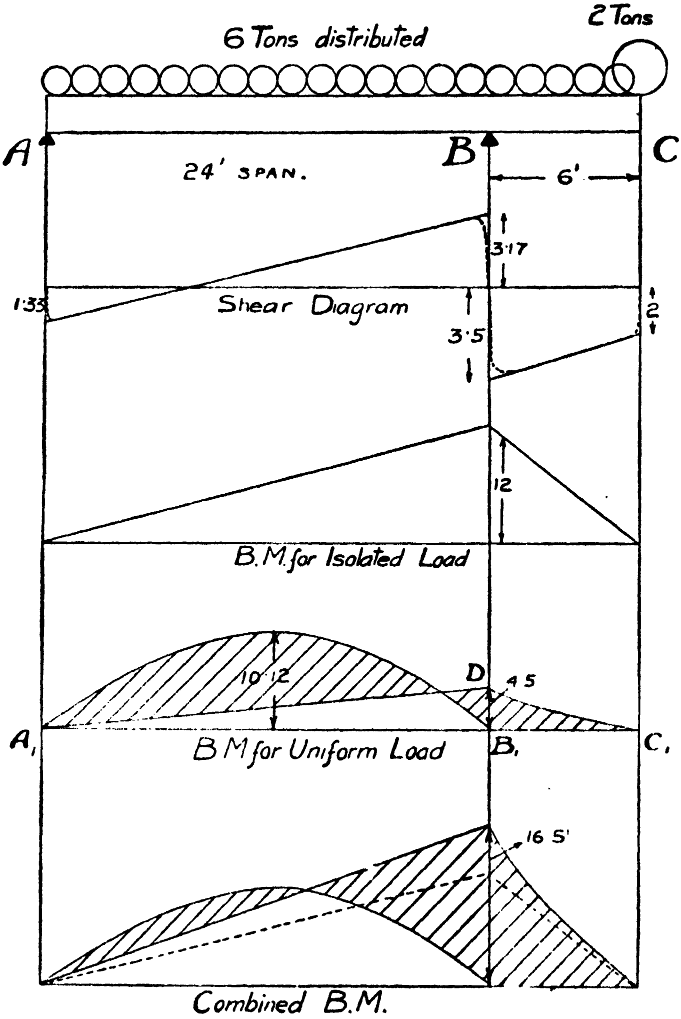

FIG. 65a.

To find the reactions take moments round A (Fig. 65a). Then

$$18\,R_B = 6 \times 12 + 2 \times 24 = 120$$

$$\therefore\ R_B = \frac{120}{18} = 6\tfrac{2}{3}\ \text{tons}$$

$$\therefore\ R_A = 8 - 6\tfrac{2}{3} = 1\tfrac{1}{3}\ \text{tons.}$$

The shear at C will be = 2 tons. It then increases until the point B is reached, when its value becomes equal to 3·5 tons. It then suddenly changes sign to a value 3·17 tons and then decreases uniformly to the end A, where the value comes 1·33. The shear diagram then curves as shown in the figure, the dotted lines indicating what occurs in practice owing to the impossibility of getting the loads and reactions concentrated on a mathematical point.

Considering first the B.M. for the isolated and uniform loads separately, the B.M. curve due to the isolated load will come as shown in the figure, the B.M. at B being equal to 6 × 2 = 12 ft. tons. Now, considering the uniform load, the diagram for the portion B C will be a parabola with vertex at C, the ordinate B_1 D at B_1 being $= \frac{wl^2}{2} = \frac{1}{4} \times \frac{6^2}{2} = 4{\cdot}5$ ft. tons. Then between B and A the B.M. curve due to this overhanging load will be the straight line A_1 D, as such overhanging load requires an isolated balancing load at A.

The B.M. curve for the portion A B will be a parabola of central height $= \frac{wl^2}{8} = \frac{1}{4} \times \frac{18^2}{8} = 10{\cdot}12$ ft. tons, the shaded portion being the resulting curve for the central and overhanging portions of the uniform load. Combining these diagrams we get the resulting B.M. curve as shown, the max. B.M. occurring at B, and being equal to 16·5 ft. tons.

Relation between Load, Shear and B.M. Diagrams.—Let A C′ D′ B, Fig. 66, represent the load curve on a span A B. Take any point P along the span, and consider a short piece C D of the load, the centre of which is at distance x from P.

Then the shear at P due to this piece of the load will be equal to the area of the portion C D of the load curve. Therefore the total shear S_p at P will be equal to the area of the load curve up to that point.

But a sum curve * is such that its ordinate at any point represents the area of the primitive curve up to that point. Therefore *the shear curve is the sum curve of the load curve.*

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* See p. 162.

Suppose B′ F E G A′ is the sum curve of the load curve. Now consider the B.M. at P.

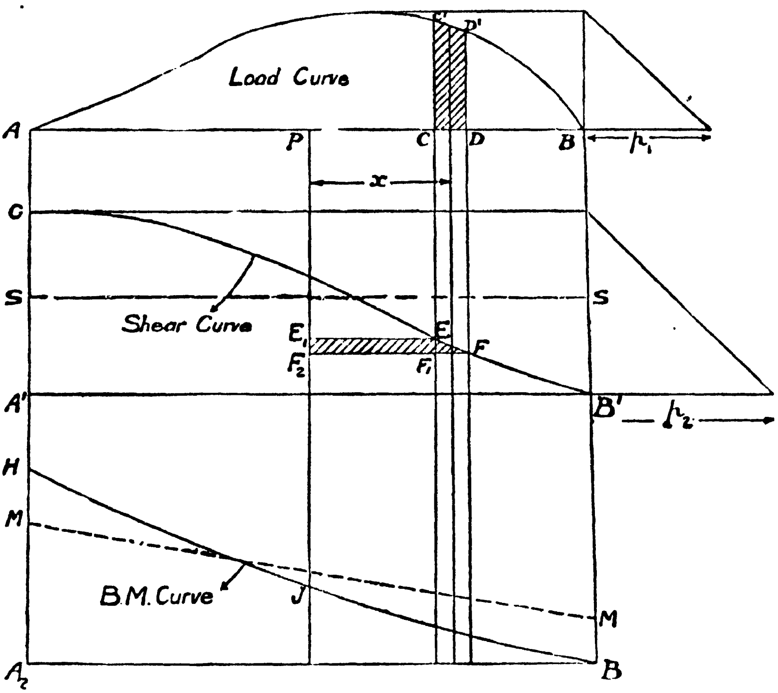


FIG. 66.—Relation between Load, Shear and B.M. Diagrams.

The B.M. at P due to the portion C D of the load

= given portion of load × x.

Now if E and F are the corresponding points on the shear curve, the difference of the ordinates at E and F gives the load on the portion C D.

∴ Load on portion C D = $E\,F_1$.

∴ B.M. at P due to portion C D = $E\,F_1 \times x$.

∴ Shaded portion $E\,F\,F_2\,E_1$ represents the B.M. at P due to the portion C D of the load.

$\therefore$ Total B.M. at P = M_P = area of shear diagram up to P.

Thus *the B.M. curve is the sum curve of the shear curve.*

So that by drawing the sum curve B J H of the shear curve we get the B.M. curve.

SCALES.—If $1'' = x$ tons per foot is the scale of the load curve, and p_1 is the polar distance measured on the space scale for obtaining the shear curve, then the scale of the shear curve $1'' = p_1 x$ tons. If p_2 is the polar distance from which the B.M. curve is obtained, measured on the space scale, the B.M. scale will be $1'' = p_2 p_1 x$ foot tons.

POINT OF MAXIMUM B.M.—If the B.M. is a maximum, the tangent to the curve at this maximum must be horizontal, and therefore the corresponding ordinate on the shear diagram must be zero in order for the line through the pole to be also horizontal.

Thus we get the rule that the maximum B.M. occurs where the shear is zero.

The base lines S S and M M of the shear and B.M. curves depend on the manner in which the ends are fixed. If one end is free, the shear and B.M. at this point are zero. If one end is freely supported the shear at this point will be equal to the reaction, and the B.M. will be zero.

The above relations are expressed mathematically as follows: Let the load at any point at distance x from the origin be $F(x)$

Then the shear at the point will be $= \int F(x)\,dx + C_1$ and the B.M. will be $= \int\int F(x)\,dx + C_1 x + C_2$.

The integration constants c_1 and c_2 depend on the manner in which the ends are fixed, and correspond to the base lines above referred to.

A Template for Bending Moment Diagrams.—FOR VARIOUS CASES OF UNIFORMLY DISTRIBUTED LOADS.—In designing beams carrying uniform loads it is necessary in order to draw the Bending Moment diagrams to draw parabolas; the usual procedure is to draw the parabolas for the special arrangement of the loads and for the particular manner in which the beams are supported, this involving a good deal of geometrical construction. A template can, however, be used to serve for a large number of cases in the following manner—

On a base A B, Fig. 67—for convenience say 5 ins. long—

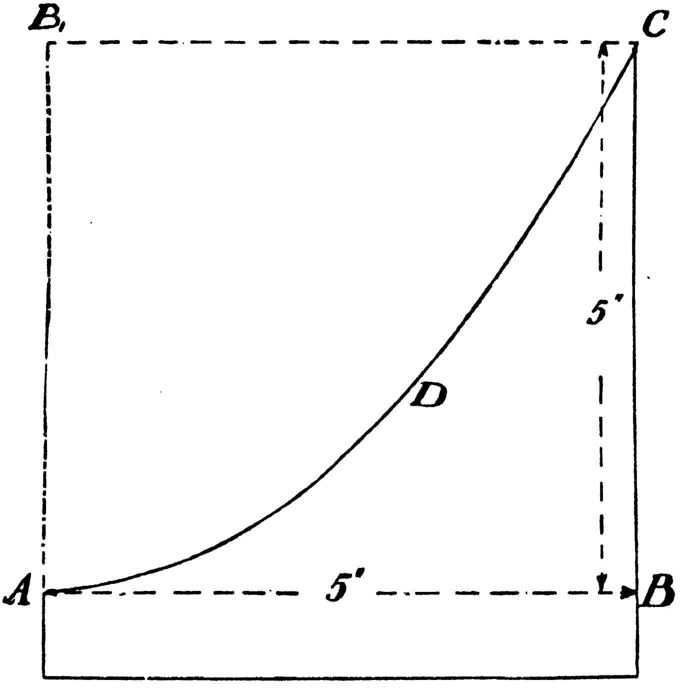


FIG. 67.

draw by the usual construction a parabola A D C with vertex at A, the height B C being for convenience equal to A B.

A template of the form A B C D can then be made, a 45° set-square being a convenient form to cut it from. A projection is preferably provided as shown to avoid a sharp point which is liable to break off. By means of this template and a suitable choice of scales, the Bending Moment (B.M.) diagrams for a large number of cases can be then drawn as follows—

CASE 1. CANTILEVER FULLY LOADED.—Draw the span E F, Fig. 68, to a suitable scale, so that E F is not larger than A B; erect a vertical at the fixed end F and place the template on the paper with the point A coinciding with the free end E and draw in the curve to the point G where it meets the vertical through B. The B.M. diagram is then as shown shaded.

Scales.—If the intensity of the load is w lbs. per foot run, then the B.M. scale will be the square of the space scale multiplied by $\frac{5w}{2}$. Take for instance the case where the space scale is

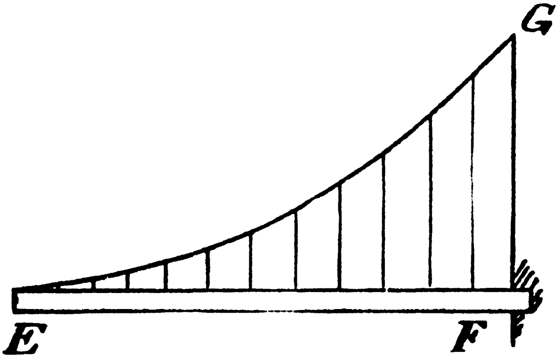


FIG. 68.

1 in. = 2 ft. and the load is 1000 lbs. per foot run; then B.M. scale is $1'' = \frac{4 \times 5 \times 1000}{2} = 10{,}000$ ft. lbs.

CASE 2. SIMPLY-SUPPORTED BEAM FULLY LOADED.—In this case, Fig. 69, we draw E_1 F_1 to represent the span and we draw as before a vertical F_1 G_1 at one end; the template is then placed on the paper with the point A coinciding with the point E_1 at the other end of the span and the curve is drawn until the vertical is cut to the point G_1. Now join G_1 E_1, the B.M. diagram then coming as shown shaded, the Bending Moment at any point H being found by projecting vertically and measuring the height x as shown.

The scales are obtained as previously described.

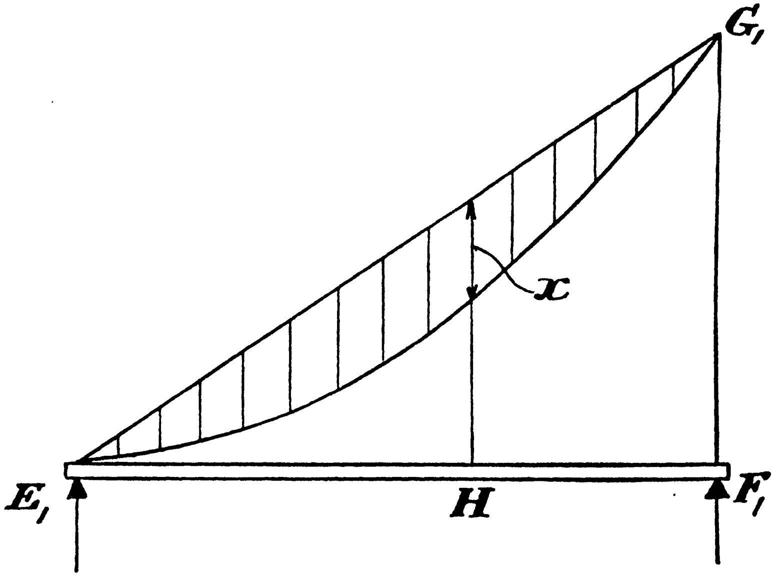


FIG. 69.

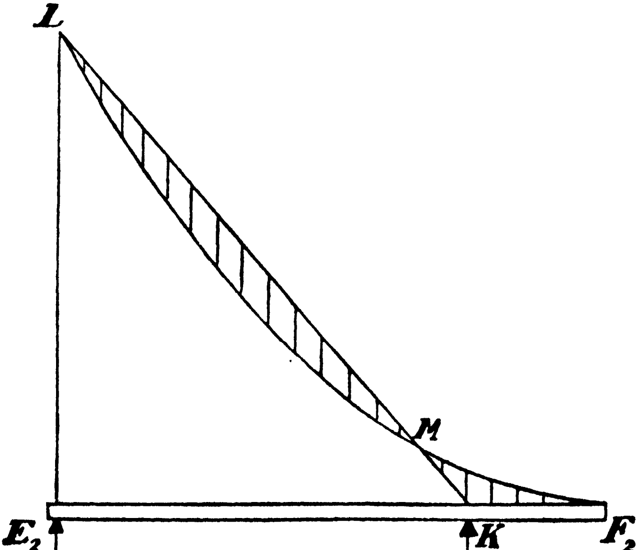


FIG. 70.

CASE 3. UNIFORMLY-LOADED BEAM OVERHANGING THE SUPPORT AT ONE END.—In this case, Fig. 70, we place the template on the paper with the vertex of the parabola at the overhanging end F_2 and draw in the curve until we meet at L the vertical through the other end E_2; then join L to the other support point K, the shaded area giving the B.M. diagram, the Bending Moment at any point being read off by projecting vertically as explained in the previous case.

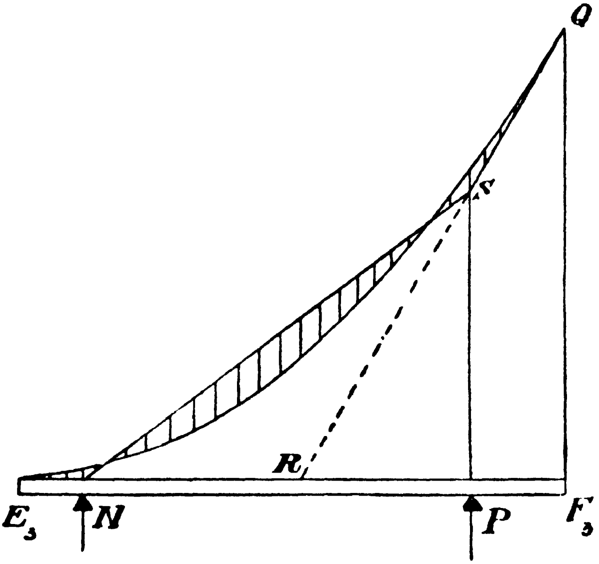


FIG. 71.

At points such as M where the B.M. diagram crosses itself, the Bending Moment changes sign; this of course corresponds to a reversal of the tension and compression flanges of the beam.

CASE 4. UNIFORMLY-LOADED BEAM OVERHANGING AT EACH END.—To obtain the B.M. diagram in this case with the aid of the template, we place the template on the paper with the vertex of the parabola coinciding with one end E_3 (Fig. 71) and draw the curve until we meet the vertical through the other end at Q.

Join Q to the mid-point R of the span, the line Q R cutting the vertical through the support P at the point S. Finally join S to the other support point N, the B.M. diagram then coming as shown shaded in the figure.

CASE 5. UNIFORMLY-LOADED CONTINUOUS BEAM OF TWO EQUAL SPANS.—We can get the B.M. diagram in this case with the aid of the template by placing it on the paper with the vertex coinciding with the end support E_4 and drawing in the curve until it intersects at the point G_3 the vertical through the centre support G_3; then by reversing the template and commencing the curve at the other outside support E_5 we shall get the reversed curve going from G_3 to E_5 as shown in Fig. 72.

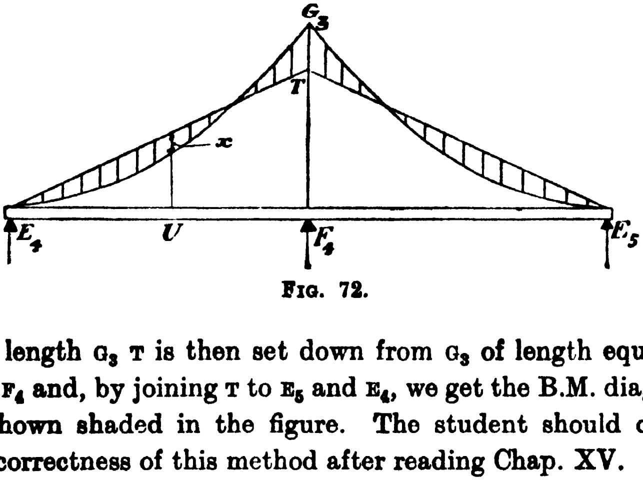


FIG. 72.

A length G_3 T is then set down from G_3 of length equal to $\frac{1}{4}$ G_3 F_4 and, by joining T to E_5 and E_4, we get the B.M. diagram as shown shaded in the figure. The student should check the correctness of this method after reading Chap. XV.

The scales are obtained as previously explained.

NUMERICAL EXAMPLE.—*Take a continuous beam of two equal spans each* 16 *ft. long, each span being covered by a load of* 1500 *lbs. per foot run.*

Taking a linear scale of 1″ = 4 ft., E_4 F_4 will be 4 ins.

Then the B.M. scale will be, as explained above,

$$1'' = \frac{4^2 \times 5w}{2} = \frac{16 \times 5 \times 1500}{2} = 60{,}000 \text{ ft. lbs.}$$

If the distance G_3 T be measured, it will be found to come equal to ·8 inch with a template of the dimensions suggested in Fig. 67.

$\therefore$ Maximum B.M. = ·8 × 60,000 = 48,000 ft. lbs.

The B.M. at any other point U can be obtained by reading off the vertical ordinate x to this scale.

In the case of a beam supported freely at one end and securely fixed at the other, the B.M. diagram will come the same as one-half of that shown in Fig. 72, the point E_4 being the freely-supported end and the point F_4 the fixed end.

A number of other cases might be given, but we think that the above are sufficient to show that a template of this kind would be of considerable assistance to draughtsmen for obtaining the B.M. diagrams for a variety of cases.

In some respects the template would be more easy to make if it were made of the shape A D O B_1, Fig. 67, because the convex curve can be somewhat more readily shaped. If, instead of the given dimensions for A B and B C, other values are taken, the rule for scales must be correspondingly amended, bearing in mind that B C should represent $\frac{AB^2}{2}$ for the B.M. scale to be equal to the square of the space scale when $w = 1$. If B C has not this value, then the B.M. scale will vary in the inverse ratio.

B.M. AND SHEAR DIAGRAMS FOR INCLINED LOADS

In all the cases that we have considered up to the present all the loading has been at right angles to the length of the beam. We will now consider some cases in which this is not the case, and will take both horizontal beams with non-vertical loads and sloping beams. The principal difference in this case is that there will be thrust in the direction of the beam, and we shall have a curve of thrust in addition to the curves of shear and B.M.

The general rule is to resolve all forces, including the reactions, along and perpendicular to the beam. From the forces along the beam a curve of thrusts can be drawn, and from the forces perpendicular to the beam the curves of shear and bending moment are drawn in the ordinary manner.

We will define the **thrust** at any point of a beam as the sum of the components in the direction of the beam of all the forces to the right of it, remembering that if the thrust is negative it becomes a pull.

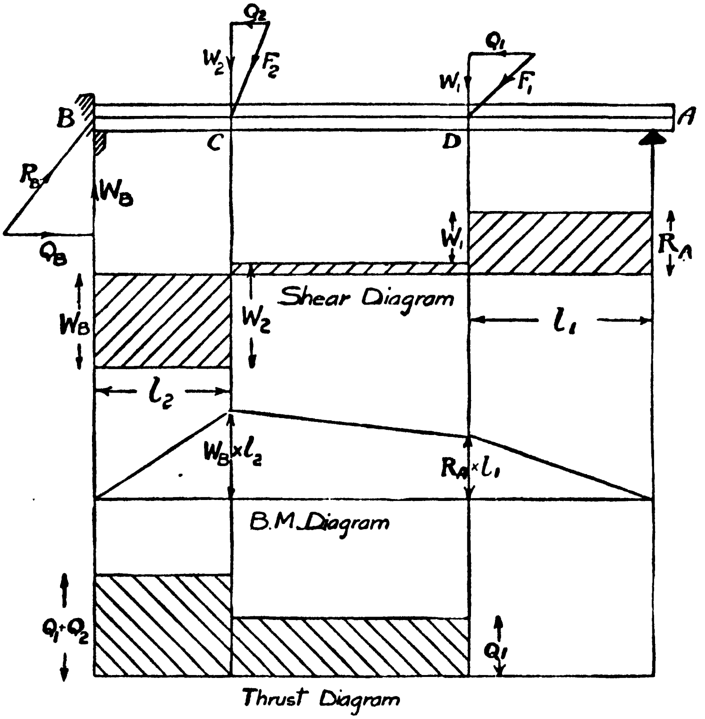


FIG. 73.—Beam with Inclined Loads.

CASE 1. HORIZONTAL BEAM FREELY SUPPORTED SUBJECTED TO INCLINED LOADS.—Let a beam A B have inclined forces F_1 and F_2 (Fig. 73) meeting the centre line in C and D. Let the end A rest on a free support and let the end B be freely supported, but prevented from longitudinal movement as shown. If the resultant of F_1 and F_2 acted towards the end A, then this end would have to be prevented from movement.

Resolve the forces F_1 and F_2 into vertical and horizontal components W_1 W_2 and Q_1 Q_2 respectively.

Then R_B will be inclined, the vertical component W_B being that found by considering the forces W_1 W_2 in the ordinary way and the horizontal component Q_B being equal to $Q_1 + Q_2$.

The reaction R_A will be vertical, and will be obtained by considering the forces W_1 and W_2 in the ordinary way.

If the resultant of F_1 and F_2 were found it would pass through the intersection of R_A and R_B, since three forces in equilibrium must pass through a point.

The shear and B.M. diagrams are then found in the usual way for weights W_1 and W_2, and are as shown.

The thrust diagram is obtained by plotting up at each point the value of the thrust, and this comes as shown. The same method applies for any number of loads, two having been chosen to give simplicity of figure.

CASE 2. INCLINED BEAM WITH VERTICAL LOADS.—REACTIONS PARALLEL.—Let an inclined beam A B (Fig. 74) be supported freely at A and pin-jointed at B. Then if it be subjected to vertical forces F_1 and F_2 at C and D, the reaction at A, and therefore also that at B, must be vertical, their values being found in the ordinary manner.

Now resolve the weights and reactions along and perpendicular to the beam, obtaining weights W_B, W_1, W_2, W_A, and thrusts Q_B, Q_1, Q_2, Q_A.

Then the B.M. diagram can be drawn either on a sloping base A B or the projected horizontal base A_1 B_1.

$$M_D = W_B \times DB$$

$$\text{but } \frac{DB}{l_1} = \frac{R_B}{W_B}$$

$$\therefore W_B \times DB = R_B\, l_1$$

∴ We see that for a sloping beam with vertical reactions the B.M. diagram is the same as for a horizontal beam of the same span as the horizontally projected length of the sloping beam.

The B.M. at a point P, for example, is obtained by drawing a vertical through it, *a b* representing the B.M.

The shear and thrust diagrams are obtained as shown, and will be easily followed from the figure.

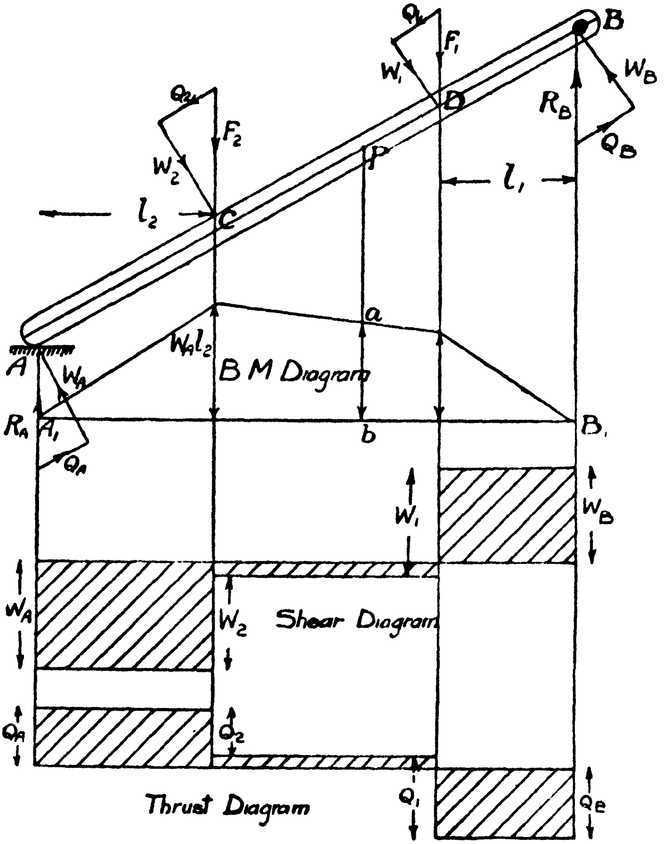


FIG. 74.—Inclined Beam with Lower End freely supported.

CASE 3. INCLINED BEAM WITH VERTICAL LOADS—TOP REACTION HORIZONTAL.—In this case the resultant load must first be found. Let this resultant act down the line X X (Fig. 75). The reaction R_B at B must be horizontal, so draw B X horizontal, then if this meets the line X X, R_A must also pass through X, so that by joining A X we get the direction of R_A. The values of R_A and R_B are then found by a triangle of forces a, b, c.

. Now resolve the weights and reactions as before along and perpendicular to A B. The perpendicular components will be

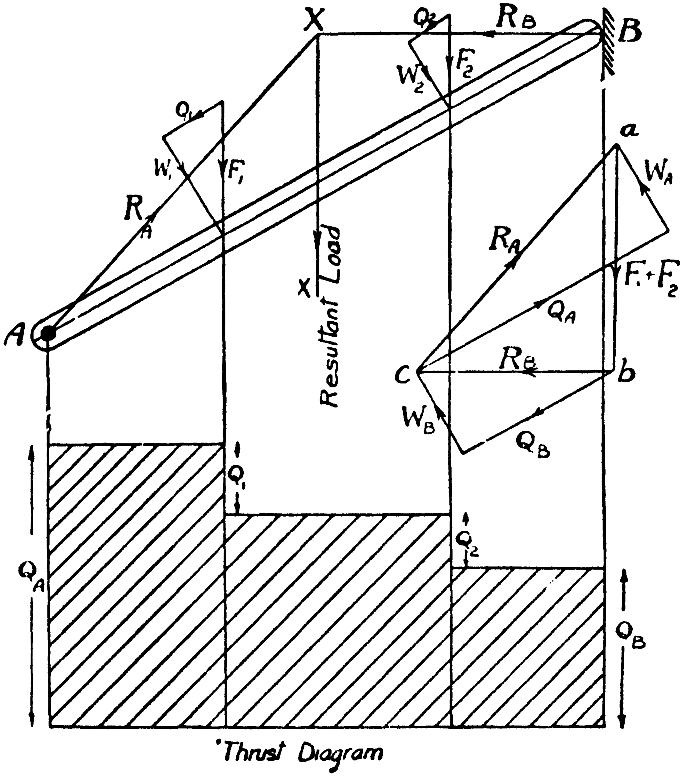


FIG. 75.—Inclined Beam with Top End freely supported.

the same as before, and so the B.M. and shear diagrams will be the same as in the previous case (Fig. 74).

The thrusts will be different, and will be as shown on the figure, which will be clearly followed.

CASE 4. SLOPING CANTILEVER.—This is worked in a similar manner. Consider, for example, a uniform load of intensity w on a cantilever of length l at an inclination θ (Fig. 76). The

B.M. curve will be a parabola. Its maximum ordinate will be $\frac{w\,l^2 \cos\theta}{2}$, because the total weight will be $w\,l$, and it acts at a distance $\frac{l \cos\theta}{2}$ from the abutment. The shear diagram

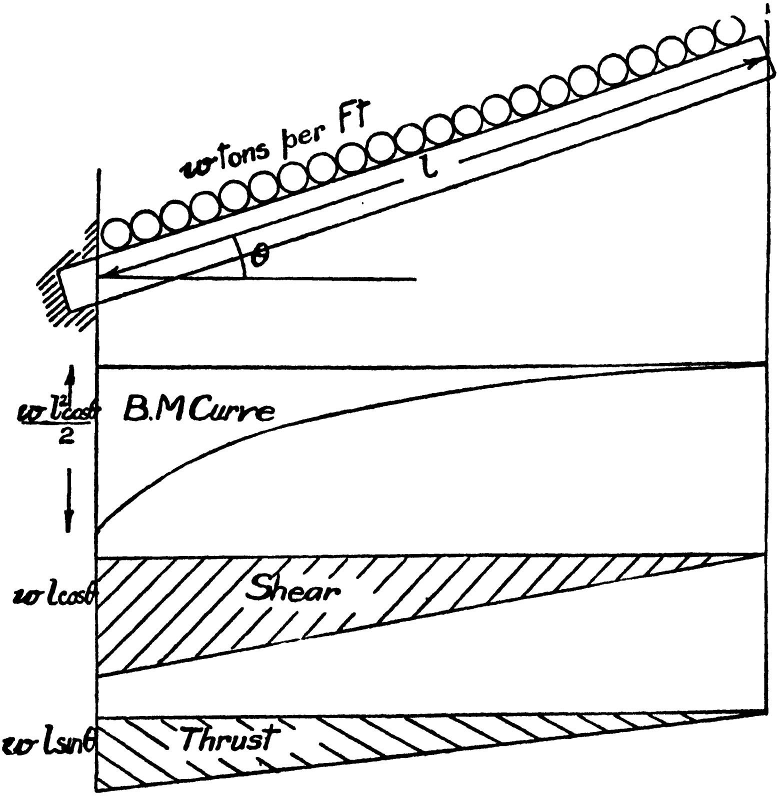


FIG. 76.

will be a sloping straight line, the maximum shear being $w\,l \cos\theta$; the thrust diagram will also be a sloping straight line, the maximum thrust being $w\,l \sin\theta$.

CHAPTER VI

GEOMETRICAL PROPERTIES OF SECTIONS—AREA, CENTROID, MOMENT OF INERTIA, AND RADIUS OF GYRATION

Before considering the relation between the Bending Moment and the stresses in a beam, we will consider some geometrical properties of sections which, as we shall find later, are involved in that relation.

The Determination of Areas.—(*a*) Mathematical Method.—If F (x) represents a function of x and the graph of the function be drawn, then the area between graph and the axis of x is given by the expression

$$A = \int F(x)\,dx$$

In practice, in the determination of areas, this method may become practically unworkable if the equation of the curve cannot be simply expressed or if the integration cannot be performed. When these conditions occur we have to rely on the planimeter or on the following.

(*b*) Graphical Method.—If a curve be plotted on a horizontal base and a new curve be drawn, such that its ordinate at any point represents the area of the given curve up to that point, the new curve is called the **Sum curve** or **Integral curve** of the given curve, which is called the **Primitive curve.**

The sum curve can be obtained graphically as follows: Let A C D, Fig. 77, be any primitive curve on a straight base A B. Divide A B into any number of parts, not necessarily equal (but for convenience of working they are generally taken as equal). These so-called base elements should be taken so small that the portion of the curve above them may be taken as a straight line. About 1 cm. or ·4 in. will usually be a suitable size and in most cases a smaller element 11 will come at the end. Find the mid-points, 1, 2, 3, etc., of each of the base elements and let the verticals through these mid-points meet the curve in 1 *a*, 2 *a*, 3 *a*, etc. Now project the points on to a vertical line A E, thus obtaining the points 1 *b*, 2 *b*, 3 *b*, etc., and join such points to a pole P on A B produced and at some convenient distance *p* from A. Across space 1 then draw A *d* parallel to P 1 *b*; *d e* across space 2 parallel to P 2 *b*, and so on, until the point *n* is reached. Then the curve A *d e* . . . *n* is the sum curve of the given curve, and to some scale B *n* represents the area of the whole curve.

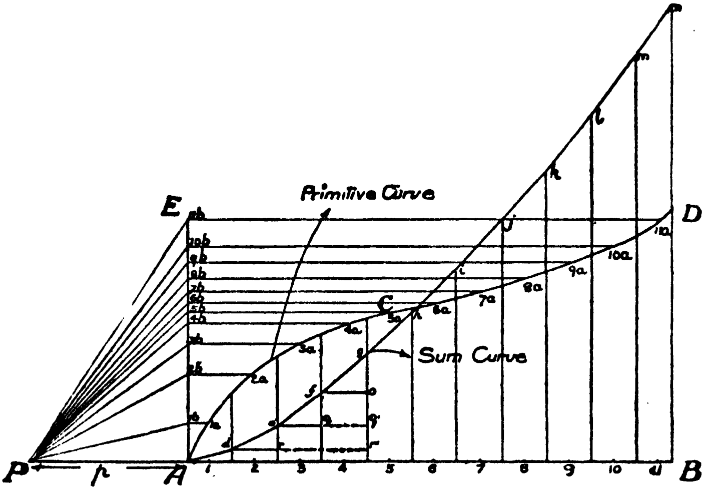


FIG. 77.—Sum Curve Construction.

PROOF.—Consider one of the elements, say 4, and draw *f o* horizontally.

Now $\triangle f, g, o$ is similar to the $\triangle$ P, 4 *b*, A

$$\therefore \frac{g\,o}{f\,o} = \frac{4\,b,\,\mathrm{A}}{\mathrm{P\,A}}$$

but $\mathrm{PA} = p$ and $4\,b,\,\mathrm{A} = 4,\,4\,a$

$$\therefore g\,o = \frac{f\,o \times 4,\,4\,a}{p} = \frac{\text{area of element 4 of curve}}{p}$$

$$\text{Similarly } fq = \frac{\text{area of element 3 of curve}}{p} \text{ and so on}$$

$$\therefore \text{ Ordinate through } g = go + fq + \ldots$$
$$= \frac{\text{area of first four elements of curve}}{p}$$

∴ The curve A d e ... n is the sum curve required.

Then if B n be measured on the vertical scale and p be measured on the horizontal scale, the area of the whole curve will be equal to $p \times$ B n.

It is obviously advisable to make p some convenient round number of units.

The sum curve obtained by this method may have the same operation performed on it, and thus the second sum curve of the primitive curve is obtained, and so on.

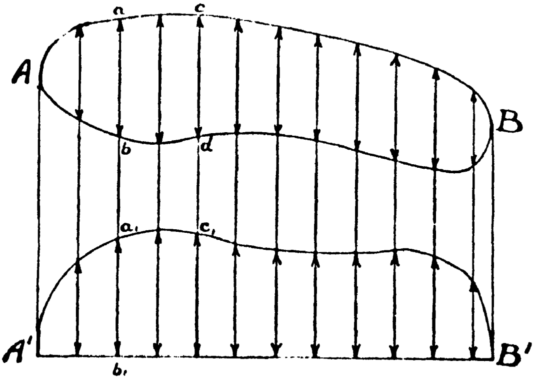


FIG. 78.

If the operation be performed on a rectangle, the sum curve will obviously come a sloping straight line, and if the sum curve of a sloping straight line be drawn, it will be found to be a parabola. In the case in which it is required to apply this construction to a curve which is not on a straight base, the curve is first brought to a straight base as follows—

Suppose A c B d, Fig. 78, is a closed curve. Draw verticals through A B to meet a horizontal base A′ B′. Divide the curve into a number of segments by vertical lines at short distances apart, and set up from the base A B lengths a_1, b_1, etc., equal to the vertical portions a, b, etc., on the curve. Joining up the points thus obtained we get the corresponding curve $A' c_1 B'$, on a straight base.

(*c*) SIMPSON'S RULE.—Divide the base into an even number of equal parts (each equal to C) and measure all the corresponding ordinates.

Then area of curve is equal to

$$\frac{1}{3}\text{C}\left\{\begin{matrix}\text{twice sum of}\\ \text{odd ordinates}\end{matrix}+\begin{matrix}\text{four times sum of}\\ \text{even ordinates}\end{matrix}-\begin{matrix}\text{sum of first}\\ \text{and last ordinates}\end{matrix}\right\}$$

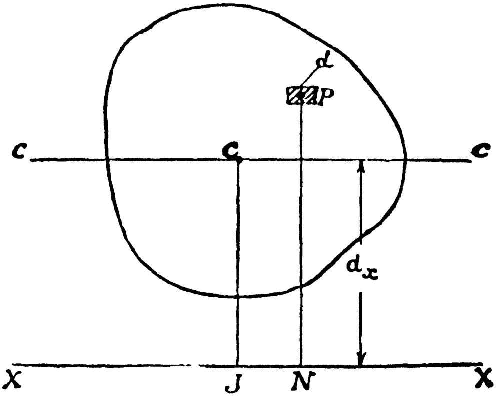


FIG. 79.—First Moment of an Area.

(*d*) PARMONTIER'S RULE.—Divide up base and measure ordinates as above, then area of curve is equal to

$$2\text{C}\times\begin{matrix}\text{sum of odd}\\ \text{ordinates}\end{matrix}-\frac{\text{C}}{6}\left\{\left(\begin{matrix}\text{second}\\ \text{ordinate}\end{matrix}-\begin{matrix}\text{first}\\ \text{ordinate}\end{matrix}\right)-\left(\begin{matrix}\text{last}\\ \text{ordinate}\end{matrix}-\begin{matrix}\text{preceding}\\ \text{ordinate}\end{matrix}\right)\right\}$$

First Moment of an Area.—Let a small element of area a of any figure be situated at the point P, Fig. 79, and let X X be any straight line or axis. Then if P N is drawn perpendicular to X X, $a \times$ P N is the first moment of the element of area about the given line. Now, if the whole figure is divided up into elements of area such as a, and the moments of each element be taken about X X and the whole of these moments be added together, the resulting sum is called the *first moment of the area.*

∴ The first moment of the whole area is the sum of quantities such as $a \times \text{PN}$. This is expressed symbolically as follows—

First moment of whole area $= \Sigma\,(a \times \text{PN})$.

Now the **centroid** or *the first moment centre* of an area is defined as the point at which the whole area can be considered concentrated, in order that its moment about any given line will be equal to the first moment of the area about the same line.

Thus if C is the centroid of the area, and CJ is drawn perpendicular to X X, and the area of the whole figure is A, we have

$$\text{A} \times \text{CJ} = \Sigma\,(a\,.\,\text{PN})\cdot$$

$$\therefore \text{CJ} = \frac{\Sigma\,(a\,.\,\text{PN})}{\text{A}}$$

This will not determine the exact position of C, but only its distance from the given line X X. If the exact position of the centroid is required we must also take moments about some other line, not parallel to X X, then the distance from the two lines will determine its position.

In connection with the centroid it should be noted that the position of the centroid depends solely on the shape of the figure, and not on the position of the axes about which moments are taken. As in the case of forces, we have positive and negative moments in areas, the moment being positive when the given element of area is above or to the right of the given axis, and negative when it is below or to the left.

First Moment about Line through Centroid.—Now consider the first moment of an area about a line C C, Fig. 80, through the centroid. The moments of elements of area above the line such as that at P will be positive, and the moments of elements of area below the line such as that at P′ will be negative.

Now in this case CJ is zero, and therefore A × CJ will also be zero, and therefore we have the rule that the first moment of any area about a line through its centroid is zero.

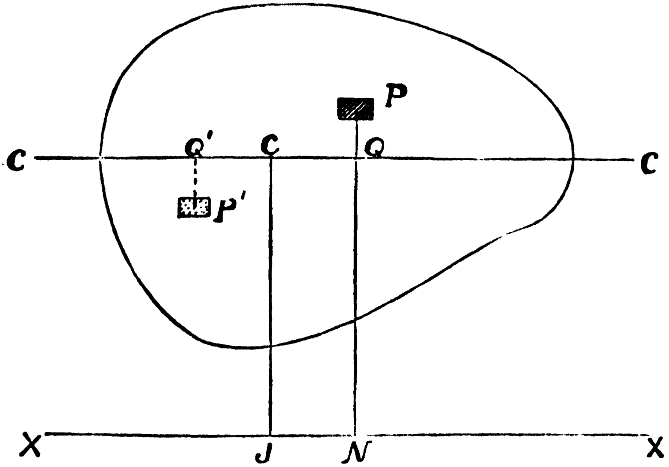


Fig. 80.

Position of Centroid with Axes of Symmetry.—Suppose an area has an axis of symmetry Y Y, Fig. 81. Then this line

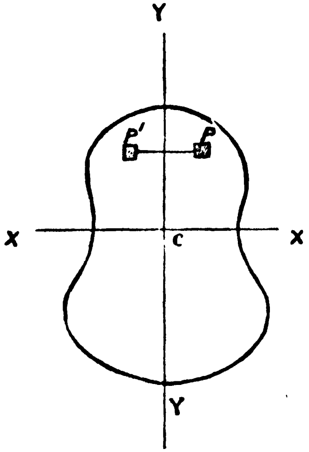


Fig. 81.

divides the area into two exactly similar halves so that corresponding to each element of area at P having a positive moment about Y Y we have an equal element at P′ having an equal negative moment about Y Y so that the total moment of the area about Y Y is zero, or Y Y passes through the centroid.

If the figure has another axis of symmetry X X, the centroid also lies on this line, or we have the rule that the centroid of a figure is at the intersection of two axes of symmetry.

For the determination of the position of the centroid for various cases see pp. 175–189.

It should be noted that the centroid of an area is the same as the centre of gravity of a template of the same shape as the area.

Second Moments or Moments of Inertia.—The product of a $\left\{\begin{matrix}\text{force } (f)\\ \text{mass } (m)\\ \text{area } (a)\\ \text{volume } (v)\end{matrix}\right\}$ by the *square* of its distance r from a given point or axis is called the *second moment* of the $\left\{\begin{matrix}\text{force}\\ \text{mass}\\ \text{area}\\ \text{volume}\end{matrix}\right\}$ about the given line or axis.

Now, in considering rotating bodies the second moment of the mass has to be considered, and this quantity has been given the name of the *moment of inertia.* In the application of the second moment to the strength of materials we shall have nothing to do with inertia, but the term *moment of inertia* has been generally adopted, and so we shall use it; but we must remember that it is really a borrowed term and quite an unsuitable one.

APPLICATION TO AREAS.—If an element of area a is situated at the point P, Fig. 82, and P N is drawn perpendicular to a line X X, then the second moment of this element of area about the line X X is equal to $a \times \text{P N}^2$. If, as in the case of the first moment, we divide the whole area up into elements and take the second moment of each, we see that the second moment of the whole area about X X is the sum of the second moments of the elements. The letter I is always used to denote the second moment, the line X X, about which the moments are taken, being indicated by writing it I_{xx}.

Thus we see $\text{I}_{xx} = \Sigma\,(a \times \text{P N}^2)$.

In the same way, considering the line Y Y, we have

$$I_{YY} = \Sigma (a \times PM^2).$$

Now suppose K is such a point that the whole area can be considered concentrated there so as to give the same second moments about X X and Y Y as the second moment of the area about these lines.

Then $A \times KQ^2 = I_{XX}$
and $A \times KR^2 = I_{YY}$.

Then the point K by analogy might be called the **secondroid** of the area with regard to the axes X X and Y Y. The point of importance with regard to the secondroid is that its position depends on the position of the lines about which the moments are taken, whereas the position of the centroid does not

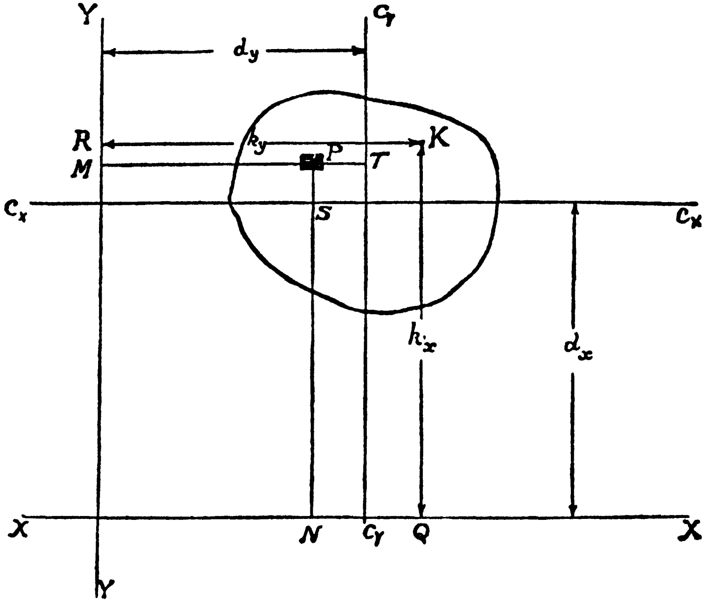


FIG. 82.—Second Moment or Moment of Inertia of an Area.

RADIUS OF GYRATION

Now, the distances of the secondroid from the lines X X and

Y Y are called the *second moment radii* or *radii of gyration* about the given lines, and are written k_x and k_y respectively.

∴ We have

$$\begin{aligned} \mathrm{A}\,k_x^2 &= \mathrm{I}_{\mathrm{XX}} = \Sigma\,(a \times \mathrm{PN}^2) \\ \text{or } k_x &= \sqrt{\frac{\mathrm{I}_{\mathrm{XX}}}{\mathrm{A}}} \\ \mathrm{A}\,k_y^2 &= \mathrm{I}_{\mathrm{YY}} = \Sigma\,(a \times \mathrm{PM}^2) \\ \text{or } k_y &= \sqrt{\frac{\mathrm{I}_{\mathrm{YY}}}{\mathrm{A}}} \end{aligned}$$

Now, in practice it is nearly always the second moment about a line through the centroid that is required, and this is obtained as follows—

GIVEN THE SECOND MOMENT OR MOMENT OF INERTIA OF AN AREA ABOUT A GIVEN LINE, TO FIND IT ABOUT A PARALLEL LINE THROUGH THE CENTROID.

Suppose we know I_{XX}.

$$\begin{aligned} \text{Now, } \mathrm{I}_{\mathrm{XX}} &= \Sigma\,(a \times \mathrm{PN}^2) \\ &= \Sigma\,[a \times (\mathrm{PS} + \mathrm{SN})^2] = \Sigma\,[a \times (\mathrm{PS} + d_x)^2] \\ &= \Sigma\,[a\,.\,(\mathrm{PS}^2 + 2\,\mathrm{PS}\,.\,d_x + d_x^2)] \\ &= \Sigma\,(a\,.\,\mathrm{PS}^2) + \Sigma\,(a\,.\,2\,\mathrm{PS}\,.\,d_x) + \Sigma\,(a\,.\,d_x^2) \end{aligned}$$

Of the terms on the right-hand side

$$\begin{aligned} \Sigma\,(a \times \mathrm{PS})^2 &= \mathrm{I}_{\mathrm{cx}} \text{ (which is required)} \\ \Sigma\,(a\,.\,2\,\mathrm{PS}\,.\,d_x) &= 2\,d_x\,\Sigma\,a\,.\,\mathrm{PS} \\ &= 2\,d_x \text{ (first moment of area about line } \mathrm{O_X\,O_X} \text{ through centroid)} \\ &= 2\,d_x \times 0 \\ &= 0 \\ \Sigma\,(a\,.\,d_x^2) &= d_x^2\,\Sigma\,a \\ &= d_x^2 \text{ (area of whole figure)} \\ &= d_x^2\,.\,\mathrm{A} \end{aligned}$$

$$\begin{aligned} \therefore \text{ We have } \mathrm{I}_{\mathrm{XX}} &= \mathrm{I}_{\mathrm{OX}} + \mathrm{A}\,d_x^2 \\ \text{or } \mathrm{I}_{\mathrm{OX}} &= \mathrm{I}_{\mathrm{XX}} - \mathrm{A}\,d_x^2 \\ \text{Similarly } \mathrm{I}_{\mathrm{OY}} &= \mathrm{I}_{\mathrm{YY}} - \mathrm{A}\,d_y^2 \end{aligned}$$

* **The Momental Ellipse or Ellipse of Inertia.**—The *principal axes* of a section are defined as two axes at right

angles through the centroid, such that the sum of quantities such as $a \times$ PM, PN, or the *product moment* as it is called, is equal to zero.

In the case of sections with an axis of symmetry, such axis determines one of the principal axes.

Let X X and Y Y, Fig. 83, be the principal axes of a section and let k_x and k_y be the radii of gyration about the two axes. With O as centre draw an ellipse, O X being equal to k_y and O Y being equal to k_x. Then this ellipse is called the *momental ellipse* or *ellipse of inertia*.

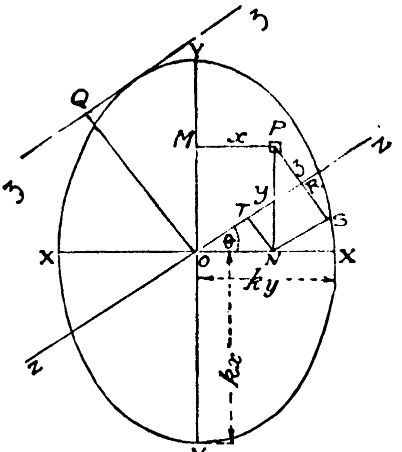


FIG. 83.—Momental Ellipse of Ellipse of Inertia.

To obtain the radius of gyration k_z about a line Z Z passing through O at an angle θ to X X, draw $z z$ a tangent to the ellipse parallel to Z Z, and draw O Q perpendicular to it.

Then $\text{O Q} = k_z$

for

$$
\begin{aligned}
I_{zz} &= \Sigma\, a \,.\, \text{P R}^2 \\
&= \Sigma\, a\, (\text{P S} - \text{S R})^2 \\
&= \Sigma\, a\, (\text{P S} - \text{N T})^2 \\
&= \Sigma\, a\, (y \cos\theta - x \sin\theta)^2 \\
&= \Sigma\, a \,.\, x^2 \sin^2\theta + \Sigma\, a\, y^2 \cos^2\theta - \Sigma\, 2\, x\, y \sin\theta \cos\theta \\
&= \sin^2\theta\, \Sigma\, a \,.\, x^2 + \cos^2\theta\, \Sigma\, a\, y^2 - 2 \sin\theta \cos\theta\, \Sigma\, x\, y
\end{aligned}
$$

Now, $\Sigma\, x\, y$ is the product moment, and as X X and Y Y are the principal axes, this is zero.

$$\therefore\; I_{zz} = \sin^2\theta\,\Sigma\,(a\,.\,x^2) + \cos^2\theta\,\Sigma\,(a\,y^2)$$
$$= I_{YY}\sin^2\theta + I_{XX}\cos^2\theta$$
$$\therefore\; A\,k_z^2 = A\,k_y^2\sin^2\theta + A\,k_x^2\cos^2\theta$$
$$k_z^2 = k_y^2\sin^2\theta + k_x^2\cos^2\theta$$

and therefore from the properties of the ellipse O Q is equal to k_z.

A rather more convenient construction (see Fig. 84) is to

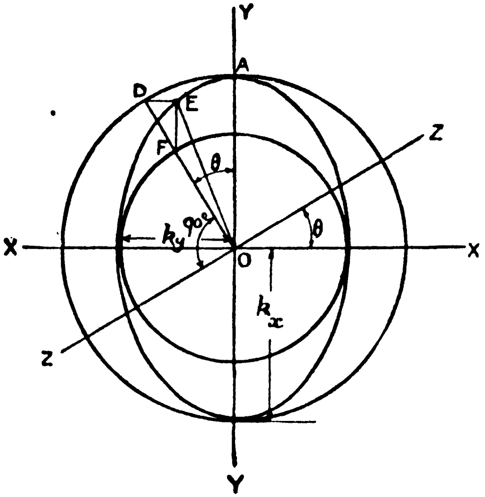


FIG. 84.

draw a circle with radius k_x and draw O D at right angles to Z Z to meet the circle in D. Draw D E horizontally to meet the ellipse in E, then O E $= k_z$. If many values are not required, the ellipse need not be drawn at all, but instead draw a second circle with radius k_y; then draw F E vertically to meet D E in E, thus fixing the point E.

To find the principal axes in the case where there is no axis of symmetry, the procedure is as follows—

(*a*) By graphical methods or by calculation first find the value of the product moment and the radii of gyration about any two axes through the centroid at right angles.

Let the product moment be A p^2 and the radii of gyration k_x and k_y.

Then the angle of inclination θ of the principal axes to k_x or k_y are given by the relation

$$\tan 2\theta = \frac{2p^2}{k_x^2 - k_y^2}$$

(*b*) By graphical methods or by calculation find the second moments of the given figure about lines X X and Y Y at right angles to each other and passing through the centroid and find it also about a third line Z Z at 45° to the other two.

Then if θ is the inclination of the principal axes to X X and Y Y

$$\tan 2\theta = \frac{I_x + I_y - 2I_z}{I_x - I_y}$$

$$\text{or } \tan 2\theta = \frac{k_x^2 + k_y^2 - 2k_z^2}{k_x^2 - k_y^2}$$

CONDITION THAT PRODUCT MOMENT IS ZERO.—It can be shown that the condition that the product moment about two lines is zero is that such lines form conjugate diameters of the momental ellipse.

A numerical example on the momental ellipse will be found on p. 242.

Second Moments about any Two Lines through the Centroid at Right Angles.—A property of the second moments of a figure that is sometimes useful is that the sum of the second moments of an area, about two lines at right angles through the centroid, is equal to the sum of the second moments about any other pair of lines at right angles through the centroid.

Second Moment or Moment of Inertia of Figure about an Axis perpendicular to its Plane.—The second moment or moment of inertia of an area about an axis O perpendicular to its plane is called the *polar second moment or moment of inertia,* and is equal to $\Sigma a \cdot PO^2$.

Let any two axes X X and Y Y at right angles be drawn through O, and let perpendiculars P N, P M be drawn to these axes, Fig. 85.

$$\text{Then } \mathrm{PO}^2 = \mathrm{PN}^2 + \mathrm{NO}^2$$
$$= \mathrm{PN}^2 + \mathrm{PM}^2$$
$$\therefore \Sigma a \,.\, \mathrm{PO}^2 = \Sigma a \,.\, \mathrm{PN}^2 + \Sigma a \,.\, \mathrm{PM}^2$$
$$= \mathrm{I}_{\mathrm{XX}} + \mathrm{I}_{\mathrm{YY}}$$

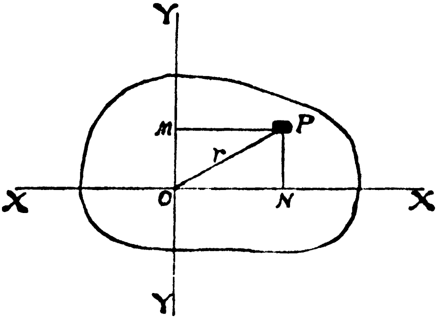


Fig. 85.—Polar Moment of Inertia.

Therefore we have the following rule—

The polar second moment, or moment of inertia, about an axis perpendicular to the plane of any area, is equal to the sums of the second moments about any two lines at right angles, drawn through the axis in the plane of the area.

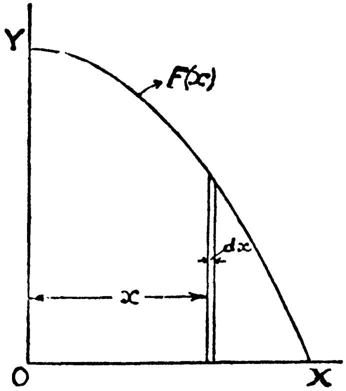


Fig. 86.

The Determination of Centroids, Moments of Inertia, and Radii of Gyration.—(*a*) Mathematical.—**Consider the curve of a function $y = F(x)$.**

Then considering a strip of width dx parallel to the axis of x, Fig. 86

$$\text{Area of curve} = \int F(x)\,dx$$

$$\text{First moment of area about OY} = \int F(x)\,dx \times x$$

$$\text{Second moment of area about OY} = \int F(x)\,dx \times x^2$$

Consider, for example, the parabola $y^2 = 4ax$, and take the area between the curve and the axis of x, Fig. 87.

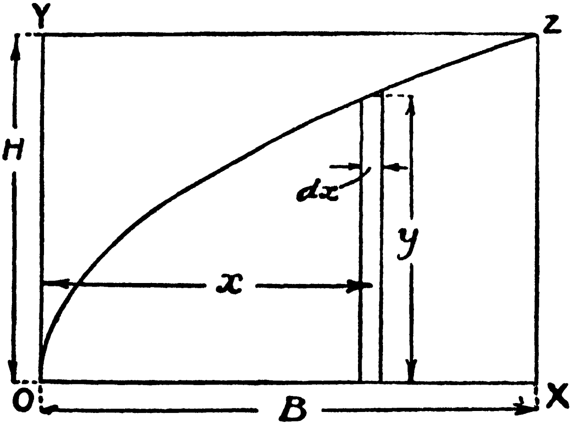


FIG. 87.

$$\text{Area of curve} = \int y\,dx \qquad = \int 2a^{\frac{1}{2}} x^{\frac{1}{2}}\,dx$$

$$= 2a^{\frac{1}{2}} \int_0^B x^{\frac{1}{2}}\,dx = \left[2a^{\frac{1}{2}} \cdot \frac{2}{3} x^{\frac{3}{2}}\right]_0^B$$

$$= \frac{4}{3} a^{\frac{1}{2}} B^{\frac{3}{2}}$$

Now $2a^{\frac{1}{2}} B^{\frac{1}{2}} = H$

$$\therefore \text{ Area of curve} = \frac{2}{3} BH, \text{ Fig. 87.}$$

$$\text{First moment about OY} = \int xy\,dx \qquad = \int 2a^{\frac{1}{2}} x^{\frac{3}{2}}\,dx$$

$$= 2a^{\frac{1}{2}} \int_0^B x^{\frac{3}{2}}\,dx = \left[2a^{\frac{1}{2}} \cdot \frac{2}{5} x^{\frac{5}{2}}\right]_0^B$$

$$= \frac{4}{5} a^{\frac{1}{2}} B^{\frac{5}{2}} = \frac{2}{5} B^2 H$$

$$\therefore \text{ distance of centroid from OY} = \frac{\frac{2}{5} B^2 H}{\frac{2}{3} BH} = \frac{3}{5} B$$

$$\text{Second moment about O Y} = \int x^2 y\,dx = \int 2\,a^{\frac{1}{2}}\,x^{\frac{5}{2}}\,dx$$

$$= 2\,a^{\frac{1}{2}}\int_{0}^{B} x^{\frac{5}{2}}\,dx = \left[2\,a^{\frac{1}{2}}\,.\,\frac{2}{7}\,x^{\frac{7}{2}}\right]_{0}^{B}$$

$$= \frac{4}{7}\,a^{\frac{1}{2}}\,\mathrm{B}^{\frac{7}{2}} = \frac{2}{7}\,\mathrm{B}^3\,\mathrm{H}$$

$$\therefore\ k_{\mathrm{Y}}^{2} = \frac{\frac{2}{7}\,\mathrm{B}^3\,\mathrm{H}}{\frac{2}{3}\,\mathrm{B}\,\mathrm{H}}$$

$$\text{or } k_{\mathrm{Y}} = \sqrt{\tfrac{3}{7}}\,\mathrm{B}.$$

If the second moment is required about the base X Z, we proceed as follows—

$$\mathrm{I}_{\mathrm{OY}} = \frac{2}{7}\,\mathrm{B}^3\,\mathrm{H}$$

$$\mathrm{I}_{\mathrm{CC}} = \mathrm{I}_{\mathrm{OY}} - \mathrm{A}\,.\,d^2$$

$$= \frac{2}{7}\,\mathrm{H}\,\mathrm{B}^3 - \frac{2}{3}\,.\,\mathrm{B}\,\mathrm{H}\,.\,\frac{9\,\mathrm{B}^2}{25}$$

$$\mathrm{I}_{\mathrm{XX}} = \mathrm{I}_{\mathrm{CC}} + \mathrm{A}\,d_1^{\,2}$$

$$= \frac{2}{7}\,\mathrm{H}\,\mathrm{B}^3 - \frac{6}{25}\,\mathrm{H}\,\mathrm{B}^3 + \frac{8}{75}\,\mathrm{H}\,\mathrm{B}^3$$

$$= \mathrm{H}\,\mathrm{B}^3\left\{\frac{150 - 126 + 56}{525}\right\} = \mathrm{H}\,\mathrm{B}^3\,.\,\frac{80}{525}$$

$$= \frac{16}{105}\,\mathrm{H}\,\mathrm{B}^3$$

A list of values of second moments, etc., for common figures will be found on p. 185.

It often happens in practice that the mathematical method is unworkable, in which case the following graphical methods are necessary.

(*b*) GRAPHICAL.—FIRST AND SECOND MOMENT CURVE METHOD.—(1) *Centroid.*—Suppose we have any area P R Q S, Fig. 88, and any two parallel lines X X and Y Y, at distance *h* apart.

Draw a thin strip of the area parallel to X X and of thickness *t* and let its centre line be P Q. From one of the ends of this centre line, say Q, draw a perpendicular Q M to Y Y and from the other end draw P N perpendicular to X X.

Join M N and let it cut P Q in Q_1 and produce M Q to cut X X in L.

Then the Δs PNQ_1, MNL are similar.

$$\therefore \frac{PQ_1}{PN} = \frac{NL}{ML} = \frac{PQ}{h}$$

$$\therefore PQ_1 = \frac{PQ \, . \, PN}{h}$$

Multiplying through by t we have

$$PQ_1 \times t = \frac{PQ \times t \times PN}{h} = \frac{\text{area of strip } PQ \times PN}{h}$$

$$\therefore \text{ Area of portion } PQ_1 \text{ of strip} = \frac{\text{First moment of strip about } xx}{h} \quad (1)$$

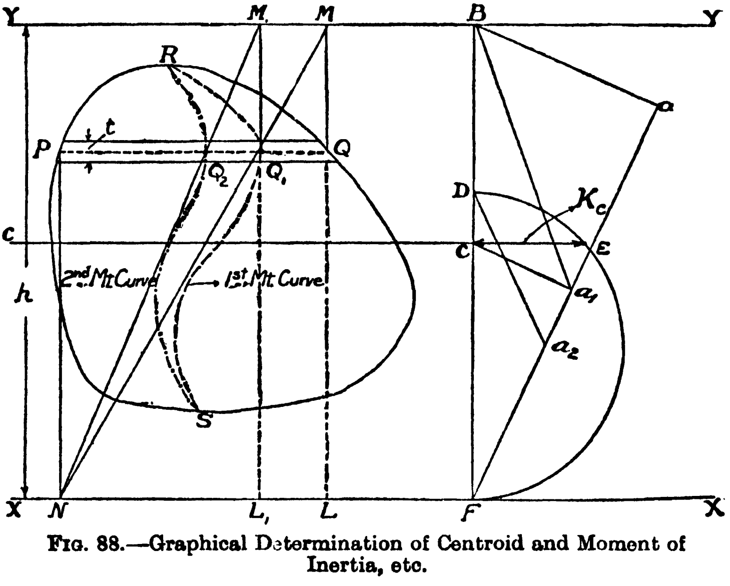


FIG. 88.—Graphical Determination of Centroid and Moment of Inertia, etc.

Now divide the whole area up into strips and join up all the points corresponding to Q_1, thus obtaining the *First Moment Curve* $B\,Q_1'\,S$.

Then the area to the left-hand side of the first moment curve will be the sum of the areas of portions of strips such as PQ_1. Call this the *First Moment Area* (A_1). Then we have

$$A_1 = \frac{\text{Sum of first moments of strips about } x\,x}{h}$$

$$= \frac{\text{First moment of whole area}}{h}$$

$$\therefore \text{ First moment of whole area} = A_1 h$$

$$\text{or distance of centroid from X X} = \frac{\text{First moment about X X}}{\text{area of figure}}$$

$$= \frac{A_1 h}{A} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Draw any vertical line F B to cut X X in F and Y Y in B, and through F draw any inclined line, on which set out F a equal on some scale to A, and F a_1 equal to A_1. Join a B and draw a_1 C parallel to it, then the centroid lies on a line through C parallel to X X or Y Y.

$$\text{For } \frac{\text{C F}}{\text{F B}} = \frac{\text{F } a_1}{\text{F } a}$$

$$\therefore \frac{\text{C F}}{h} = \frac{A_1}{A}$$

$$\text{or C F} = \frac{A_1 h}{A}$$

And this by relation (2) above gives the distance of the centroid from X X.

(2) *Second Moment.*—If the second moment is required about the line X X draw Q_1 M_1 perpendicular to Y Y and join M_1 N, cutting P Q in Q_2 and let M_1 Q_1 produced cut X X in L_1.

Then the Δs P N Q_2, M_1 N L_1 are similar.

$$\therefore \frac{\text{P } Q_2}{\text{P N}} = \frac{\text{N } L_1}{M_1 L_1} = \frac{\text{P } Q_1}{h}$$

$$\therefore \text{P } Q_2 = \frac{\text{P } Q_1 \times \text{P N}}{h}$$

Multiply through by t, then we have

$$\text{P } Q_2 \times t = \frac{\text{P } Q_1 \times t \times \text{P N}}{h}$$

$$\text{But we have seen that P } Q_1 \times t = \frac{\text{area of strip P Q} \times \text{P N}}{h}$$

$$\therefore \text{P } Q_2 \times t = \frac{\text{area of strip P Q} \times \text{P N}^2}{h^2}$$

$$\therefore \text{Area of portion P } Q_2 \text{ of strip} = \frac{\text{second moment of strip P Q about X X}}{h^2} \ldots (3)$$

Now repeat this construction for each of the strips and join up all the points corresponding to Q_2, thus obtaining the *second moment curve* R Q_2 S.

Then the area to the left-hand side of the second moment curve will be the sum of the areas of portions of strips such as P Q_2. Call this the *second moment area* (A_2). Then we have

$$A_2 = \frac{\text{Sum of second moments of strips about X X}}{h^2}$$

$$= \frac{I_{xx}}{h^2}$$

$$\therefore I_{xx} = A_2 h^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Some care is required in determining which area to read as A_1 or A_2. It does not matter whether the verticals are drawn downward from P or from Q, but when the moments are required about one of the lines, say X X, read, for the first moment area, the area on that side of the first moment curve from which the perpendiculars are drawn to X X, and in drawing the second moment curve draw from the first moment points, such as Q_1, perpendiculars to the other line Y Y, again reading the area to the side from which the perpendiculars were drawn to X X.

Now, on the line F a set out F a_2 equal to A_2 on the same scale to which the other areas were drawn, and join a_1 B, drawing a_2 D parallel to it.

On D F describe a semicircle, and draw a line C E parallel to X X to meet it in E.

Then C E will be equal to k, the radius of gyration about C C.

PROOF—

$$\frac{FD}{FB} = \frac{F\,a_2}{F\,a_1} = \frac{A_2}{A_1}$$

$$\therefore FD = \frac{h \times A_2}{A_1} = \frac{\dfrac{h \times I_{xx}}{h^2}}{\dfrac{A \times CF}{h}} = \frac{I_{xx}}{A\,.\,CF} = \frac{k_x^2}{CF}$$

Now $$\frac{FD}{FE} = \frac{FE}{FC}$$

$$\therefore FD\,.\,FC = FE^2$$

$$\therefore FD = \frac{FE^2}{CF}$$

$$\therefore FE = k_x$$

Now $$\text{F E}^2 = \text{F C}^2 + \text{C E}^2$$

$$\therefore k_x^2 = \text{C E}^2 + d_x^2$$

$$\therefore \text{C E}^2 = k_x^2 - d_x^2$$

But we have already shown that

$$k_c^2 = k_x^2 - d_x^2$$

$$\therefore \text{C E} = k_c.$$

Numerical Example.—*Graphical Determination of Radius of Gyration of Rail Section about Centroid.*

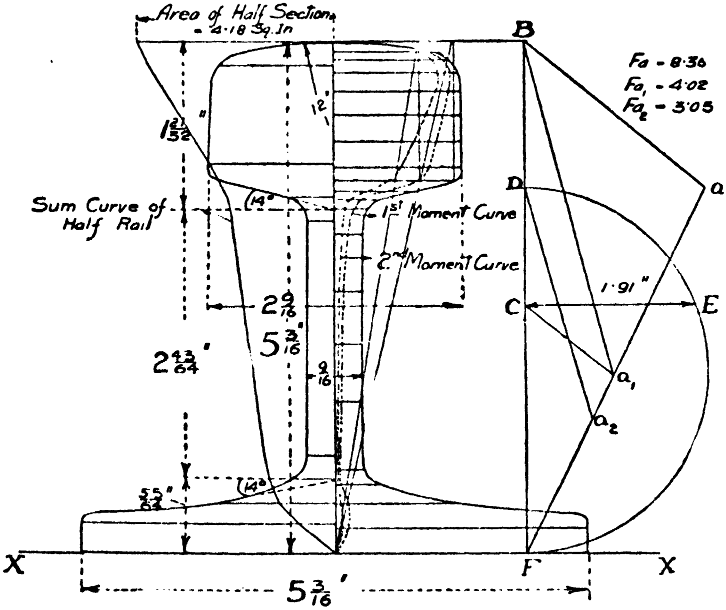


Fig. 89.—Rail Section.

Fig. 89 shows the graphical determination of the radius of gyration about the centroid parallel to the base of a British Standard 85 lb. flat rail section.

Since the section is symmetrical about a vertical centre line, the first and second moment curves need be drawn only for half the section, this simplifying the construction considerably. The lines X X and Y Y are taken as the horizontal lines, touching the section at top and bottom.

The areas A, A_1, A_2 are next found by planimeter or by sum-curve construction. (To avoid complication of figures, the sum curves for the first and second moment curves are omitted.) The first and second moment areas are to the left of the curves.

When multiplied by two, because only half the section was considered, we get

A = 8·36 sq. ins. A_1 = 4·02 sq. ins. A_2 = 3·05 sq. ins.

To the side of the figure a vertical F B is drawn between the X X and Y Y lines, and the points a, a_1, a_2 obtained as shown.

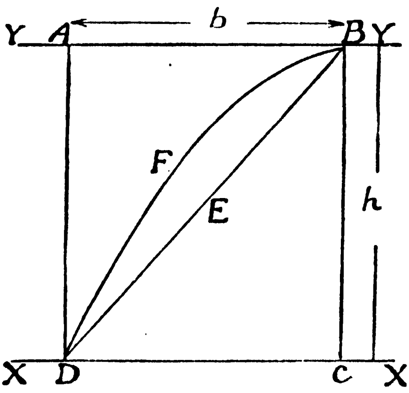


FIG. 90.—Moment of Inertia of Rectangle.

Then by joining a B and drawing a_1 C parallel, we get the point C, C F giving the distance of the centroid from X X; and by joining a_1 B and drawing a_2 D parallel, we get the point D.

On D F a semicircle is drawn, and C E is drawn horizontal to meet the semicircle in E.

Then C E = k, which on measuring will be found to be 1·91 ins.

This construction should be gone through as an exercise.

APPLICATION OF ABOVE METHOD TO CASE OF RECTANGLE.—Let A B C D, Fig. 90, be a rectangle of base b and height h, and take the lines X X and Y Y through C D and B A respectively. Then the first moment curve will be the diagonal B E D, while the second moment curve will come a parabola B F D, so that

$$A_1 = \frac{b\,h}{2}$$

$$A_2 = \frac{b\,h}{3}$$

$$\therefore\ I_{xx} = \frac{b\,h}{3} \times h^2 = \underline{\frac{b\,h^3}{3}}$$

$$d_x = \frac{A_1 \times h}{A} = \frac{b\,h}{2} \cdot \frac{h}{b\,h} = \frac{h}{2}$$

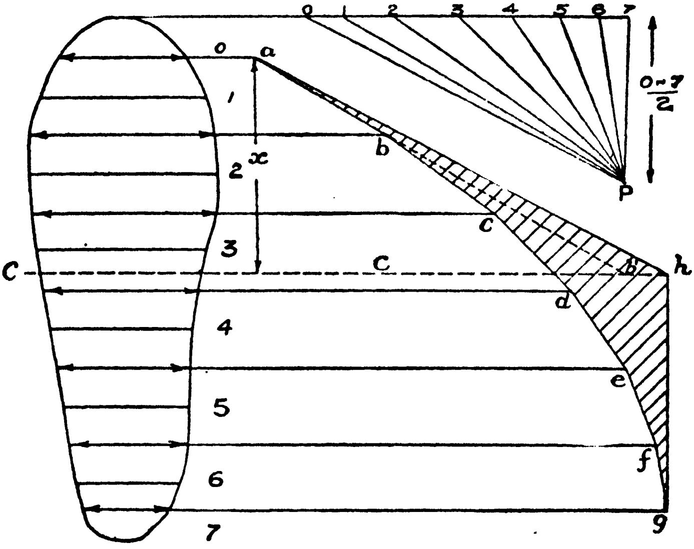


Fig. 91.—Mohr's Construction for Moment of Inertia.

$$\therefore I_{cc} = I_{xx} - A\,d_x^2$$

$$= \frac{b\,h^3}{3} - \frac{b\,h\,.\,h^2}{4} = \underline{\frac{b\,h^3}{12}}$$

Alternative Graphical Construction—Mohr's Method.—The following graphical method for obtaining the second moment about the centroid is in some cases more convenient in use than the one previously given. Divide the area, Fig. 91, into a number of small strips of equal breadth, parallel to the direction about which moments are taken, and draw the centre line of each of said strips. Then if the strips are sufficiently small (we have only taken a few strips in the figure to avoid complication) the lengths of these centre lines represent the areas of the separate strips. Now, on a vector line, to some scale, set out 0, 1, 1, 2,... 6, 7 to represent the area of each strip, and take a pole P at distance = ½ total area 0, 7 from this vector line. Then anywhere across space 0 draw and produce a line *a h* parallel to 0, P; across space 1 draw *a b* parallel to P 1; across space 2, *b c* parallel to P 2, and so on until the point *g* is reached. Then draw the last link *g h* parallel to the last line P 7 to meet *a h* in *h*.

Then the line C C through the centroid passes through *h*, and if *a* is the area of the shaded area, and A is the area of the figure,

$$I_{cc} \text{ of figure} = A \times a.$$

PROOF.—Consider one of the elemental areas, say 0, 1, and produce *a b* to meet the horizontal through *h* in b^1. Then, by the law of the link and vector polygon construction, treating the areas of the elements as forces,

$$b'h = \text{moment of first force about C C} \times \frac{1}{\text{polar distance}}$$

$$= 0,1 \times x \times \frac{1}{\frac{1}{2}\text{ total area}}$$

$$= 0,1 \times x \times \frac{2}{A}$$

$$\text{Area of } \Delta\, a\, b'\, h = \tfrac{1}{2}\, b'\, h \times x$$

$$= \frac{0,1 \times x^2 \times 2}{2A}$$

$$= \frac{\text{second moment of element about C C}}{A}$$

$$\therefore \text{ Area of shaded figure} = a = \frac{\text{second moment of figure about CC}}{A}$$

i. e. A *a* = second moment of figure about C C.

The proof that h determines the centroid is based upon the fact that in the link and vector polygon construction the meet of the first and last links determines the resultant, and in this case the centroid is where the resultant of the separate areas considered as forces act.

*** Equivalent Centroid and Second Moment of Heterogeneous Sections.**—Suppose that the cross section of a beam is composed of two materials for which Young's modulus is not the same, and let Young's modulus for one material B be m times Young's modulus for the second material C. Then in the case of direct stress we have seen that the material B behaves as if it were replaced by m times its area of the material C. In the case of a beam the same relation holds, so that we may replace the material B by an area m times as wide, the width being taken parallel to the line about which moments are taken.

Then if A is the area of material B, and A_1 that of material C, the equivalent area of homogeneous material C is given by

$$A_2 = A_1 + m\,A$$

To obtain the distance d of the equivalent centroid from a line x x, take first moments of the separate areas about x x and let them be M and M_1 respectively.

Then equivalent first moment of the second material is

$$M_2 = M_1 + m\,M$$

$$\therefore d = \frac{M_1 + m\,M}{A_1 + m\,A}$$

To obtain the equivalent second moment about a line x x, take the separate second moments about x x and let them be I and I_1 respectively, then the equivalent second moment of the second material is given by

$$I_2 = I_1 + m\,I$$

We shall give numerical examples and further explanation of this when dealing with flitched beams and reinforced beams.

The above reasoning may be shown graphically as follows—

Let A B C D (Fig. 92) represent any area which has embedded in it two bars X and Y of different material. For considering the moments about any line such as D B shown dotted, make a strip E F of the same depth as X, and of area equal to (m − 1) area of X and also a strip G H of area equal to (m − 1) area of Y.

Then the equivalent first and second moments of the heterogeneous section about the given line will be the same as a homogeneous section of form A E F B G H C D.

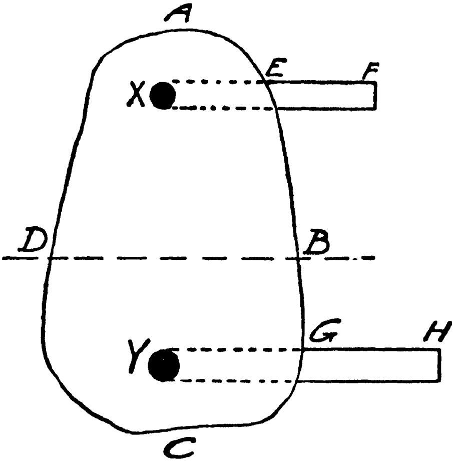


FIG. 92.

We take E F = (m − 1) area of X because the bar already occupies an area equal to its area, so that equivalent area of second material = [(m − 1) + 1] area of X = m × area of X.

Calculation of Moment of Inertia and Radii of Gyration of Sections used in Constructional Work.—The moments of inertia of sections composed of sections of known moment of inertia are found by adding up the moments of the separate parts, or subtracting when the area consists of the difference of known areas.

AREA, POSITION OF CENTROID, AND MOMENT OF INERTIA OF COMMON FIGURES

(*See* FIG. 93)

No.	Figure.	Area	Position of Centroid		Moment of Inertia			
			From X X	From Y Y	I_{XX}	I_{YY}	I_{UU}	I_{ZZ}
1	Rectangle	$b\,h$	$\frac{h}{2}$	$\frac{b}{2}$	$\frac{b\,h^3}{3}$	$\frac{h\,b^3}{3}$	$\frac{b\,h^3}{12}$	$\frac{h\,b^3}{12}$
2	Parallelogram	$b\,h$	$\frac{h}{2}$	—	$\frac{b\,h^3}{3}$	—	$\frac{b\,h^3}{12}$	—
3	Triangle	$\frac{b\,h}{2}$	$\frac{h}{3}$	$\frac{b}{2}$	$\frac{b\,h^3}{12}$	$\frac{7\,h\,b^3}{48}$	$\frac{b\,h^3}{36}$	$\frac{h\,b^3}{48}$
4	Trapezium (where $a = n\,b$)	$\frac{h\,b}{2}\left(1+n\right)$	$\frac{h}{3}\left(\frac{2\,n+1}{n+1}\right)$	—	$\frac{b\,h^3}{12}\left(3\,n+1\right)$	—	$\frac{b\,h^3}{36}\frac{(n^2+4\,n+1)}{(n+1)}$	$\frac{b\,h^3}{12}\left(n+3\right)$
5	Square	b^2	—	—	—	—	$\frac{b^4}{12}$	$\frac{b^4}{12}$
6	Circle	$\frac{\pi\,d^2}{4}$	$\frac{d}{2}$	—	$\frac{5\,\pi\,d^4}{64}$	—	$\frac{\pi\,d^4}{64}$	—
7	Ellipse	$\frac{\pi\,d_1\,d_2}{4}$	—	—	—	—	$\frac{\pi\,d_2{}^3\,d_1}{64}$	$\frac{\pi\,d_1{}^3\,d_2}{64}$
8	Parabolic segment (interior)	$\frac{2}{3}\,b\,h$	$\frac{2}{5}\,h$	$\frac{3}{8}\,b$	$\frac{16}{105}\,b\,h^3$	$\frac{2}{15}\,h\,b^3$	$\frac{8}{175}\,b\,h^3$	—
9	Parabolic segment (exterior)	$\frac{1}{3}\,b\,h$	$\frac{3}{10}\,h$	$\frac{b}{4}$	—	—	—	—

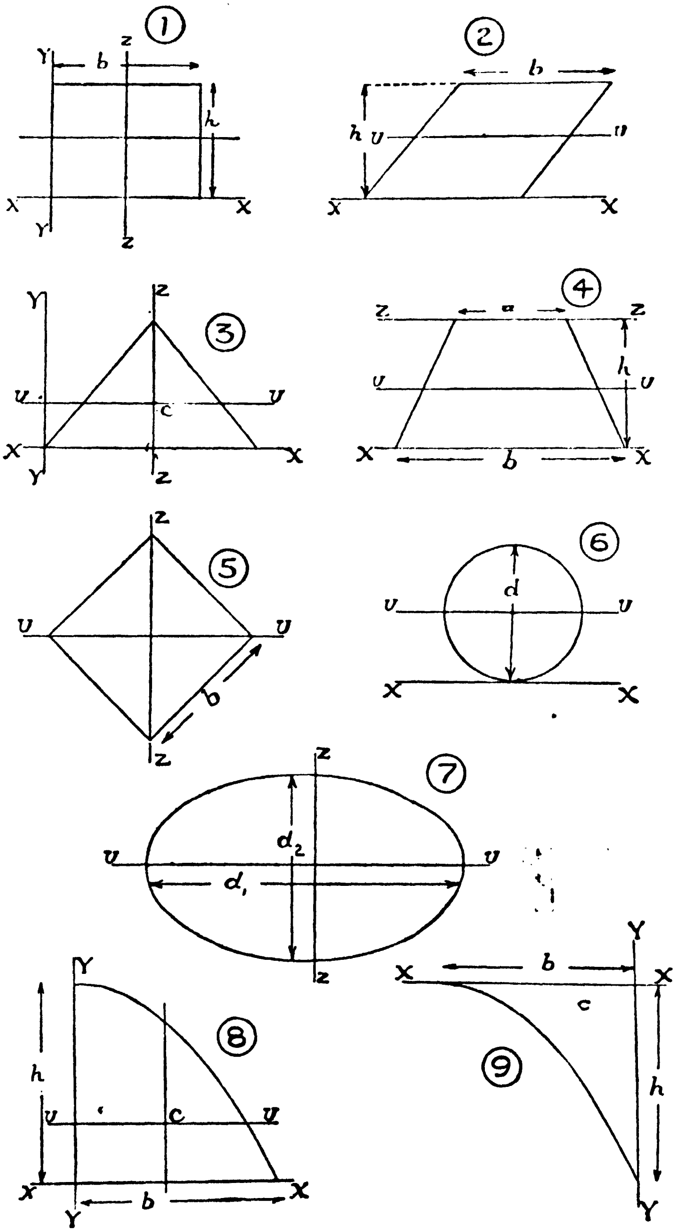


FIG. 93.—Properties of Common Figures.

For the properties of British Standard Steel Sections, see Appendix.

The following examples should make the method of calculation clear for any such case. See Figs. 93 and 94.

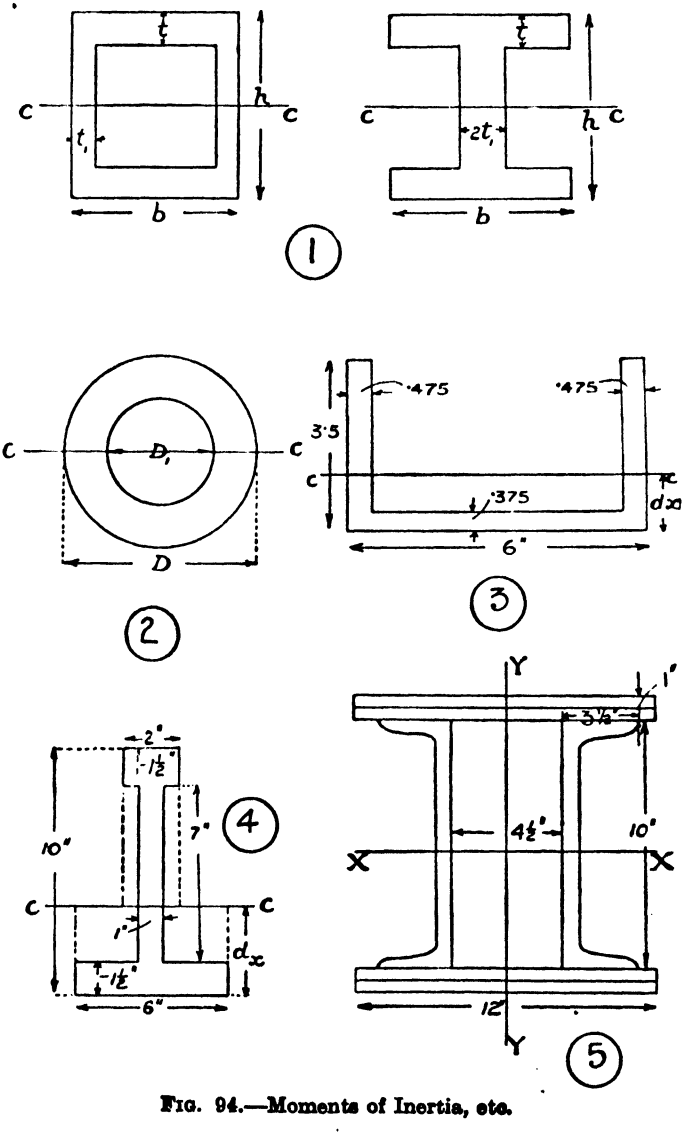


FIG. 94.—Moments of Inertia, etc.

(1) **Box or I Section.**—These are geometrically equivalent as far as the line o o is concerned, because if the box section be cut in half vertically and the two halves be turned back to back, we get the **I** section.

$$\text{Then } I_{oo} = \frac{b\,h^3 - (b - 2\,t_1)\,(h - 2\,t)^3}{12}$$

(2) **Hollow Circular Section.**

$$I_{oo} = \frac{\pi}{64}\,(D^4 - D_1^{\,4})$$

When the thickness of metal is small and equal to t, this approximates to $I_{oo} = \dfrac{\pi\,D^3\,t}{8}$

(3) **Channel Section** (neglecting inclination of sides and rounded corners).—Consider the section shown in Fig. 94 (3).

$$\begin{aligned}\text{Area} = A &= 3{\cdot}5 \times {\cdot}475 + 5{\cdot}05 \times {\cdot}375 + 3{\cdot}5 \times 475\\ &= 5{\cdot}219 \text{ sq. ins.}\end{aligned}$$

To obtain distance d_x of centroid from **x x** take first moments about **x x**. Then

$$\begin{aligned}&A \times d_x\\ &= 3{\cdot}5 \times {\cdot}475 \times \frac{3{\cdot}5}{2} + \frac{5{\cdot}05 \times {\cdot}375 \times {\cdot}375}{2} + \frac{3{\cdot}5 \times {\cdot}475 \times 3{\cdot}5}{2}\\ &= 2{\cdot}910 + {\cdot}354 + 2{\cdot}910 = 6{\cdot}175\\ &\therefore d_x = \frac{6{\cdot}175}{5{\cdot}219} = 1{\cdot}185 \text{ ins.}\end{aligned}$$

Second moment about **x x** $= I_{xx}$

$$\begin{aligned}&= \frac{{\cdot}475 \times 3{\cdot}5^3}{3} + \frac{5{\cdot}05 \times {\cdot}375^3}{3} + \frac{{\cdot}475 \times 3{\cdot}5^3}{3}\\ &= 6{\cdot}775 + {\cdot}089 + 6{\cdot}775 = 13{\cdot}639 \text{ in. units.}\end{aligned}$$

$$\begin{aligned}\therefore I_{oo} &= I_{xx} - A\,d_x^2\\ &= 13{\cdot}639 - (5{\cdot}219 \times 1{\cdot}185^2)\\ &= 13{\cdot}639 - 7{\cdot}323 = \underline{6{\cdot}316 \text{ in. units.}}\end{aligned}$$

$$\therefore k_{oo} = \sqrt{\frac{I}{A}} = 1{\cdot}010 \text{ ins.}$$

(4) **Cast-Iron Beam Section.**

$$\text{Area} = A = 2 \times 1\tfrac{1}{2} + 7 \times 1 + 6 \times 1\tfrac{1}{2} = 19 \text{ sq. ins.}$$

Moments round base

$$A\,d_x = 3 \times 9{\cdot}25 + 7 \times 5 + 9 \times {\cdot}75$$

$$= 27{\cdot}75 + 35 + 6{\cdot}75 = 69{\cdot}5 \text{ in. units.}$$

$$\therefore d_x = \frac{69{\cdot}5}{19} = 3{\cdot}658 \text{ ins.}$$

$$\therefore I_{oo} = \frac{6 \times 3{\cdot}658^3}{3} - \frac{5 \times 2{\cdot}158^3}{3} + \frac{2 \times 6{\cdot}342^3}{3} - \frac{1 \times 4{\cdot}892^3}{3}$$

$$= \underline{219{\cdot}95 \text{ in. units.}}$$

(5) **Built-up Mild Steel Column Section.**—Composed of two 10 × 3½ × 28·21 channels and four 12 in. × ½ in. plates. Required to find k_x and k_y. From the Table of Standard Sections we obtain the following information concerning the Channel Sections—

Area of each 8·296 sq. ins.
I about centroid parallel to X X = 117·9 in. units
I „ „ „ Y Y = 8·194 „ „
Distance of centroid from web = ·933 in.

∴ Total area of section = (4 × 12 × ½) + (2 × 8·296) = 40·592 sq. ins.

MOMENT OF INERTIA ABOUT X X.

2 channels, 117·9 each = 235·8

2 pairs of 12 × ½ in. plates about centroid $= \dfrac{2 \times 12 \times 1^3}{12} =$ 2·0

A × d^2 for two pairs of plates = 2 × 12 × 5·5² = 726·1

Total .. .. .. = $\underline{963{\cdot}9 \text{ in. units}}$

$$\therefore k_x = \sqrt{\frac{963{\cdot}9}{40{\cdot}592}} = 4{\cdot}90 \text{ ins.}$$

MOMENT OF INERTIA ABOUT Y Y.

4 plates 12 × ½ about centroid $= \dfrac{4 \times \frac{1}{2} \times 12^3}{12} =$ 288·0

2 channels about centroid = 2 × 8·194 = 16·4

A × d^2 for each channel = 2 × 8·296 × 3·183² = 168·5

Total .. .. = 472·9

$$\therefore k_y = \sqrt{\frac{472{\cdot}9}{40{\cdot}592}} = 3{\cdot}41 \text{ ins.}$$

(6) **Built-up Beam Section.**—Composed of two 14 in. × 6 in. × 46 lb. **I** beams and four 14 in. × $\frac{5}{8}$ in. plates (Fig. 95). Required I_{xx}.

From the Standard Section Tables we obtain the following information concerning the **I** beams—

Area of each = 13·53

I_{xx} „ „ = 440·5

Mean thickness of each flange = ·698 in.

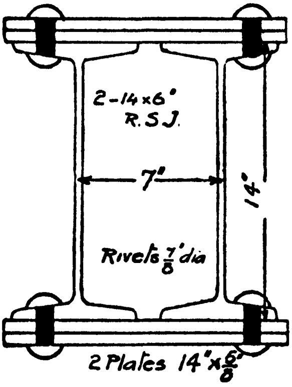


FIG. 95.

I_{xx} OF WHOLE SECTION (NOT ALLOWING FOR RIVETS).—

I_{xx} of two **I** beams = 2 × 440·5 = 881

$$\text{I of two pairs of plates about centroid} = \frac{2 \times 14}{12} \times \left(\frac{5}{4}\right)^3 = 4{\cdot}8$$

$$A\,d^2 \text{ for two pairs of plates} = 4 \times 14 \times \frac{5}{8} \times 7{\cdot}625^2 = 2035$$

Total .. .. .. = 2920·8

ALLOWANCE FOR RIVETS (neglect I of each rivet-hole about its centroid).

$$\text{Area of each hole} = \left(2 \times \frac{5}{8} + {\cdot}698\right)\frac{7}{8} = 1{\cdot}704$$

Dist. of centroid from x x = 7·276

$\therefore I_{xx} = 4 \times 1{\cdot}704 \times 7{\cdot}276^2 = 360{\cdot}8$

$\therefore$ Nett $I_{xx} = 2920{\cdot}8 - 360{\cdot}8 = \underline{2560}$

(7) **Built-up Sections—Approximate Method.**—The moment of inertia of built-up sections can be found approximately by adding the moment of inertia of the **I** beams or channels to A d^2 for the plates, d being taken as the distance from the centre of one set of plates to x x and the *nett area of the plates* being taken for A.

Taking the section of the previous example, we then get I_{xx} as follows—

$$I_{xx} \text{ of two I beams} = 2 \times 440{\cdot}5 = 881$$

$$A\,d^2 \text{ for plates} = 4 \times \frac{5}{8}\left(14 - 2 \times \frac{7}{8}\right) \times 7{\cdot}625^2 = 1781$$

$$\text{Total approximate } I_{xx} = \overline{\underline{2662}} \text{ in. units}$$

CHAPTER VII

STRESSES IN BEAMS

We have seen in a previous chapter how the bending moment and shearing force at different points along a beam, loaded in various manners, can be found; our next problem is to find the relations between these quantities and the stresses occurring in the beam.

We shall get a good preliminary idea of the stresses occurring in beams by considering a model devised by Professor Perry. Suppose that a beam fixed at one end carries a weight, W (Fig. 96), at the other end, and that it is cut through at a certain section. Then the right-hand portion can be kept in equilibrium by attaching a rope to the top and passing over a pulley, a weight W being attached to the other end of the rope, and by placing a block B at the lower portion of the section and a chain A at the upper portion. Then the pull in the rope overcomes the shearing force; and the block B carries a compressive force C, and the chain A carries a tensile force T. Since these are the only horizontal forces, they must be equal and opposite, and thus form a *couple*. Then the moment of this couple must be equal and opposite to the couple, due to the loading, which we have called the bending moment.

In the actual beam, owing to the deflection which takes place, the material on one side of the beam will be stretched, and the material on the other side will be compressed, so that at some point between the two sides the material will not be strained at all, and the axis in the section of the beam at which no strain occurs is called the **neutral axis** (N.A.). We see, therefore, that: *The neutral axis is the line in the section of a beam along which no strain, and therefore no stress, occurs.*

In an elevation of a beam there is also a line of no strain or stress, which may also be termed a neutral axis. These two axes are really the traces of a *neutral surface.*

If we know the manner in which the strain varies from the neutral axis to the outer sides of the beam, from a knowledge of the relation between stress and strain we can find the stresses at different points across the beam, remembering that the total compressive stress must be equal to the total tensile stress, and the moment of their couple must be equal to the bending moment. The moment of the couple due to the stresses is often called the *moment of resistance.*

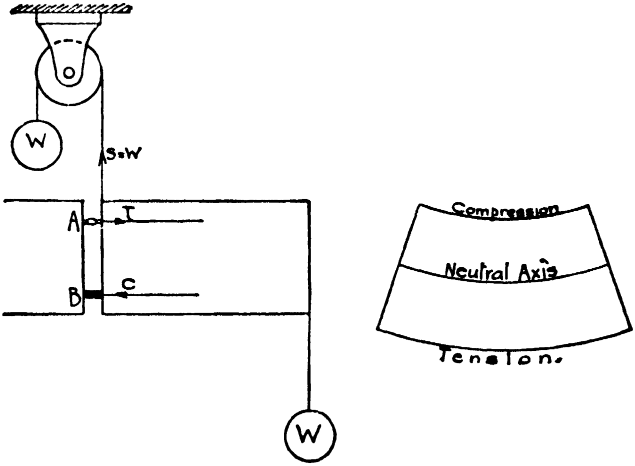


FIG. 96.—Stresses in Beams.

Assumptions in Ordinary Beam Theory.—We will first make the following assumptions with regard to the bending of beams, and from such assumptions we will deduce a relation between the maximum stresses, due to bending at any cross section and the bending moment—

(*a*) That for the material the stress is proportional to the strain, and that Young's modulus (E) is equal for tension and compression.

(*b*) That a cross section of the beam which is plane before bending remains plane after bending.

(*c*) That the original radius of curvature of the beam is very great compared with the cross-sectional dimensions of the beam.

We will also for the present restrict our investigation to the case of *simple* bending, *i. e.* that in which the following conditions hold—

(1) There is no resultant thrust or pull across the cross section of the beam.

(2) The section of the beam is symmetrical about an axis through the centroid of the cross section parallel to the plane in which bending occurs.

To get a clear idea of the stresses in beams it is absolutely necessary to have a clear idea of the assumptions involved in formulating any particular theory, and of the effect of such assumptions on the results.

Let A B, Fig. 97, represent the cross section of a beam which has been bent (the amount of bending having been exaggerated). Before bending, the line A B had the position of A_1 B_1, so that B B_1 represents the maximum tensile strain, and A A_1 the maximum compression strain. From our assumption (*b*), called *Bernoulli's assumption*, A_1 B_1 and A B are both straight lines. The neutral axis then passes through C, the point of no strain, and it follows from the above assumptions that the strains are proportional to the distances from the N.A. From assumption (*a*) it follows that the diagram of intensity of stress is also a sloping straight line, A_2 B_2, the portions B_2 C and CA_2 being continuous, because Young's modulus is equal in tension and compression.

It is clear that the maximum stresses in compression and tension occur at the points A and B, and let these be f_c and f_t respectively, d_c and d_t being the distances A C and B C.

Position of Neutral Axis.—Now consider an element of area a at a point P at distance P N from the N.A.

Then the stress at the points P is equal to $P_1 P_2$

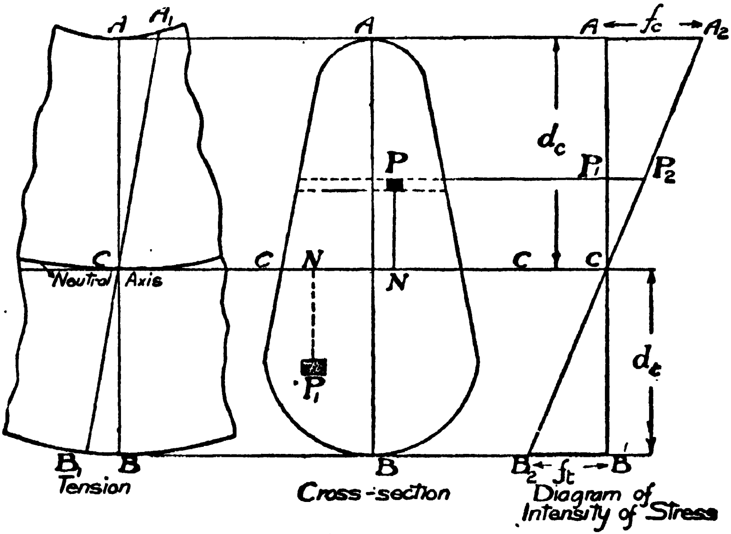


FIG. 97.—Stresses in Beams.

$$\text{But } \frac{P_1 P_2}{P_1 O} = \frac{A A_2}{A C} = \frac{f_c}{d_c}$$

$$\therefore P_1 P_2 = \frac{f_c}{d_c} \times P_1 O$$

$$= \frac{f_c}{d_c} \times P N$$

$$\therefore \text{Stress carried by the element} = a \times \frac{f_c}{d_c} \times P N$$

$$\therefore \text{Total stress carried by section above N.A.} = \Sigma a \times \frac{f_c}{d_c} \times P N$$

$$= \frac{f_c}{d_c} \Sigma a \times P N$$

$$= \frac{f_c}{d_c} \times \text{first moment of area above N.A. about N.A.}$$

Similarly if an element of area at a point P_1 be considered, we see that

Total stress carried by section below N.A.

$$= \frac{f_t}{d_t} \times \text{first moment of area below N.A. about N.A.}$$

But we have seen that the total tension T must be equal to the total compression C, and it follows from assumptions (*a*) (*b*) that

$$\frac{f_c}{d_c} = \frac{f_t}{d_t}$$

$\therefore$ we see that the first moment of the areas above and below the N.A. about the N.A. are equal and opposite in sign. Therefore, the total first moment of the whole area about the N.A. is zero. But we have seen that the first moment of an area is zero about a line through the centroid.

Therefore, *in simple bending with the given assumptions, the neutral axis passes through the centroid.*

The Moment of Resistance (M.R.)—We have proved that the stress carried on an element a of area about a point P is equal to $a \times \frac{f_c}{d_c} \times \mathrm{PN}$

The moment of this stress about the N.A.

$$= \text{stress} \times \mathrm{PN}$$

$$= a \times \frac{f_c}{d_c} \times \mathrm{PN}^2$$

$\therefore$ Total moment of all the stresses over the cross section

$$= \Sigma\, a \,.\, \frac{f_c}{d_c} \times \mathrm{PN}^2$$

$$= \frac{f_c}{d_c} \Sigma (a \times \mathrm{PN}^2)$$

$$= \frac{f_c}{d_c} (\text{second moment of whole area about the N.A.})$$

$$= \frac{f_c \mathrm{I}}{d_c}$$

But the total moment of all the stresses is the moment of the couple which we have called the moment of resistance.

$$\therefore \text{ we see that M.R.} = \frac{f_c \mathrm{I}}{d_c} \text{ or } \frac{f_t \mathrm{I}}{d_t}$$

The moment of resistance must, as has already been shown, be equal to the bending moment, which we will call M.

$$\therefore\ \mathrm{M} = \frac{f_c \mathrm{I}}{d_c} \text{ or } \frac{f_t \mathrm{I}}{d_t} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

It will be seen that I, d_c and d_t depend merely on the shape of the cross section, and $\frac{\mathrm{I}}{d_c}$ and $\frac{\mathrm{I}}{d_t}$ are called the *compression modulus* and *tension modulus* respectively of the section, and are written Z_c and Z_t.

Thus our relation becomes

$$\mathrm{M} = f_c \mathrm{Z}_c = f_t \mathrm{Z}_t \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

In practice we usually want to know f_c and f_t which give the *maximum* stresses across the section, and so we will write the result as

$$f_c = \frac{\mathrm{M}}{\mathrm{Z}_c} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$f_t = \frac{\mathrm{M}}{\mathrm{Z}_t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

In the case where the section is symmetrical about the N.A., d_c is equal to d_t, so that Z_c and Z_t are equal. In this case, therefore, $f_c = f_t$, and we may write the relation as

$$f = \frac{\mathrm{M}}{\mathrm{Z}}.$$

For values of section moduli for British Standard Beam Sections, see Appendix.

Unit Section Modulus; Beam Factor.—If instead of taking the section modulus as Z we took $\frac{\mathrm{Z}}{\mathrm{A}}$ and called it the "unit section modulus," the quantity would be rather more useful and probably more easy to conceive to those who find difficulty in purely mathematical conceptions.

We should then have

$$\text{Unit compression section modulus} = z_c = \frac{\mathrm{Z}_c}{\mathrm{A}} = \frac{\mathrm{I}}{\mathrm{A}\,d_c} = \frac{k^2}{d_c}$$

$$\text{,, tension ,, ,,} = z_t = \frac{k^2}{d_t}$$

z_c and z_t then become lengths

∴ equations (3) and (4) give—

$$f_c . A = \frac{M}{z_c}$$

$$f_t . A = \frac{M}{z_t}$$

z_c and z_t can be found graphically by setting out C D, Fig. 97*a*, horizontally to represent k, the radius of gyration

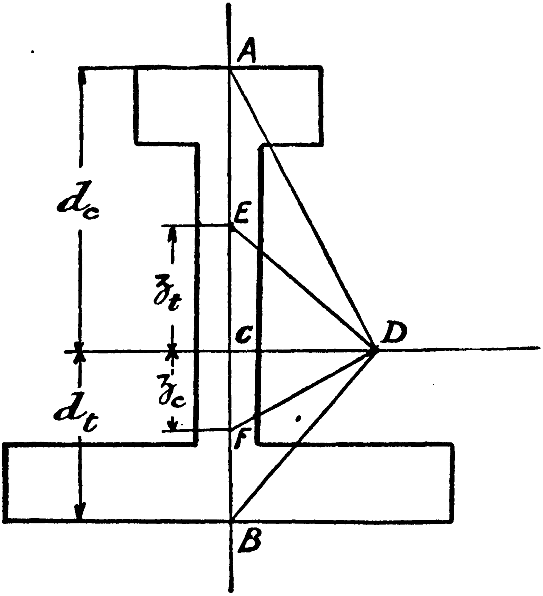


FIG. 97*a*.

of the section about the N.A., joining B D, A D and drawing lines D E, D F respectively at right angles to them. Then C E, C F are z_t and z_c respectively.

Take, for example, the Δs C E D, B C D; they are similar.

$$\therefore \frac{CD}{CE} = \frac{BC}{CD} \qquad \therefore CE = \frac{CD^2}{BC} = \frac{k^2}{d_t} = z_t.$$

Beam factor.—The quantity $\frac{z}{d} = \frac{Z}{A d}$, for a symmetrical section, or $\frac{Z \text{ min.}}{A\,d}$, for a beam of asymmetrical section whose safe bending stresses are the same in compression and tension, is a measure of the value of the section as a beam and may be called the *beam factor*. Take, for instance, the 15″ × 6″ standard **I** beam (Appendix). $Z = 83{\cdot}9$ and $A = 17{\cdot}3$.

$$\therefore \text{ beam factor} = \frac{83{\cdot}9}{15 \times 17{\cdot}3} = {\cdot}323$$

For the 10″ × 8″ standard beam $Z = 69{\cdot}0$, $A = 20{\cdot}2$.

$$\therefore \text{ beam factor} = \frac{69{\cdot}0}{10 \times 20{\cdot}2} = {\cdot}342$$

$\therefore$ the 10″ × 8″ is a more economical section.

A rectangular section would give a beam factor $= \frac{1}{6} = {\cdot}167$.

Numerical Examples.—The following numerical examples will make it clear how the stresses in beams loaded in given manners can be found, and how a safe load can be found for a beam of given span and section.

(1) *The five sections a, b, c, d, e, Fig. 98, have each an area of 4 sq. ins. Find their relative strengths as beams for the same span, if they are of the same material.*

We have seen that $M = f Z$. Now if all the beams are loaded in the same way, M will be proportional to the load they can carry, and as f is the same for each, we see that their relative strengths as beams depend on their values of the modulus of each section. For table of second moments, see p. 185.

Section a.

$$I = \frac{b h^3}{12} = \frac{2 \times 2^3}{12}$$

$$Z = \frac{I}{d} = \frac{I}{1}$$

$$\therefore Z = \frac{2 \times 2^3}{12 \times 1} = \frac{2 \times 8}{12}.$$

$$= 1{\cdot}33 \text{ in. units.}$$

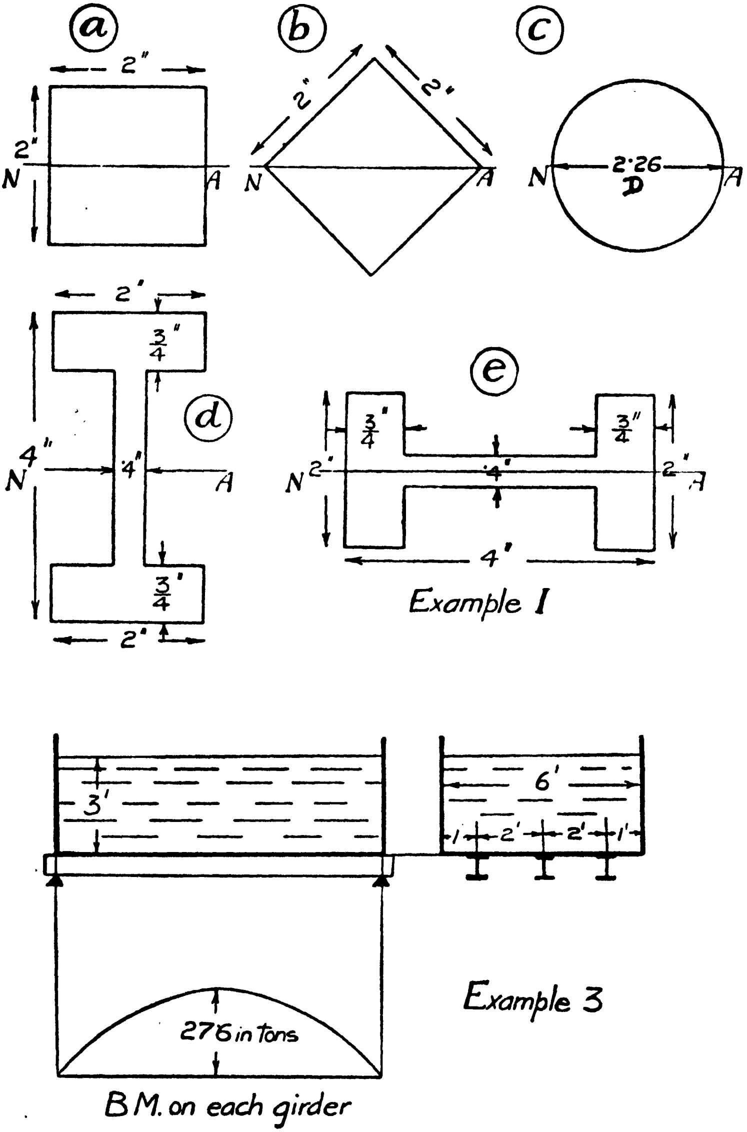


Fig. 98.—Examples of Beams.

Section b. This is composed of two triangles.

$\therefore \mathrm{I} = 2 \times \frac{bh^3}{12}$, *h* in this case being the height of the triangle.

$$\therefore \mathrm{I} = \frac{2 \times 2{\cdot}828 \times 1{\cdot}414^3}{12}$$

$$d = 1{\cdot}414$$

$$\therefore Z = \frac{2 \times 2{\cdot}828 \times 1{\cdot}414^3}{1{\cdot}414 \times 12} = \frac{2{\cdot}828}{3}$$

$$= {\cdot}943 \text{ in. units.}$$

Section c.

$$\mathrm{I} = \frac{\pi D^4}{64} = \frac{\pi \times 2{\cdot}26^4}{64}$$

$$d = 1{\cdot}13$$

$$\therefore Z = \frac{\pi \times 2{\cdot}26^4}{64 \times 1{\cdot}13}$$

$$= 1{\cdot}13 \text{ in. units.}$$

Section d.

$$\mathrm{I} = \frac{2 \times 4^3}{12} - \frac{2 \times {\cdot}8 \times 2{\cdot}5^3}{12}$$

$$= 10{\cdot}67 - 2{\cdot}08 = 8{\cdot}59$$

$$d = 2''$$

$$\therefore Z = \frac{8{\cdot}59}{2}$$

$$= 4{\cdot}29 \text{ in. units.}$$

Section e. This is composed of three rectangles.

$$\therefore \mathrm{I} = \frac{{\cdot}75 \times 2^3}{12} + \frac{2{\cdot}5 \times {\cdot}4^3}{12} + \frac{{\cdot}75 \times 2^3}{12}$$

$$= {\cdot}5 + {\cdot}013 + {\cdot}5$$

$$= 1{\cdot}013$$

$$d = 1''$$

$$\therefore Z = 1{\cdot}013 \text{ in. units.}$$

We see, therefore, that the order of the sections, from strongest to weakest, is *d, a, c, e, b.*

We may take it, as a rule, that the strongest beam for a given area of cross section is that which has a depth as great as is practically possible, and which has as much as possible of the metal at the outer portions of the beam.

(2) *A girder of 20 ft. span carries a uniformly distributed load of 10 tons, and a central load of 4 tons. Find a suitable British standard beam section for the girder if the maximum stress is to be 7 tons per sq. in.*

Its maximum B.M. due to the uniform load will be equal to $\frac{Wl}{8}$ (see Figs. 59, 59*a*, Cases 2 and 3)

$$= \frac{10 \times 20 \times 12}{8} \text{ in. tons}$$
$$= 300 \text{ in. tons.}$$

The maximum B.M. due to the central load $= \frac{W_1 l}{4}$

$$= \frac{4 \times 20 \times 12}{4}$$
$$= 240 \text{ in. tons.}$$

These both occur at the same point, so that the maximum B.M. due to both loads = 540 in. tons.

Now
$$M = fZ$$
$$i.e.\ 540 = 7Z$$
$$\therefore Z = \frac{540}{7} = 77{\cdot}14 \text{ in. units.}$$

On referring to the table of standard sections (Appendix), we see that the section having the nearest modulus to this is a 14 × 6 × 57 lb. section for which Z = 76·12, and we will adopt this section as being sufficiently strong.

(3) *A tank which weighs ½ ton and measures 10′ × 6′ × 3 is filled with water, and carried on three girders placed lengthwise, so that each girder takes an equal weight. If the girders are 6″ × 3″ × 12 lb. Standard Beams find the maximum stress in each. (A.M.I.C.E. Altered slightly.)*

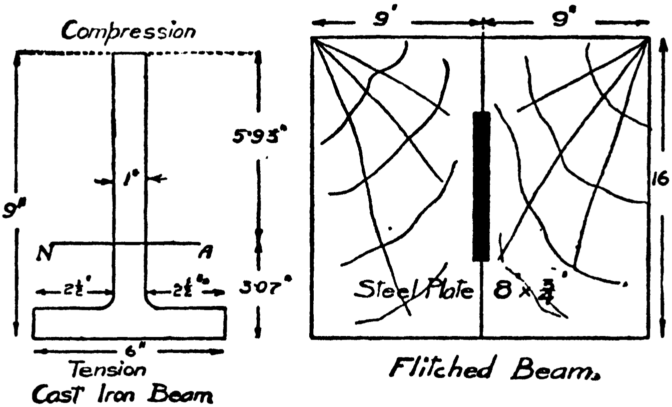
$$\text{Weight of water in tank} = \frac{10 \times 6 \times 3 \times 62{\cdot}5}{2240} \text{ tons}$$
$$= 5{\cdot}02 \text{ tons.}$$

$\therefore$ Total weight carried by girders $= 5{\cdot}02 + {\cdot}5 = 5{\cdot}52$ tons

$$\therefore \text{ Maximum B.M. on each girder} = \frac{5{\cdot}52}{3} \times \frac{10 \times 12}{8}$$
$$= 27{\cdot}6 \text{ in. tons.}$$

Z for a 6″ × 3″ × 12 lb. beam is 6·736 in. units

$$\therefore f = \frac{27{\cdot}6}{6{\cdot}736} = 4{\cdot}1 \text{ tons per sq. in.}$$



Cast Iron Beam

Flitched Beam

FIG. 99.

(4) *A cast-iron beam is the shape of an inverted* T, 9 *ins. deep over all, width of flange* 6 *ins., thickness of web and flange* 1 *in. If its length is* 12 *ft. find what weight at the centre will cause a tensile stress of* 1 *ton per sq. in. in the flange. What would the maximum compressive stress then be?* (*A.M.I.C.E.*)

First find the centroid and second moment of the section. (See Fig. 99.)

Area of section $= A = 9 \times 1 + 5 \times 1 = 14$ sq. in.

$$\text{1st Moment about base} = A\,d = (9 \times 1) \times \frac{9}{2} + 2\,(2\tfrac{1}{2} \times 1) \times \frac{1}{2}$$
$$= 40{\cdot}5 + 2{\cdot}5 = 43$$
$$\therefore d = \frac{43}{14} = 3{\cdot}07 \text{ ins.}$$

$$\text{2nd Moment about base} = \mathrm{I}_x = \frac{1 \times 9^3}{3} + \frac{2 \times 2\tfrac{1}{2} \times 1^3}{3}$$

$$= 243 + 1{\cdot}67 = 244{\cdot}67$$

$\therefore$ 2nd Moment about parallel line through centroid

$$= \mathrm{I}_c = \mathrm{I}_x - \mathrm{A}\,d^2$$

$$= 244{\cdot}67 - 14 \times 3{\cdot}07^2$$

$$= 244{\cdot}67 - 132{\cdot}07$$

$$= 112{\cdot}6 \text{ in. units.}$$

$$\therefore \mathrm{Z}_c = \frac{112{\cdot}6}{9 - 3{\cdot}07} = \frac{112{\cdot}6}{5{\cdot}93}$$

$$= 18{\cdot}99 \text{ in. units.}$$

$$\mathrm{Z}_t = \frac{112{\cdot}6}{3{\cdot}07} = 36{\cdot}67 \text{ in. units.}$$

$$\therefore \text{Safe B.M. in tension} = f_t \times \mathrm{Z}_t$$

$$= 36{\cdot}67 \text{ in. tons.}$$

Neglecting weight of beam itself, if central load is W, the maximum B.M. is $\frac{\mathrm{W}\,l}{4}$

$$\therefore \text{Maximum B.M.} = \frac{\mathrm{W}\,l}{4} = \frac{\mathrm{W} \times 12 \times 12}{4} = 36\,\mathrm{W} \text{ in. tons.}$$

$$\therefore \mathrm{W} = \frac{36{\cdot}67}{36} = \underline{1{\cdot}02 \text{ tons.}}$$

$$\text{The compression stress} = \frac{f_t \times d_c}{d_t} = \frac{1 \times 5{\cdot}93}{3{\cdot}07} = 1{\cdot}93 \text{ tons per sq. in.}$$

(5) *A flitched beam consists of two timbers, each* 9 *ins. thick and* 16 *ins. deep, and a steel plate placed symmetrically between them, the steel plate being* 8 *ins. deep and* ¾ *in. thick. If* E *for timber is* 1,500,000 *lbs. per sq. in. and for steel* 30,000,000 *lbs. sq. per in. find the maximum tensile stress in the steel plate when the maximum ensile stress in the timber is* 1000 *lbs. per sq. in.*

Determine also for the same intensity of stress in the timber the percentage increase of load the flitched beam will carry as compared with the two timbers when not reinforced with the steel plate. (*B.Sc. Lond.*)

Using the notation given on p. 183, we see that

$$m = \frac{30{,}000{,}000}{1{,}500{,}000} = 20 \text{ (see Fig. 99).}$$

∴ The steel plate is equivalent to a timber 20 times as wide, *i. e.* a timber 15 × 8 ins.

∴ For the equivalent section of timber for the whole flitched beam I_2

$$= \frac{2 \times 9 \times 16^3}{12} + \frac{(15 - \frac{3}{4})\, 8^3}{12}$$

$$= 6{,}144 + 608$$

$$= 6{,}752 \text{ in. units.}$$

For the timber beam not reinforced I = 6,144.

When the stress in the timber at the outside of the section is 1000 lbs. per sq. in., that 4 ins. below the N.A., *i.e.* at the maximum depth of the equivalent timber plate, will be

$$\frac{4}{8} \times 1000 = 500 \text{ lbs. per sq. in.}$$

But steel carries 20 times the stress in the timber for the strain.

∴ Stress in steel = 20 × 500 = 10,000 lbs. per sq. in.

For the flitch beam the equivalent modulus is

$$\frac{6752}{8} = 844 \text{ in. units.}$$

$$\therefore \text{ Safe B.M. in ft. lbs.} = \frac{844 \times 1000}{12} = 70{,}333$$

$$\text{For the plain timber beam } Z = \frac{6{,}144}{8} = 768 \text{ in. units}$$

$$\therefore \text{ Safe B.M. in ft. lbs.} = \frac{768 \times 1000}{12} = 64{,}000$$

∴ Increased B.M. carried by flitched beam = 6,333

$$\therefore \% \text{ increase} = \frac{6{,}333}{6{,}4000} \times 100 = 9{\cdot}9\ \%$$

We shall have further numerical examples on the stresses in beams at various points in the book.

Approximate Value of Modulus of I Sections.—In practice girders are usually made of **I** section, because the most economical section is that in which as much as possible of the metal is placed in the edges or flanges. In this case an approximate formula for the modulus of the section can be found as follows; Let D (Fig. 100) be the distance between the centre of flanges of the section, the thickness of the flanges being t. Then if B is the breadth of the flanges, and t_1 the thickness of the web, we have

$$I = \frac{B\,(D + t)^3}{12} - \frac{(B - t_1)\,(D - t)^3}{12} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$\therefore\ 12\,I = B\,(D^3 + 3\,D^2t + 3\,Dt^2 + t^3) - (B - t_1)\,(D^3 - 3\,D^2t + 3\,Dt^2 - t^3)$$

$$= B\,(6\,D^2t + 2\,t^3) + t_1\,(D^3 - 3\,D^2t + 3\,D\,t^2 - t^3)$$

$$\therefore\ \frac{12\,I}{D^2} = 6\,B\,t\left(1 + \frac{2\,t^2}{6\,D^2}\right) + t_1\left(D - 3\,t + \frac{3\,t^2}{D} - \frac{t^3}{D^2}\right) \ldots\ldots(2)$$

Now if t is small compared with D, t^2 and t^3 are negligible,

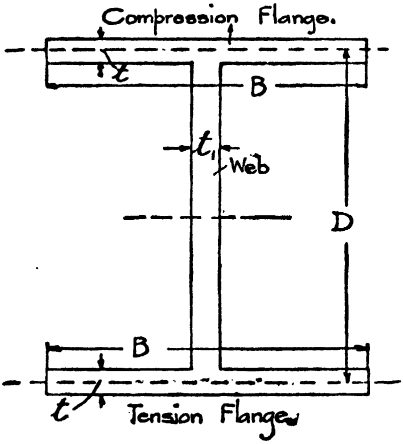


FIG. 100.

$$\therefore\ \frac{12\,I}{D^2} = 6\,B\,t + t_1\,D\left(1 - \frac{3\,t}{D}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\text{or } I = \frac{D^2}{12}\left\{(6\,B\,t + t_1\,D\left(1 - \frac{3\,t}{D}\right)\right\}$$

$$\text{Now } Z = \frac{I}{\frac{D + t}{2}} = \frac{2\,I}{D + t} = \frac{2\,I}{D\left(1 + \frac{t}{D}\right)} = \frac{2\,I}{D}\left(1 - \frac{t}{D}\right) \text{ nearly}$$

since t is small compared with D.

$$\therefore \; Z = \frac{D^2}{6D}\left\{6Bt + t_1 D\left(1 - \frac{3t}{D}\right)\right\}\left(1 - \frac{t}{D}\right)$$

$$= \frac{D}{6}\left\{6Bt - \frac{6Bt^2}{D} + t_1 D - 3tt_1 - tt_1 + \frac{3t_1t^2}{D}\right\} \quad ..(4)$$

$$= \frac{D}{6}\left\{6Bt + t_1(D - t)\right\} \text{ to a first approximation,}$$

neglecting all remaining terms containing t^2 or $t\,t_1$.

Now $B \times t$ = area of one flange = A

and $t_1(D - t)$ = area of the web = a

$$\therefore \; Z = DA + \frac{aD}{6}$$

$$= D\left(A + \frac{a}{6}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Therefore we get the following rule: *The modulus of an* I *section beam is approximately equal to the depth between the centres of the flanges multiplied by the area of one flange plus one-sixth of the area of the web.*

Discrepancies between Theoretical and Actual Strengths of Beams.—Many practical men have expressed considerable surprise that in testing beams the actual and theoretical breaking strengths do not agree. A number of beams are tested, and a tension test is also made from the same material, and it is found that the load which, on the ordinary bending theory, should cause the breaking stress in the beam, does not cause fracture, the amount of additional load depending on the shape of the cross section. This was the origin of the old "beam paradox," it being thought that the material must be stronger in bending than in tension. In fact, for cast-iron beams, an old erroneous theory which, for a rectangular beam, made $M = \frac{f \times b h^2}{4}$ instead of $\frac{f \times b h^2}{6}$ agrees considerably better with the breaking test than the modern theory.

Now this discrepancy in the case of ductile metals is due to the fact that the ordinary bending theory is not applicable to breaking stresses, and no one who appreciated the value of the assumptions made in obtaining such theory would expect the theoretical and actual breaking strengths to agree. This is because the stress is not proportional to strain after the elastic limit is reached.

Some experimenters who have measured the deflections of beams have stated that for mild steel the stresses at the elastic limit do not agree, but that is due to a confusion between the elastic limit and the yield point, and to the fact that the deflections were not measured with sufficient accuracy. In Chap. I we saw that for a tension test of mild steel the elastic limit and yield point were quite close to each other; but in bending this is not the case, the yield point occurring at a considerably later point than the elastic limit. Considerable error, therefore, arises if the yield point in bending be taken instead of the elastic limit. If the latter be carefully measured it will be found that the stresses in tension and bending at the elastic limit agree very closely. This point is proved, incidentally, in the Andrews-Pearson paper on Stresses in Crane Hooks, referred to in Chap. XIX. The reason for the yield point coming some distance after the elastic limit in bending is that only the material at the extreme edges has been stressed up to the yield point, and the whole section will not yield until the material nearer the centre has become stressed up to the yield point.

We see, therefore, that there is no discrepancy between theory and tests so long as the conditions laid down in formulating the theory are fulfilled. If those conditions do not hold beyond a certain point, then, after that point, we must get a new theory if we wish to calculate the stresses.

These so-called discrepancies between theoretical and actual strengths of beams point to the desirability of choosing the working stresses for ductile metals in terms of the stress at the elastic limit, and not of the breaking stress—as we pointed out in Chap. III.—because if the working stress in a beam is, say, one-half of the stress at the elastic limit in tension, then twice the load on the beam will cause the elastic limit in the beam; if, however, the working stress be taken as one-fourth of the breaking stress in tension, four times the load will not cause failure, the exact load to do this being more, and depending on the shape of the section.

Cast-Iron Beams.—The discrepancy in the case of cast-iron beams is due to the fact that the stress-strain diagram is not a straight line except for very low stresses, and that for given strain the stress is appreciably less in tension than in compression. The result of this is that the stress diagram becomes more like that shown in Fig. 101; the neutral axis becomes raised above the centroid and the diagram is curved. In this figure the actual stress is shown about one-half of the calculated stress. This effect will be most marked in sections such as rounds or diagonal squares with a large amount of metal in the web; it is least in I sections with the compression flange smaller than the tensile, *i. e.* the discrepancy between theory and practice is least in a well-designed section.

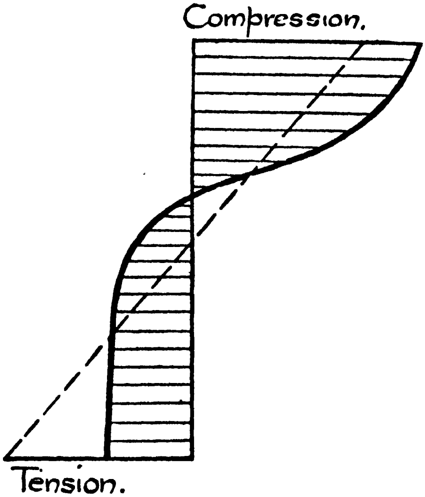


Fig. 101.

* **Diagonal Square Sections.**—It is an interesting fact that we can obtain as follows the apparently paradoxical

result that by cutting away part of a beam of diagonal square section we increase it strength.

Referring to Fig. 101A and the table on p. 185

$$I_{\text{N.A.}} \text{ of whole section} = \frac{D^4}{(\sqrt{2})^4 \,.\, 12}$$
$$= \frac{D^4}{48}$$

$$\therefore \text{ Z of whole section} = \frac{D^4}{48} \div \frac{D}{2} = \frac{D^3}{24}$$

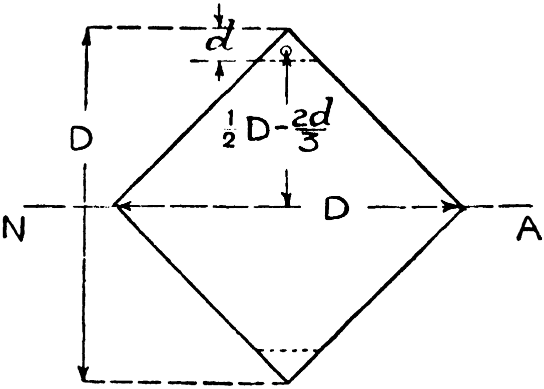


FIG. 101A.

$I_{\text{N.A}}$ of two triangles removed $= 2 \times \frac{2d^4}{36}$ (about their own centroids) $+ 2\,.\,d^2\left(\frac{D}{2} - \frac{2d}{3}\right)^2$ (to bring to N.A.)

$$\therefore \text{ I of remaining section} = \frac{D^4}{48} - \frac{d^4}{9} - 2d^2\left(\frac{D}{2} - \frac{2d}{3}\right)^2$$
$$= \frac{D^4}{48}\left\{1 - \frac{16d^4}{3D^4} - \frac{2d^2D^2 \,.\, 48}{4D^4}\left(1 - \frac{4d}{3D}\right)^2\right\}$$

∴ Z of remaining section

$$= \frac{D^4}{\frac{48(D-2d)}{2}}\left\{1 - \frac{16d^4}{3D^4} - \frac{24d^2}{D^2}\left(1 - \frac{4d}{3D}\right)^2\right\}$$

$$\therefore a = \frac{\text{Z of remaining section}}{\text{Z of original section}}$$
$$= \frac{D}{D-2d}\left\{1 - \frac{16d^4}{3D^4} - \frac{24d^2}{D^2}\left(1 - \frac{4d}{3D}\right)^2\right\}$$

Now let $\frac{d}{D} = x$

$$
\begin{aligned}
\text{Then } a &= \frac{1}{(1-2x)}\left\{1 - \frac{16x^4}{3} - 24x^2\left(1 - \frac{4x}{3}\right)^2\right\} \\
&= \frac{1}{(1-2x)}\{1 - 24x^2 + 64x^3 - 48x^4\} \\
&= 1 + 2x - 20x^2 + 24x^3 \\
&= (1-2x)(1+4x-12x^2) \\
&= (1-2x)^2(1+6x)
\end{aligned}
$$

This is a maximum when $\frac{da}{dx} = 0$

$$
\begin{aligned}
&\textit{i.e. } 2 - 40x + 72x^2 = 0 \\
&\textit{i.e. } 1 - 20x + 36x^2 = 0 \\
&\textit{i.e. } (1-18x)(1-2x) = 0
\end{aligned}
$$

From this $x = \frac{1}{18}$ or $\frac{1}{2}$ and the maximum is for $x = \frac{1}{18}$ because clearly $x = \frac{1}{2}$ reduces the section to zero. From this we see that the strongest section is obtained by removing one-eighteenth of the depth from the top and bottom, *i. e.* one-ninth of the depth in all.

When $\qquad x = \frac{1}{18}$

$$a = \left(\frac{8}{9}\right)^2 \cdot \frac{4}{3} = \frac{256}{243} = 1{\cdot}053 \text{ nearly}$$

Therefore the section $\frac{8}{9}$ of the depth of the original section appears to have a strength 1·053 times as much, *i.e.* about 5·3 % increase. The additional strength is, however, only apparent, because when failure starts at the edges we arrive at the stronger section.

We do not know of any accurate tests that have been made to find to what extent this result holds in the actual beam. In a cast-iron beam of the original section under test a small crack will start on the tension side which will have the effect of cutting off one edge only. A complete study of this problem on a modified theory for cast-iron beams will be found in a paper by Mr. Clark in *Proc. Inst. C.E.* (1901–2).

*** Influence of Shearing Force on Stresses in Beams.**—It must be remembered that up to the present we have considered only the tensile and compression stresses due to the bending moment. Besides these stresses there are tangential stresses due to the shearing force. The resultant stress at any internal point of the beam is the resultant or principal stress of the tangential and direct stresses, which resultant is found as shown in Chap. I. We shall deal in a subsequent chapter with the distribution of the shearing stresses across the section of the beam, but for the present we will assume that the shear stress is a maximum at the centroid and diminishes to zero at the extremities. Fig. 102 shows diagrammatically the shear and direct stresses across the cross section of the beam and also the resultant stresses which, as will be seen,

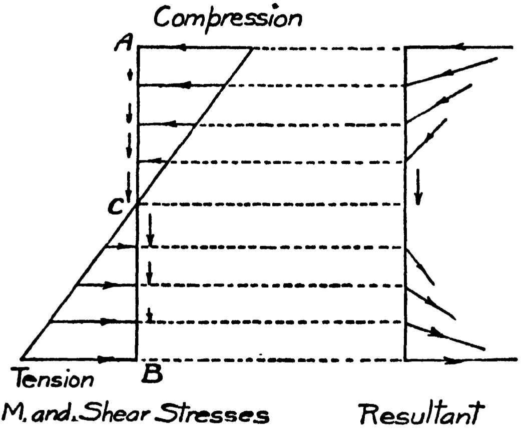


FIG. 102.—Principal Stresses in Beams.

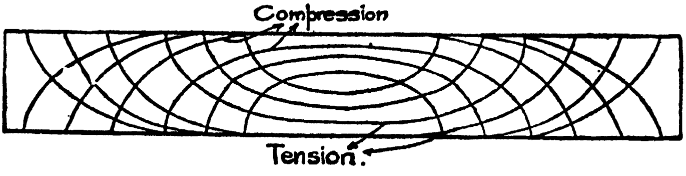


FIG. 102A.

are parallel to the centre line of the beam at the extremities and are perpendicular to it at the centroid.

If the principal stresses at various depths be found for a number of cross sections at various points along the span, and the directions of principal stress be joined up by a curve, we get a number of lines showing the manner in which the directions of principal stresses vary from one point to another. Such curves will be found in Rankine's *Applied Mechanics*, and are of the form shown in Fig. 102A.

In practice it will be found that, except for very short beams carrying heavy loads, the maximum tensile or compressive stress due to bending moment is usually greater than the maximum shear stress, so that the consideration of stress due to bending moment is, as a rule, considerably more important than that of the shear stresses.

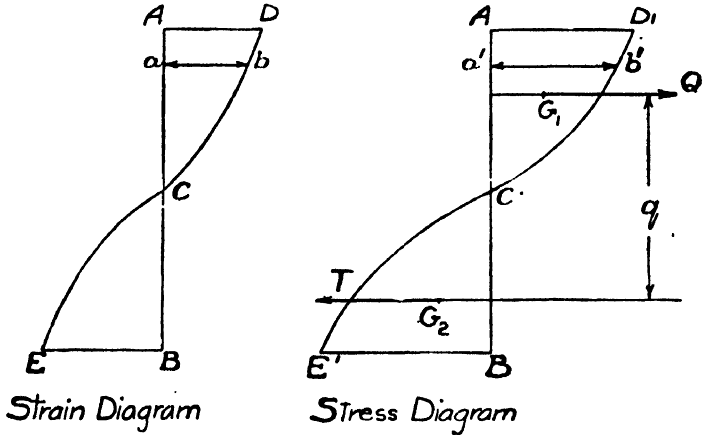


FIG. 103.

*** Moment of Resistance in General Case.**—To follow the correct theory of beams it is not necessary to make any of the assumptions previously given, and we will now find the moment of resistance in the most general case. To investigate this, we must suppose that we know by experimental or other means the shape after distortion which is taken up by a cross section of the beam which was originally plane. We must also know the relation between stress and strain for the material of which the beam is composed.

Let A B, Fig. 103, represent the elevation of a cross section of a beam which after bending is strained to the shape D C E. Then from the stress-strain curve and from the shape of the cross section draw a curve of stress D′ C E′. This is obtained as follows: let *a b* be any ordinate of the strain diagram; then from the stress-strain curve find the stress corresponding to this strain, and multiply the stress by the breadth of the beam at the given point, and plot this equal to *a′ b′* to some convenient scale; joining up points such as *b′* we get the stress diagram.

Now let the area of the stress diagrams be Q and T and their centroids G_1 and G_2. Then, of course, in simple bending Q and T will be equal, and if q is the perpendicular distance between the centroids, the moment of resistance will be equal to $T \times q$ or $Q \times q$.

If the reader fully follows this general method with regard to the stresses in beams, he should not have the difficulty commonly experienced in following the more particular theories.

CHAPTER VIII

STRESSES IN BEAMS—(*continued*)

*REINFORCED CONCRETE BEAMS

There are many formulæ for the strength of reinforced concrete beams, such formulæ being deduced from certain assumptions with reference to the distribution of stress in the bent beam.

We will consider three methods of calculating the stresses in reinforced concrete beams, working in each case the case of a rectangular section, this being most common, and in all three we will make the following assumptions—

(1) That a section of the beam which is plane before bending remains plane after bending. (Bernoulli's assumption (see p. 194).)

(2) That the beam is subjected to pure bending, *i.e.* that the total compressive stress is equal to the total tensile stress.

Standard Notation.—Throughout the treatment we will adopt the following notation (see Fig. 104).

$$m = \frac{\text{Young's modulus for steel or other metal}}{\text{Young's modulus for concrete}} = \frac{E_s}{E_c}$$

t = Tensile stress per sq. in. in reinforcement.

t_c = ,, ,, ,, concrete.

c = Compressive stress per sq. in. in concrete

A_r = Area of cross section of reinforcement.

A_c = ,, ,, ,, concrete.

b = Breadth of beam.

d_t = Total depth of beam.

d = Depth of beam to centre of reinforcement.

n = Depth from compressive edge to neutral axis (N.A.).

$(d-n)$ = „ „ centre of reinforcement to neutral axis.

n_1 = Ratio $\frac{n}{d}$.

r = Proportional area of reinforcement to area above it $= \frac{A_r}{b\,d}$

I_e = Equivalent moment of Inertia of section.

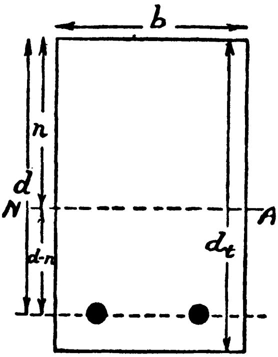


FIG. 104.—Notation for Reinforced Concrete Beams.

First Method—Ordinary Bending Theory.—The first method which we will consider is one which is not much used in practice because it gives safe loads which are lower than tests show to be necessary. It is, however, the general method applicable to beams formed of two elastic materials, and serves as a useful and instructive introduction to the subject.

According to this method, we assume that the reinforced beam behaves exactly as an ordinary homogeneous beam with the reinforcement replaced by a narrow strip m times the area of the reinforcement, and at constant distance from the N.A. We showed how to find the centroid, moment of inertia, and radius of gyration of such an equivalent homogeneous section on p. 183.

In the general case, let n (Fig. 105) be the distance to the neutral axis (equivalent centroid), and I_E the equivalent moment of inertia about the centroid.

$$\text{Then } t_c = \frac{M(d_t - n)}{I_E} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$c = \frac{M n}{I_E} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$t = \frac{m M (d - n)}{I_E} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where M is the bending moment.

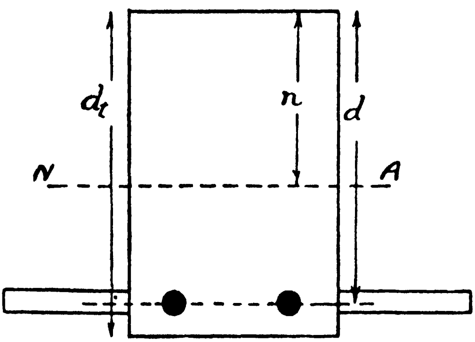


FIG. 105.—Reinforced Concrete Beams. Method 1.

In the case of the rectangular beam we then get the following results—

Equivalent area of section $= b\,d_t + (m - 1)\,A_T$......(4)

As explained on p. 184, it is $(m - 1)\,A_T$, because when we take away the reinforcement and replace it by m times its area of concrete, we have first to fill up the hole in which the reinforcement was, and this takes once A_T, so that remaining additional area $= (m - 1)\,A_T$.

Take moments round the top, then we have

$$n\,\{b\,d_t + (m - 1)\,A_T\} = \frac{b\,d_t^2}{2} + (m - 1)\,A_T\,d$$

$$\therefore n = \frac{\dfrac{b\,d_t^2}{2} + (m - 1)\,A_T\,d}{b\,d_t + (m - 1)\,A_T} \ldots\ldots (5)$$

This fixes the position of the neutral axis.

Taking second moments about the neutral axis, we have

$$I = \frac{b\,n^3}{3} + \frac{b\,(d_t - n)^3}{3} + (m - 1)\,A_r\,(d - n)^2 \ldots\ldots (6)$$

In this formula we neglect the second moment of reinforcement about its own axis.

NUMERICAL EXAMPLE.—*Take the case of a beam* 6 *ins. wide and* 12 *ins. deep, the centre of the reinforcement being* 2 *ins. from the bottom and the area of reinforcement* = 1·44 (*see Fig.* 106).

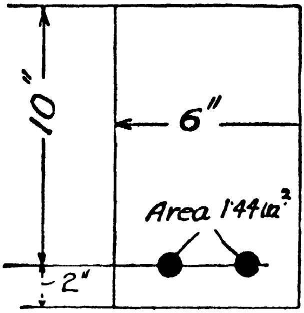


FIG. 106.

Taking $m = 15$, we get

$$n = \frac{6 \times 144 + 2 \times 14 \times 1{\cdot}44 \times 10}{2\,(72 + 14 \times 1{\cdot}44)}$$

$$= 6{\cdot}87 \text{ ins.}$$

$$\therefore (d_t - n) = 12 - 6{\cdot}87 = 5{\cdot}13$$

$$I = \frac{6 \times (6{\cdot}87)^3}{3} + \frac{6 \times (5{\cdot}13)^3}{3} + 14 \times 1{\cdot}44 \times 3{\cdot}13^2$$

$$= 648 + 270 + 197 = 1115 \text{ nearly.}$$

∴ Taking a safe stress of 100 lbs. per sq. in. in tension for the concrete,

$$\text{Safe B.M.} = \frac{100 \times 1115}{5{\cdot}13 \times 12} = 1820 \text{ ft. lbs.}$$

Then t_c = comp. stress in concrete $= \dfrac{100 \times 6{\cdot}87}{5{\cdot}13} = 134$ lb./in.²

Then t = Tensile stress in steel $= \dfrac{15 \times 100 \times 3{\cdot}13}{5{\cdot}13} = 915$ lb./in.²

It will be seen that we have taken $t_c = 100$, which is higher than usually allowed for concrete in tension; but if the concrete cracked the steel would still hold, and so we are justified in using a higher stress.

The above example shows that on this method of calculation the beam is not very economical, as the steel is very little stressed and the concrete has only a small stress in compression.

For this reason it is usual in practice to neglect the tensile stresses in the concrete, that is to say that it does not matter if the concrete does crack. Practice shows that such cracks, if present, do not matter so long as the adhesion between steel and concrete is good, and the tensile stress in the steel and the compressive stress in the concrete are within safe limits.

We should like in this connection to point out that to neglect the tensile stresses in the concrete does not, as some writers state, increase the factor of safety. We shall see later that neglecting such stresses we get a much larger safe B.M. on the beam, and thus *reduce* the factor of safety.

STRENGTH OF SAME BEAM NOT REINFORCED.—To serve as a useful comparison we will find the strength of a 12″ × 6″ concrete beam without reinforcement.

$$\text{If not reinforced, } t_c = \frac{M \times 6}{\frac{6 \times 12^3}{12}} = \frac{M}{144}$$

In this case we must take safe $t_c = 50$ lb./in.²

$$\therefore \text{ Safe B.M.} = \frac{50 \times 144}{12} = 600 \text{ ft. lbs.}$$

Therefore, calculating by our first method, the reinforced beam is roughly three times as strong. It would cost roughly twice as much, so that we see there is 50 % saved.

Second Method—Straight-line, No-tension Method.—This method we name as above, because the additional assumptions are indicated by such name.

We will now make the following additional assumptions—

(*a*) All the tensile stress is carried by the reinforcement.

(*b*) For the concrete the stress is proportional to the strain.

(*c*) The area of reinforcement is so small that we may assume the stress constant over it.

Fig. 107 shows the section, strain diagram, and stress diagram.

We will first give the usual treatment which is based upon argument from first principles.

In accordance with our first assumption a vertical plane section becomes an inclined plane section A′ B′, the neutral axis (N.A.) being at the point C.

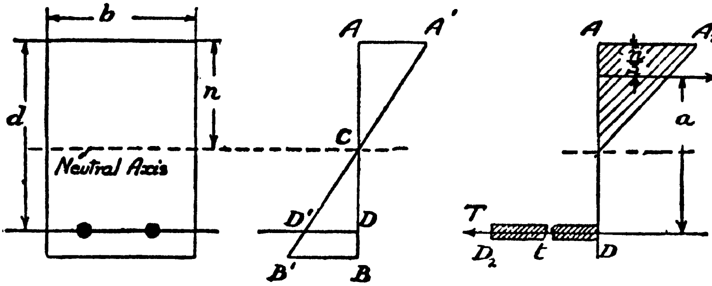


FIG. 107.—Reinforced Concrete Beams. Method 2.

What we first require to determine is the position of the N.A.

Now A A′ and D D′ represent the maximum strains in the concrete and the steel respectively, and since the line A′ B′ is assumed straight, these strains are proportional to their distances from the neutral axis.

$$\therefore \text{ We have } \frac{\text{max. strain in concrete}}{\text{max. strain in steel}} = \frac{n}{(d-n)} \cdots\cdots\cdots\cdots (7)$$

$$\text{but max. strain in concrete} = \frac{c}{E_c}$$

$$\text{and max. strain in steel} = \frac{t}{E_s}$$

$$\therefore \frac{n}{(d-n)} = \frac{c}{t} \cdot \frac{E_s}{E_c} = \frac{mc}{t}$$

$$\therefore nt = mc(d-n) \cdots\cdots\cdots\cdots\cdots\cdots (8)$$

$$d - n = \frac{n\,t}{m\,c}$$

$$\therefore\ d = n\left(1 + \frac{t}{m\,c}\right)$$

$$\therefore\ n = \frac{d}{1 + \frac{t}{m\,c}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

This determines the distance from the N.A. when both *c* and *t* are known; but this will not always be the case. If the reinforcing bars are of given size, then *t* will depend on that size, and to determine the position of the neutral axis, we proceed as follows: The stress diagram shows the distribution of stress in the cross section. Since we have assumed the stress proportional to the strain, the stress diagram for the concrete will be a triangle. It will be seen therefore that the mean compressive stress is $\frac{c}{2}$, and since the compression area is $b\ n$, we see that the total compressive stress is $\frac{1}{2}\,c\,b\,n$.

As the stress in the steel is assumed uniform, we get that the total tensile stress in the steel is $t\,A_T$, and if the beam is subjected to pure bending these must be equal.

$$\therefore\ t\,A_T = \frac{1}{2}\,c\,b\,n \quad \ldots\ldots\ldots\ldots\ldots(10)$$

$$i.e.\ \frac{c}{t} = \frac{2\,A_T}{b\,n}.$$

Comparing this with equation (8) we get

$$\frac{2\,A_T}{b\,n} = \frac{n}{m\,(d - n)}$$

$$\therefore\ b\,n^2 = 2\,m\,A_T\,(d - n) \quad \ldots\ldots\ldots\ldots(11)$$

$$\therefore\ b\,n^2 = 2\,m\,A_T\,d - 2\,m\,A_T\,n$$

$$\therefore\ b\,n^2 + 2\,m\,A_T\,.\,n - 2\,m\,A_T\,d = 0.$$

The real solution of the quadratic equation gives

$$n = \frac{m\,A_T}{b}\left\{-1 + \sqrt{\left(1 + \frac{2\,b\,d}{m\,A_T}\right)}\right\} \quad \ldots\ldots\ldots(12)$$

Since all the quantities in this expression are given, this fixes the position of the neutral axis.

We may write this

$$\frac{n}{d} = \frac{m\,A_T}{b\,d}\left\{\sqrt{1 + \frac{2\,b\,d}{m\,A_T}} - 1\right\}$$

$$\text{i.e. } \frac{n}{d} = n_1 = r\,m\left\{\sqrt{1 + \frac{2}{r\,m}} - 1\right\}$$

$$= \left\{\sqrt{m^2 r^2 + 2\,m\,r} - m\,r\right\} \quad \text{..........(13)}$$

For $m = 15$. This gives—

$r =$	$\frac{n}{d} = n_1$
·007	·365
·010	·417
·015	·483
·020	·530

These values are plotted on Fig. 108.

Moment of Resistance.—We can now find comparatively simply the moment of resistance. The resultant compression acts at the centre of gravity of its triangle.

Therefore the distance between the resultant compressions and tensions is $d - \frac{n}{3}$.

∴ If C and T represent these resultant compressions and tensions, we have that the moment of the couple due to the resisting stresses, which is called the moment of resistance, is given by

$$\text{M R} = \text{C}\left(d - \frac{n}{3}\right)$$

$$= \frac{1}{2}c\,b\,n\left(d - \frac{n}{3}\right) \quad \text{............(14)}$$

$$\text{or, M R} = \text{T}\left(d - \frac{n}{3}\right)$$

$$= t\,A_T\left(d - \frac{n}{3}\right) \quad \text{..............(15)}$$

And this moment of resistance must be equal to the maximum bending moment for the loading.

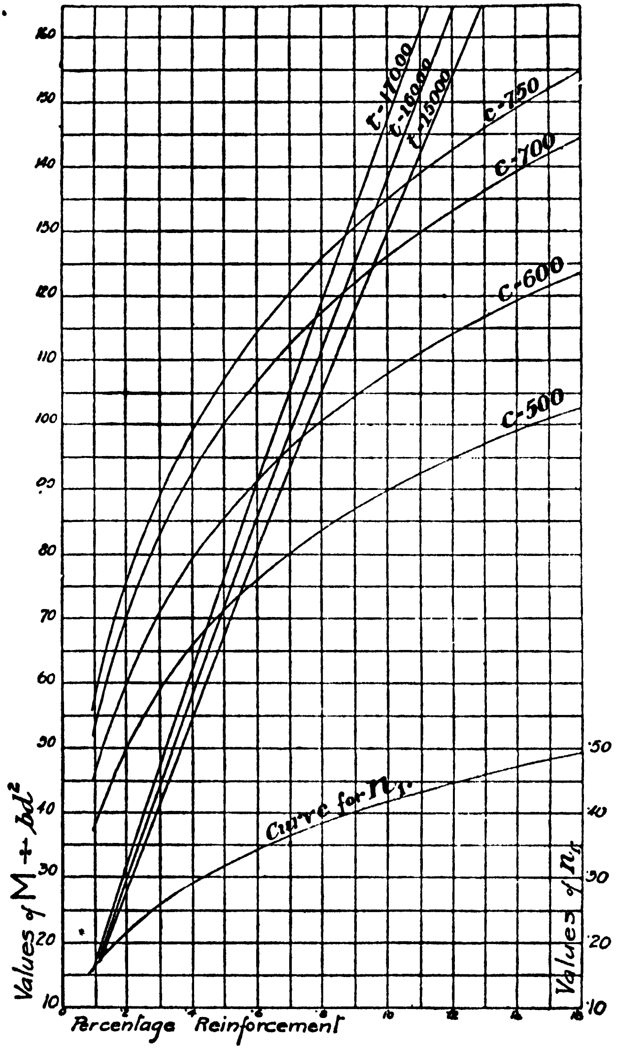


FIG. 108.—Rectangular Reinforced Concrete Beams.

Numerical Example.—*Take the same section as worked by the previous formula (see Fig.* 106); and take $c = 500$ lbs. per in.^{2}

Then equation (12) gives

$$n = \frac{1\cdot44 \times 15}{6}\left\{\sqrt{1 + \frac{2 \times 6 \times 10}{1\cdot44 \times 15}} - 1\right\}$$

$$= 5\cdot61 \text{ inches.}$$

$$d - n = 10 - 5\cdot61 = 4\cdot39 \text{ inches.}$$

Then M.R. or safe B.M. considering the concrete is equal to

$$\frac{500 \times 6}{2} \times 5\cdot61\left\{4\cdot39 + \frac{2}{3}5\cdot61\right\} \text{ in. lbs.}$$

$$= \frac{1,500}{12} \times 5\cdot61 \times 8\cdot13 = 5,700 \text{ ft. lbs. nearly.}$$

Comparing this with the safe B.M. by the first method we see that the present is more than three times as much.

Stress in steel is then equal to $\dfrac{\text{M.R.}}{A_T\left(d - \frac{n}{3}\right)}$

$$= \frac{5700 \times 12}{1\cdot44 \times 8\cdot13} = 5720 \text{ lbs. per in.}^2$$

Assuming a span of 10 ft., the max. B.M. if the load is uniformly distributed is $\dfrac{W \times 10}{8}$ ft. lbs.

$$\therefore \frac{W \times 10}{8} = 5700 \qquad \therefore W = 4560 \text{ lbs.}$$

This includes the weight of the beam, which is roughly

$$\frac{10 \times 12 \times 6}{144} \times 150 \text{ lbs.} = 750 \text{ lbs.}$$

$$\therefore \text{Safe load uniformly distributed} = 4,560 - 750$$
$$= 3,810 \text{ lbs.}$$

It will be seen from the stress in the steel that the area of reinforcement is more than it need have been. By combining equations (9) and (10) we could have found the value of A_T to give the stress in the steel, say 16,000 lbs. per sq. in., when the compressive stress in the concrete is 500 lbs. per sq. in.

The above formula gives results which are in fairly good agreement with tests, and is the one most largely used in practice.

Alternative Treatment for Straight-line, No-Tension Method.—The following treatment follows more nearly the ordinary method of dealing with beams than the above, but it is not nearly so often used in this country. We shall, however, find it very useful in the case of beams other than rectangular ones with tension reinforcement only and so we give it here; its value has been scarcely sufficiently appreciated. Fig. 109 shows the section of the beam and also the equivalent section.

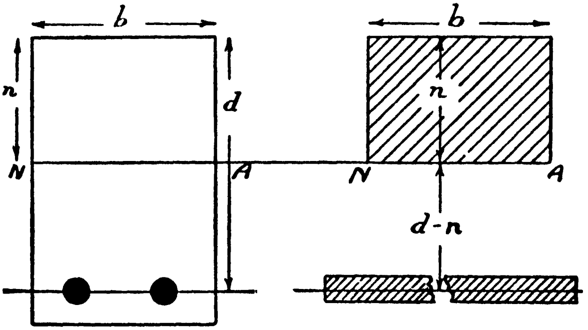


FIG. 109.

We find the position of the neutral axis for the equivalent section by the rule given on p. 196 that the total first moment of the cross section of the beam about the N.A. is zero.

$\therefore b\,n \times \frac{n}{2}$ = 1st moment of compression area about N.A.

$- m\,A_t\,(d - n)$ = 1st moment of equivalent tension area about N.A.

$$\therefore \frac{b\,n^2}{2} - m\,A_t\,(d - n) = 0 \;\ldots\ldots\ldots\ldots (15)$$

$$\text{or } b\,n^2 + 2\,m\,A_t\,n - 2\,m\,A_t\,d = 0 \text{ [cf. p. 221]} \ldots (12)$$

This is the same relation as before.

Equivalent moment of Inertia of the section

$$= I_E = \frac{b\,n^3}{3} + m\,A_T\,(d - n)^2$$

$$= \frac{b\,n^3}{3} + \frac{b\,n^2}{2}(d - n) \quad \text{[from 15 above]}$$

$$= \frac{b\,n^2}{2}\left(\frac{2\,n}{3} + d - n\right)$$

$$= \frac{b\,n^2}{2}\left(d - \frac{n}{3}\right)$$

$$\text{Now } M = \frac{c\,I_E}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15a)$$

[compare ordinary bending formula p. 197]

$$= \frac{c\,b\,n^2\left(d - \frac{n}{3}\right)}{2\,n}$$

$$= \frac{1}{2}\,c\,n\,b\left(d - \frac{n}{3}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

This agrees exactly with our previous result.

Considering the stress on the reinforcement

$$M = \frac{t\,I_E}{m\,(d - n)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15b)$$

$$\therefore \frac{c}{n} = \frac{t}{m\,(d - n)}$$

$$\text{or } \frac{n}{d} = \frac{1}{1 + \frac{t}{m\,c}} \quad \text{as before.}$$

This is used if A_T is not given.

Case in which Stresses are given and Area of Steel has to be found.

In the case we have $n_1 = \dfrac{1}{1 + \frac{t}{m\,c}}$

Suppose for instance $t = 16{,}000$ and $c = 600$ and $m = 15$

$$n_1 = \frac{1}{1 + \frac{16{,}000}{600 \times 15}} = \frac{9}{25} = \cdot 36$$

$$\therefore n = \cdot 36\,d,$$

Then from equation 10 we can calculate the necessary area of reinforcement by the relation

$$A_r = \frac{c\,b\,n}{2\,t}$$

$$\therefore \frac{A_r}{b\,d} = r = \frac{c\,n_1}{2\,t} = \frac{600 \times \cdot 36}{2 \times 16{,}000} = \cdot 00675$$

In this case the moments of resistance given by equations (14) and (15) will be equal and

$$\text{M.R.} = \frac{600\,b \times \cdot 36\,d\,(d - \cdot 12\,d)}{2}$$

$$= 95\,b\,d^2$$

$$\therefore \text{Safe B.M.} = 95\,b\,d^2$$

The coefficient 95 is called the **resistance modulus** and can be plotted in very convenient form as in Fig. 108. These curves may be likened to the tables of section moduli for steel sections.

NUMERICAL EXAMPLE.—*A reinforced concrete beam is required to carry a bending moment of* 240,000 *in. lbs. Design the section for stresses* $c = 600$, $t = 16{,}000$, *assuming that the breadth of the beam is* 10 *inches.*

By equation (11) $d = \sqrt{\frac{B}{95\,b}}$

$$= \sqrt{\frac{240{,}000}{95 \times 10}}$$

$$= 15 \cdot 9 \text{ inches.}$$

$$\therefore \frac{A_r}{d\,b} = \cdot 00675$$

$$A_r = \cdot 00675 \times 15 \cdot 9 \times 10$$

$$= 1 \cdot 07 \text{ sq. inches.}$$

Adopt 2 — $\frac{7}{8}''$ bars [giving an area 1·2].

Third Method—General No-tension Method.—In this method we will, as before, assume that all the tensile stress is taken by the steel, but we will assume that the stress-strain curve for concrete is not straight but some other curve.

In this way we get the stress diagram, Fig. 110, from the strain diagram.

Suppose that its area $= k \, . \, c \, . \, n$ and that its centroid is at distance y from the top.

Then, as in equations (8), (9) we get

$$n = \frac{(d - n)\, m \, . \, c}{t}$$

$$\therefore n = \frac{d}{1 + \frac{t}{m\,c}}$$

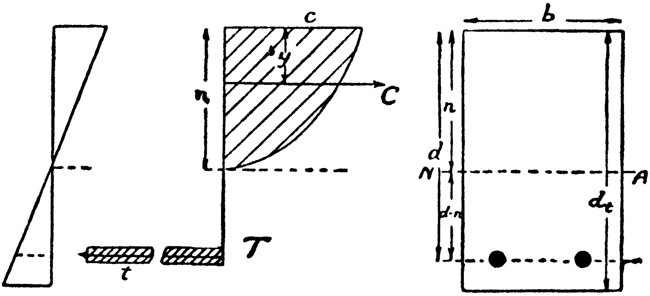


FIG. 110.—Reinforced Concrete Beams. Method 3.

Now, since the total compressive stress must be equal to the total tensile stress, we have

$$t\,A_T = k\,b\,n\,c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

$$\therefore n = \frac{(d - n)\, m\, A_T}{k\,b\,n}$$

$$\therefore k\,b\,n^2 = m\,A_T\,(d - n)$$

$$\therefore k\,b\,n^2 + m\,A_T\,n - m\,A_T\,d = 0 \quad \ldots\ldots\ldots (17)$$

$$n = \frac{A_T\,m}{2\,k\,b}\left\{\sqrt{1 + \frac{4\,d\,k\,b}{m\,A_T}} - 1\right\} \quad \ldots\ldots (18)$$

$$\text{or, } \frac{n}{d} = \frac{r\,m}{2\,k}\left\{\sqrt{1 + \frac{4\,k}{r\,m}} - 1\right\} \quad \ldots\ldots\ldots (19)$$

Then moment of resistance

$$= \text{M.R.} = t\,A_T\,(d - y) \text{ for tension} \ldots\ldots\ldots (20)$$

$$= k\,b\,n\,c\,(d - y) \text{ for compression} \quad \ldots (21)$$

Numerical Example with Stress-strain Curve Parabola.—Take the section that we have worked for the previous formulæ. (See Fig. 106.)

If the stress-strain curve is a parabola tangential at the compression edge we have

$$k = \frac{2}{3}$$

$$y = \frac{3n}{8}$$

For the given section $n = \dfrac{1{\cdot}44 \times 15}{2 . \frac{2}{3} . 6}\left\{\sqrt{1 + \dfrac{4 . 2 . 10 . 6}{15 . 1{\cdot}44 . 3}} - 1\right\}$

$$= 5{\cdot}12''$$

$$\therefore d - n = 10 - 5{\cdot}12 = 4{\cdot}88$$

$\therefore$ Safe M.R. for concrete

$$= \frac{2}{3} \times 6 \times 5{\cdot}12 \times 500\,(4{\cdot}88 + 3{\cdot}20) \text{ in. lbs.}$$

$$= \frac{2}{3} \times \frac{6}{12} \times 5{\cdot}12 \times 500 \times 8{\cdot}08 \text{ ft. lbs.}$$

$$= 6{,}900 \text{ ft. lbs. nearly}$$

Then stress in reinforcement

$$= t = \frac{6{,}900 \times 12}{1{\cdot}44 \times 8{\cdot}08} = 7{,}120 \text{ lbs. per sq. in.}$$

It will be seen that this method gives higher values still for the safe bending moments. The stress-strain curve for concrete, although nearly parabolic, would not have the vertex of the parabola at a stress of 500 lbs. per sq. in.

From the above we think that it should be clear that there is not much difficulty in finding the stress in reinforced concrete beams so long as we know accurately the properties of the concrete, and are clear as to what assumptions we are making.

Reinforced Concrete T Beams.—Reinforced concrete floors usually consist of reinforced slabs with reinforced beams at definite intervals in a longitudinal direction, the whole being monolithic. Fig. 111 shows a section of such a floor, which may be regarded as a number of T beams. The reinforcing bars A in a transverse direction in the slabs are arranged as shown to take the tension at the top where the bending moment reverses, due to the slabs being continuous.

It is usual to take the effective breadth of the flanges of the T beams as less than B—½ to ¾ B—because the concrete between the beams acts as a short beam in a direction at right angles, and so the centre portion is comparatively highly stressed for this reason.

We will now consider the stress in the beam, adopting the no-tension, straight-line method.

CASE 1. If $d_s > n$ we get the same rules as given in method (2) for rectangular beams, b_s being substituted for b.

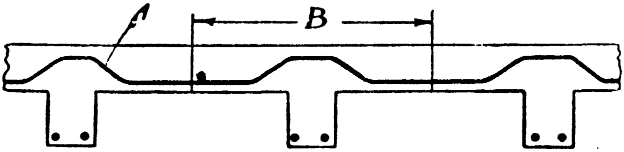


FIG. 111.

CASE 2. If $d_s < n$ we proceed as follows—

As before we have from a consideration of the strain diagram

$$n = \frac{(d - n)\,m\,c}{t}$$

$$n = \frac{d}{1 + \dfrac{t}{m\,c}}$$

Now consider the total stress diagram, Fig. 112, *i.e.* horizontal lengths of compression figure = compressive stress per sq. in. × breadth of beam.

Now total compressive stress on the section

$$= \mathrm{C} = \text{area } (\mathrm{KDH} - \mathrm{HFG})$$

$$= \frac{c\,b_s\,n}{2} - \frac{(b_s - b_r)\,x}{2} \times \frac{x}{n} \,.\, c$$

$$\mathrm{C} = \frac{c}{2}\left\{b_s\,n - \frac{(b_s - b_r)\,x^2}{n}\right\}$$

$$\text{But } \mathrm{C} = \mathrm{T} = t\,\mathrm{A}_T.$$

$$\therefore\ t\,A_T = \frac{c}{2}\left\{b_s n - \frac{(b_s - b_r)\,x^2}{n}\right\} \quad \ldots\ldots\ldots(22)$$

$$\therefore\ \frac{c}{t} = \frac{2\,A_T}{\left\{b_s n - \frac{(b_s - b_r)\,x^2}{n}\right\}}$$

$$\therefore\ n = \frac{2\,A_T\,m\,(d - n)}{\left\{b_s n - \frac{(b_s - b_r)\,x^2}{n}\right\}}$$

$$n\left\{b_s n - \frac{(b_s - b_r)\,(n - d_s)^2}{n}\right\} = 2\,A_T\,m\,(d - n)$$

$$n\left\{b_r n + 2\,d_s\,(b_s - b_r) + \frac{d_s^{\,2}}{n}\,(b_r - b_s)\right\} = 2\,A_T m\,(d - n)$$

$$\text{i.e. } b_r n^2 + 2\,n\,\{A_T\,m + d_s\,(b_s - b_r\} = 2\,A_T\,m\,d + (b_s - b_r)\,d_s^{\,2} \quad (23)$$

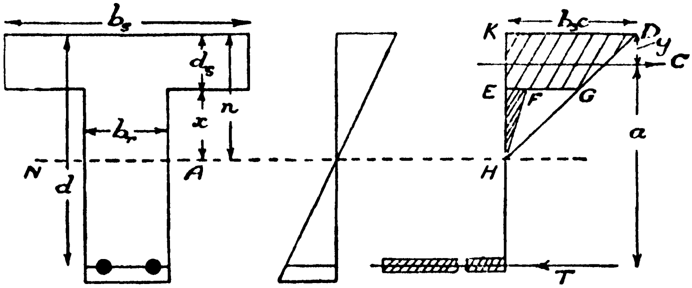


Fig. 112.—Reinforced T Beams.

From this quadratic the value of n can be found.

Then if the centroid of the compressive stress-strain curve area is at distance a from the centre of reinforcement

$$\text{Safe B.M.} = \text{C} \times a$$

Let the centroid of the compressive stress strain diagram be at distance y from the top.

Now this centroid is the same as the centre of pressure on a similar body subjected to fluid pressure, the N.A. being the water line. In this case it is easily shown that

$$y = \frac{\text{2nd Mt. of area above N.A. about top.}}{\text{1st Mt. of area above N.A. about top.}}$$

$$= \frac{\frac{b_s n^3}{3} - \frac{(b_s - b_r)\,x^3}{3}}{\frac{b_s n^2}{2} - \frac{(b_s - b_r)\,x^2}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24)$$

This enables us to find a.

Many writers neglect the rib, *i.e.* neglect the portion F E H of the stress diagram, and others further assume $y = \cdot 4\,d$ $\left(\text{the extreme limits between which } y \text{ must lie are } \frac{d_s}{3} \text{ and } \frac{d_s}{2}\right)$. This avoids the quadratic equation and makes the calculation much easier.

We may put $b_r = 0$ in equation (23)

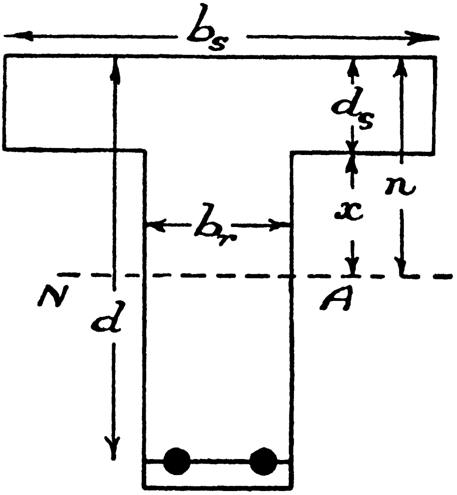


FIG. 112*a*.

We then get at once

$$n = \frac{2\,m\,A_T\,d + b_s\,d_s^2}{2\,(m\,A_T + b_s\,d_s)} \quad \ldots\ldots\ldots\ldots\ldots (25)$$

a then becomes equal to $d - \frac{d_s}{3}\left(\frac{3\,n - 2\,d_s}{2\,n - d_s}\right)$

Considering compression we have
Safe bending moment $= \mathrm{C} . a$

$$= \frac{c}{2}\,b_s\,d_s\,\frac{(2n - d_s)}{n}\left\{d - \frac{d_s}{3}\left(\frac{3n - 2d}{2n - d_s}\right)\right\} \quad \ldots\ldots (26)$$

Alternative Treatment.—Applying the method of the equivalent section we have (Fig. 113) a much simpler treatment.

Moment of equivalent section about N.A. = 0

$$\text{i.e. } b_s \frac{n^2}{2} - (b_s - b_r)\frac{(n - d_s)^2}{2} = m\,A_T\,(d - n)$$

This gives the same quadratic as equation (23)

$$I_s = \frac{b_s n^3}{3} - \frac{(b_s - b_r)(n - d_s)^3}{3} + m\,A_T\,(d - n)^2 \quad \ldots\ldots(27)$$

neglecting the rib we should have

$$b_s \frac{n^2}{2} - \frac{b_s (n - d_s)^2}{2} = m\,A_T\,(d - n)$$

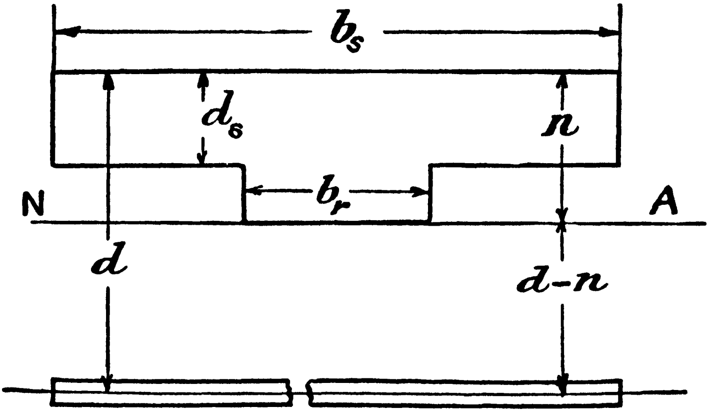


Fig. 113.—Reinforced Concrete T Beams.

This gives as before

$$n = \frac{2\,m\,A_T\,d + b_s d_s^2}{2\,(b_s d_s + m\,A_T)}$$

Having found I_s we have as before

$$\text{Safe B.M.} = \frac{c\,I_s}{n} \text{ for concrete}$$

$$= \frac{t\,I_s}{m\,(d - n)} \text{ for reinforcement}$$

Numerical Example of T Beam.—Take the T beam of section shown in Fig. 114. In this case we will not assume the area of reinforcement (A_r) to be given, but will calculate it so as to give

$$c = 600 \text{ lbs. per sq. in.}$$

$$t = 16{,}000 \text{ lbs. per sq. in.}$$

$$m = 15$$

Then we have $n = \dfrac{d}{1 + \dfrac{t}{m\,c}}$

$$= \frac{15}{1 + \dfrac{16{,}000}{15 \times 600}} = 5{\cdot}4 \text{ ins.}$$

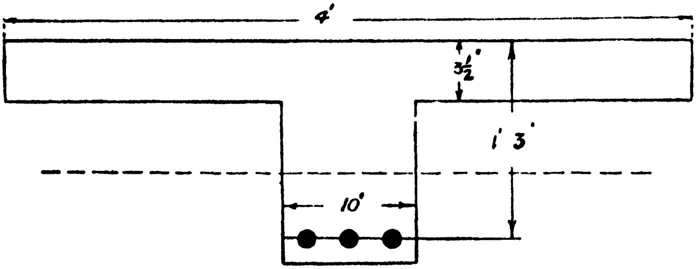


Fig. 114.

∴ From equation (22)

$$16{,}000\,A_r = \frac{600}{2}\left\{5{\cdot}4 \times 48 - \frac{38 \times 1{\cdot}9^2}{5{\cdot}4}\right\}$$

$$\therefore A_r = 4{\cdot}38 \text{ sq. ins.}$$

∴ Adopt, say, 3 bars $1\frac{3}{8}''$ diameter.

Then working by the equivalent moment of Inertia

$$I_s = 15 \times 4{\cdot}49 \times 9{\cdot}6^2 + 48 \times \frac{5{\cdot}4^3}{3} - \frac{38 \times 1{\cdot}9^3}{3}$$

$$= 8{,}641$$

$$\therefore \text{Safe B.M.} = \frac{600 \times 8{,}641}{5{\cdot}4} = 960{,}000 \text{ in. lbs.}$$

For other cases of reinforced concrete beams the reader is referred to the author's *Elementary Principles of Reinforced Concrete Construction* (Scott Greenwood & Son, London).

COMBINED BENDING AND DIRECT STRESSES

If the loading on a beam is such as to cause a direct stress in addition to bending stresses, then the resultant stresses across the section will be obtained by adding together the separate stresses. Let B D, Fig. 115, represent the elevation of a section of a beam, C being the centroid of the section whose area is A and whose compression and tensile moduli are Z_c and Z_t, D being the compression side and B the tension side.

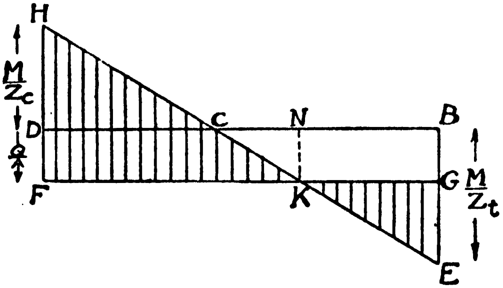


FIG. 115.

Then, if the direct force is a thrust Q, there will be a uniform compression stress of $\frac{Q}{A}$ over the section. If the bending moment is equal to M, the maximum compression and tensile stresses due to bending are equal respectively to $\frac{M}{Z_c}$ and $\frac{M}{Z_t}$. Therefore we have

$$\text{Resultant maximum compressive stress} = f_c = \frac{Q}{A} + \frac{M}{Z_c} \quad \text{..(1)}$$

$$\text{Resultant maximum tensile stress} = f_t = \frac{M}{Z_t} - \frac{Q}{A} \quad \text{..(2)}$$

The distribution of the combined stresses across the section is then as shown in Fig. 115, F H representing the maximum compressive stress, and G E the maximum tensile stress. The neutral axis then is at the point N, where the stress is zero.

If the direct force is a pull T instead of a thrust Q, we have

$$\text{Resultant maximum tensile stress} = f_t = \frac{\mathrm{T}}{\mathrm{A}} + \frac{\mathrm{M}}{\mathrm{Z}_t} \quad ..(3)$$

$$\text{Resultant maximum compressive stress} = f_c = \frac{\mathrm{M}}{\mathrm{Z}_c} - \frac{\mathrm{T}}{\mathrm{A}} \quad ..(4)$$

Stresses obtained from Line of Pressure.—If the resultant force across the cross section is R, Fig. 116, and the line of pressure cuts D B produced in L, the *load point*, then resolving R along and perpendicular to the cross section we get a shearing force S and a thrust Q.

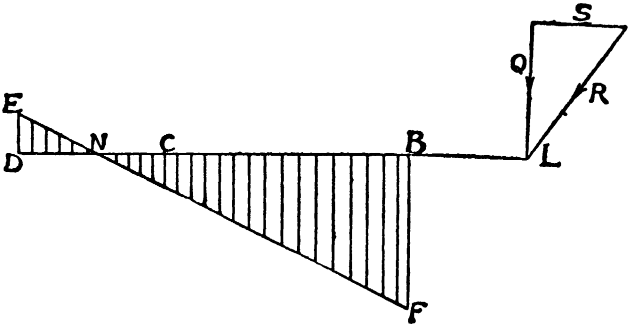


FIG. 116.

In this case $\mathrm{M} = \mathrm{Q} \times \mathrm{CL} = \mathrm{Q} \times x$

and if $\mathrm{CD} = d_t$ and $\mathrm{CB} = d_c$

we have
$$\mathrm{Z}_c = \frac{\mathrm{I}}{d_c} = \frac{\mathrm{A}k^2}{d_c}$$
$$\mathrm{Z}_t = \frac{\mathrm{I}}{d_t} = \frac{\mathrm{A}k^2}{d_t}$$

where k is the radius of gyration about a line through the centroid parallel to the neutral axis.

∴ We have from equations (1) and (2)

$$f_c = \frac{\mathrm{Q}}{\mathrm{A}} + \frac{\mathrm{Q} \, . \, x \, d_c}{\mathrm{A}k^2}$$

$$= \frac{Q}{A}\left(1 + \frac{x\,d_c}{k^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$f_t = \frac{Q\,.\,x\,d_t}{A\,k^2} - \frac{Q}{A}$$

$$= \frac{Q}{A}\left(\frac{x\,d_t}{k^2} - 1\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Or if the resultant normal component is a pull T, equations (3) and (4) become

$$f_t = \frac{T}{A}\left(1 + \frac{x\,d_t}{k^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

$$f_c = \frac{T}{A}\left(\frac{x\,d_c}{k^2} - 1\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Position of the Neutral Axis.—The position of the neutral axis N can be found as follows—

Let it be at distance y from C.

$$\text{Then stress due to bending} = \frac{M\,y}{I}$$

$$= \frac{Q\,.\,x\,y}{A\,k^2}$$

At this point the stress due to bending is exactly equal to the direct stress.

$$\therefore \frac{Q\,.\,x\,y}{A\,k^2} = \frac{Q}{A}$$

$$\text{or} \qquad x\,y = k^2$$

$$\text{i.e.} \qquad y = \frac{k^2}{x} \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

The following numerical examples will make the question of combined direct and bending stresses clear; further examples will occur in the course of the book.

Numerical Examples.—(1) *A tension rod is a flat bar* 8 *inches wide and* 1 *inch thick: owing to bad fitting, the line of pull, instead of passing along the geometrical axis of the bar, lies* ¼ *of an inch to one side of it, in the plane which bisects the thickness of the rod. Determine the maximum and minimum stresses set up in this bar in a section at right angles to the line of pull when the pull is* 36 *tons.*

Show by a sketch the actual distribution of the stress across the section. (*B.Sc. Lond.*)

In this case the direct stress $= \frac{T}{A} = \frac{36}{8 \times 1} = 4{\cdot}5$ tons per sq.in.

The B.M. is equal to $T \times x$, and the second moment is equal to $\frac{1 \times 8^3}{12} = \frac{128}{3}$

$$\therefore k^2 = \frac{I}{A} = \frac{128}{3} \times \frac{1}{8} = \frac{16}{3}$$

$$\therefore f_t = \frac{T}{A}\left(1 + \frac{x\,d_t}{k^2}\right)$$

$$= 4{\cdot}5\left(1 + \frac{1}{4} \times 4 \times \frac{3}{16}\right)$$

$$= 4{\cdot}5 \times 1\frac{3}{16} = 5{\cdot}344 \text{ tons per sq. in.}$$

$$f_c = \frac{T}{A}\left(\frac{x\,d_c}{k^2} - 1\right)$$

$$= 4{\cdot}5\left(\frac{3}{16} - 1\right)$$

$$= -4{\cdot}5 \times \frac{13}{16} = -3{\cdot}656 \text{ tons per sq. in.}$$

The distribution of the stress is then as shown in Fig. 117.

(2) *A hollow circular column has a projecting bracket on which a load of* 1 *ton rests. The centre of this load is* 2 *feet from the centre of the column. External diameter of column is* 10 *inches, and thickness* 1 *inch. What is the maximum compression stress?* (*A.M.I.C.E.*)

In this case $A = \frac{\pi}{4}(10^2 - 8^2) = 28{\cdot}28$

$$I = \frac{\pi}{64}(10^4 - 8^4) = 289{\cdot}8 \text{ in. units}$$

$$\therefore k^2 = \frac{289{\cdot}8}{28{\cdot}28} = 10{\cdot}25$$

$$\therefore f_c = \frac{Q}{A}\left(1 + \frac{x\,d_c}{k^2}\right)$$

$$= \frac{1}{28{\cdot}28}\left(1 + \frac{24 \times 5}{10{\cdot}25}\right)$$

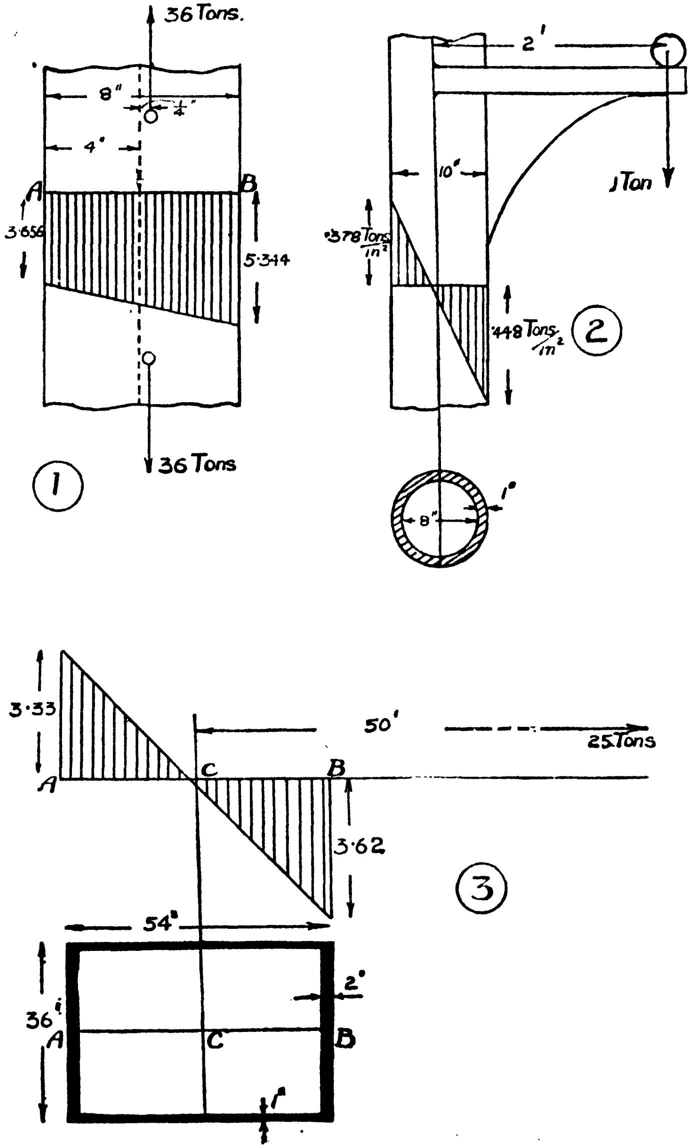


Fig. 117.—Combined Bending and Direct Stress.

$$= \frac{12{\cdot}7}{28{\cdot}28} = {\cdot}448 \text{ ton per sq. in.}$$

$$f_t = \frac{Q}{A}\left(\frac{x\,d_t}{k^2} - 1\right)$$

$$= \frac{10{\cdot}7}{28{\cdot}28} = {\cdot}379 \text{ ton per sq. in.}$$

The distance of the N.A. from the centre of the section is then given by $y = \frac{k^2}{x}$

$$= \frac{10{\cdot}25}{24} = {\cdot}427 \text{ in.}$$

The distribution of stresses is then as shown in Fig. 117.

(3) *A built-up crane jib is in the form of a curved girder, and a horizontal section near the base is a hollow rectangle. The outside dimensions of this rectangle are* 54 *and* 36 *inches, and the larger and shorter sides are* 1 *inch and* 2 *inches thick respectively. Find the maximum tensile and compressive stresses induced in the material when a load of* 25 *tons is suspended from the end of the crane, the horizontal distance of the load from the centre of the section being* 50 *feet. Show by a sketch how the intensity of stress varies across the section.* (*B.Sc. Lond.*)

It will be noted that in this question no means are given to connect the plates of the rectangle, such means being necessary in practice.

Proceeding as in the previous example, we see that

$$A = 2\,(72) + 1\,(100) = 244 \text{ sq. ins.}$$

$$I = \frac{36 \times 54^3}{12} - \frac{34 \times 50^3}{12} = 118{,}200$$

$$\therefore k^2 = \frac{118{,}200}{244} = 484{\cdot}5$$

$$\therefore f_c = \frac{25}{244}\left(1 + \frac{600 \times 27}{484{\cdot}5}\right)$$

$$= \frac{25}{244} \times 34{\cdot}5 = 3{\cdot}62 \text{ tons per sq. in.}$$

$$f_t = \frac{25}{244}\left(\frac{600 \times 27}{484{\cdot}5} - 1\right) = 3{\cdot}33 \text{ tons per sq. in.}$$

Fig. 116 shows the manner in which the stresses are distributed.

* BEAMS WITH OBLIQUE LOADING

In obtaining our formulæ for the stresses in beams, we assumed that "the section of the beam is symmetrical about an axis through the centroid of the cross section parallel to the plane in which bending occurs."

We saw in dealing with moments of inertia, or second moments, that an axis of symmetry is called a principal axis of the section. Our assumption, therefore, is equivalent to saying that one of the principal axes lies in the plane of loading of the beam.

When such is not the case the loading is said to be oblique and we proceed as follows or in the alternative method given on p. 244. Draw the momental ellipse for the beam, X X and Y Y (Fig. 118) being the principal axes, and let Z Z be the trace of the plane of loading. *Then the neutral axis will be the diameter of the ellipse conjugate to the plane of loading. The plane of bending will be at right angles to the neutral axis.*

This is proved as follows—

Consider an element of area at the point P of a section (Fig. 118), and let P N and P M be drawn perpendicular to the plane of loading and neutral axis respectively. Then the intensity of stress at P is proportional to P M, the distance from the neutral axis, so that if c is a constant we may write

$$f_r = c \times \text{PM}.$$

∴ The moment of the load over the area about Z Z is equal to $f_r \times a \times \text{PN} = c \times a \times \text{PM} \times \text{PN}.$

Now since Z Z is the plane of loading, the moment of all the stresses over the section about Z Z must be zero, since the couple to the stresses must also be in plane Z Z.

$$\therefore \Sigma f_r \times a \times \text{PN} = 0$$

$$i.e.\ \Sigma c \times a \times \text{PM} \times \text{PN} = 0$$

$$i.e.\ \Sigma a . \text{PM} \times \text{PN} = 0$$

But $\Sigma a . \text{PM} . \text{PN}$ is what we have previously called the *product moment*, and it can be shown that if the product moment of an area about two lines is equal to zero, such lines must be conjugate diameters of an ellipse.

Therefore to find the neutral axis draw a chord the diameter conjugate to z z. To do this draw a chord parallel to z z and bisect it and join c to the point of bisection.

Now suppose the radius of gyration about the N.A. is $k_{N.A.}$,

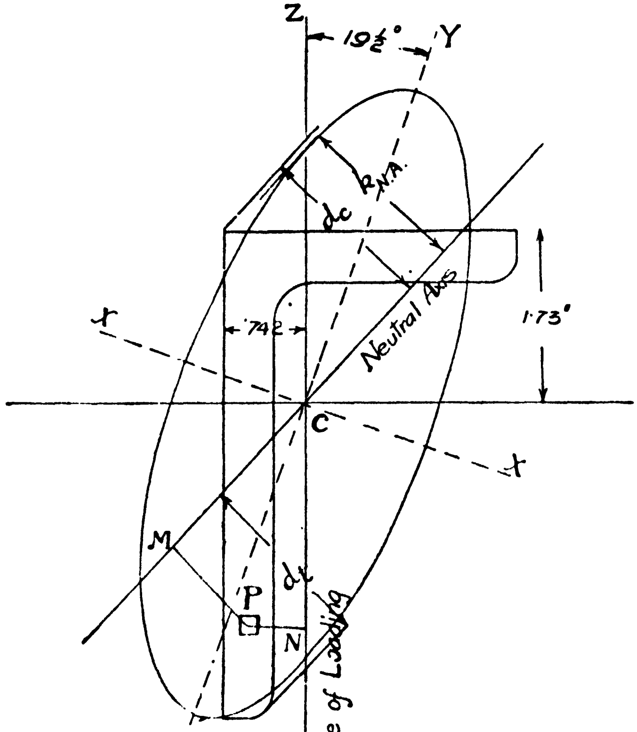


FIG. 118.—Stresses due to Oblique Loading.

and d_c and d_t are the distances from the extreme points of the section to the compression and tension sides respectively.

Then the moduli are

$$Z_c = \frac{A\,k_{N.A.}^2}{d_c} = \frac{I_{N.A.}}{d_c}$$

$$Z_t = \frac{A\,k_{N.A.}^2}{d_t} = \frac{I_{N.A.}}{d_t}$$

Then the maximum compression and tension stresses are obtained by the relations

$$f_c = \frac{M}{Z_c}$$

$$f_t = \frac{M}{Z_t}$$

Numerical Example.—*A* 5″ × 3″ × ½″ *unequal angle section is loaded on the small side with the long leg downward. Find the safe bending moment for a stress of* 7 *tons per square inch.*

From the tables of standard sections we see that for this section the maximum and minimum values of the radius of gyration are 1·69 and ·65 inches, the principal axes being at 19½° to the vertical line **z z**, which is the trace of the plane of loading.

The momental ellipse is now drawn (to twice the scale in Fig. 118), the major axis being equal to twice k_{xx}, and the minor axis equal to twice k_{yy}.

By the construction previously given we get the diameter of the ellipse conjugate to **z z**. This gives the neutral axis. To obtain $k_{N.A}$ draw a tangent to the ellipse parallel to the N.A. and draw a line from o perpendicular to this axis. This will be found to be ·88 inch. Now measure the distances d_c d_t from the neutral axis to the extreme fibres of the section and these will be found to be 1·80 and 1·83 inches respectively. The area of the section is 3·75 sq. ins. Therefore we see

$$Z_c = \frac{3{\cdot}75 \times 88^2}{1{\cdot}80} = 1{\cdot}61 \text{ in. units}$$

$$Z_t = \frac{3{\cdot}75 \times {\cdot}88^2}{1{\cdot}83} = 1{\cdot}59 \text{ in. units}$$

∴ If safe stress $= f_c = f_t = 7$ tons per square inch

$$\text{Safe B.M.} = 7 \times 1{\cdot}59 = \underline{11{\cdot}13 \text{ in. tons}} \ldots\ldots\ldots\ldots (1)$$

If we had taken the N.A. at right angles to the plane of loading, as in the case of a symmetrical beam, we should have had $k = 1{\cdot}60$, $d_c = 1{\cdot}73$, and $d_t = 3{\cdot}27$.

$$\text{This would give } Z_c = \frac{3\cdot75 \times 1\cdot60^2}{1\cdot73} = 5\cdot55 \text{ in. units}$$

$$Z_t = \frac{3\cdot75 \times 1\cdot60^2}{3\cdot27} = 2\cdot94 \text{ in. units}$$

$$\therefore \text{ Safe B.M.} = 7 \times 2\cdot94 = \underline{20\cdot58 \text{ in. tons}} \ldots\ldots\ldots (2)$$

(In finding the safe B.M. we, of course, consider only the least modulus if the working stresses are the same in tension and compression.)

We see from comparing results (1) and (2) that a very large error is made by failing to find the true neutral axis. This error is very commonly made by practical designers.

A similar allowance should be made for symmetrical sections where one of the principal axes does not coincide with the plane of loading. Such cases occur in practice in plate girders where the wind is blowing on one side while the load is crossing, and in sloping bridges where the cross girders are placed with their flanges at the same inclination as the main girders.

* **Alternative Treatment for Oblique Loading.**—In some cases it is much simpler to proceed by what is known as the "*principle of superposition.*"

Let λ be the angle of inclination of the plane of loading to the principal axis and let x, y be the co-ordinates referred to the principal axes of the point at which the stress is required.

$$\text{Then } f = \frac{M \sin \lambda \,.\, y}{I_{xx}} + \frac{M \cos \lambda \,.\, x}{I_{YY}} \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

We shall show later that this comes to a simple result in a common case.

Let N N (Fig. 119) be the neutral axis and let n be the perpendicular distance of a point P from it; then f_p = stress at P $= m \,.\, n$ where m is a constant.

$$\text{Then } M \sin \lambda = \text{component of M about X X} = \Sigma f \,.\, a \,.\, y$$

$$= \Sigma m \,.\, n\, y\, a \; (1)$$

$$\text{but } n = \text{PS} = \text{PR} - \text{SR} = \text{PR} - \text{QT} = y \cos a - x \sin a$$

$$\therefore M \sin \lambda = m \cos a\, \Sigma\, y^2 a - m \sin a\, \Sigma\, x\, y \,.\, a \ldots\ldots\ldots (2)$$

$$\text{but } \Sigma\, x\, y \,.\, a = \text{product moment about principal axes} = 0$$

$$\therefore M \sin \lambda = m \cos a\, \Sigma\, y^2\, a = m \cos a\, I_{xx} \ldots\ldots\ldots\ldots (3)$$

$$
\begin{aligned}
\text{Similarly } M\cos\lambda &= \text{component of } M \text{ about } \mathrm{Y\,Y}\\
&= \Sigma f a \,.\, x\\
&= \Sigma m n x \,.\, a\\
&= \Sigma m\,(y\cos\alpha - x\sin\alpha)\,x \,.\, a\\
&= m\cos\alpha\,\Sigma x y a - m\sin\alpha\,\Sigma x^2 a
\end{aligned}
$$

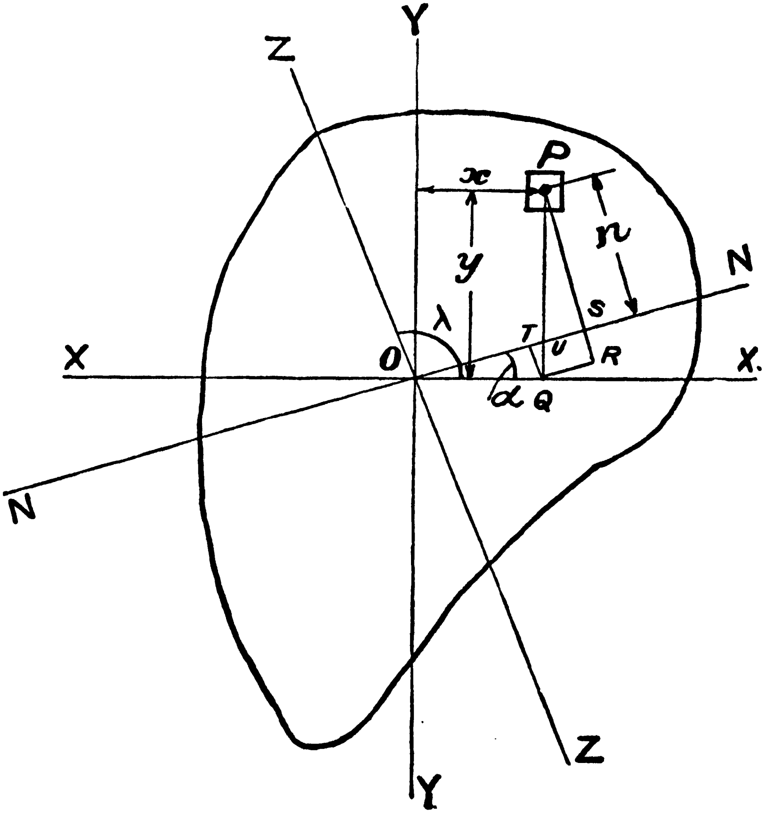


Fig. 119.—Oblique Loading of Beams.

$$
\begin{aligned}
&= o - m\sin\alpha\, I_{\mathrm{YY}}\\
&= -\, m\sin\alpha\, I_{\mathrm{YY}} \quad \dots\dots\dots\dots\dots\dots\dots (4)
\end{aligned}
$$

∴ Dividing (3) by (4)

$$\tan\lambda = \frac{-\cot\alpha\, I_{\mathrm{XX}}}{I_{\mathrm{YY}}}$$

$$\textit{i.e. } \cot\alpha = \frac{-\, I_{\mathrm{YY}}\tan\lambda}{I_{\mathrm{XX}}}$$

This enables us to calculate the position of the neutral axis.

From (3) and (4) $m = \dfrac{M \sin \lambda}{I_{xx} \cos a} = \dfrac{-M \cos \lambda}{I_{YY} \sin a}$

$$\therefore f = m n = m (y \cos a - x \sin a)$$
$$= m y \cos a - m x \sin a$$
$$= \frac{M \sin \lambda \, y \cos a}{I_{xx} \cos a} - \left(\frac{-M \cos \lambda \, x \sin a}{I_{YY} \sin a}\right)$$
$$= \frac{M \sin \lambda \, . \, y}{I_{xx}} + \frac{M \cos \lambda \, . \, x}{I_{YY}} \quad \ldots\ldots\ldots(7)$$

Numerical Example.—*Take, for instance, the obliquely loaded column shown in Fig.* 120. *Loads of* 30 *and* 40 *tons respectively are transmitted at* A *and* B, *the resultant of which is a load of* 70 *tons acting at* D.

The following are the properties of the section—

$$A = 28{\cdot}59 \text{ sq. ins.}$$
$$k_{xx} = 4{\cdot}45 \text{ ins.}$$
$$k_{YY} = 3{\cdot}39 \text{ ins.}$$

Measurement gives $ZOX = \lambda = 110{\cdot}4°$

$$M \sin \lambda = 70 \times OD \sin \lambda = 70 \times EO = 70 \times \frac{4 \times 5{\cdot}5}{7}$$
$$= 220 \text{ in. tons}$$

(This is the same as 40 × O B)

$$M \cos \lambda = 70 \times OD \cos \lambda = -70 \times DE = 70 \times \tfrac{3}{7} \times 2{\cdot}72$$
$$= -81{\cdot}6 \text{ in. tons}$$

(This is the same as 30 × O A)

The maximum stress occurs at the top left-hand corner for which $y = 5{\cdot}5$, $x = -6$ ins.

$$\therefore \text{ Max. bending stress} = f = \frac{220 \times 5{\cdot}5}{28{\cdot}59 \times 4{\cdot}45^2} + \frac{81{\cdot}6 \times 6}{28{\cdot}59 \times 3{\cdot}39^2}$$
$$= 2{\cdot}14 + 1{\cdot}49$$
$$= 3{\cdot}63 \text{ tons per sq. in.}$$

$$\text{Direct stress} = \frac{70}{28{\cdot}59} = 2{\cdot}45 \text{ tons per sq. in.}$$

$\therefore$ combined stress = 6·08 tons per sq. in.

Simplified Result in Special Case.—In the case, as the above, where the oblique loading is caused by two bending moments in the principal axes, M sin λ and M cos λ will be the separate bending moments in the two axes and we thus get the following rule—

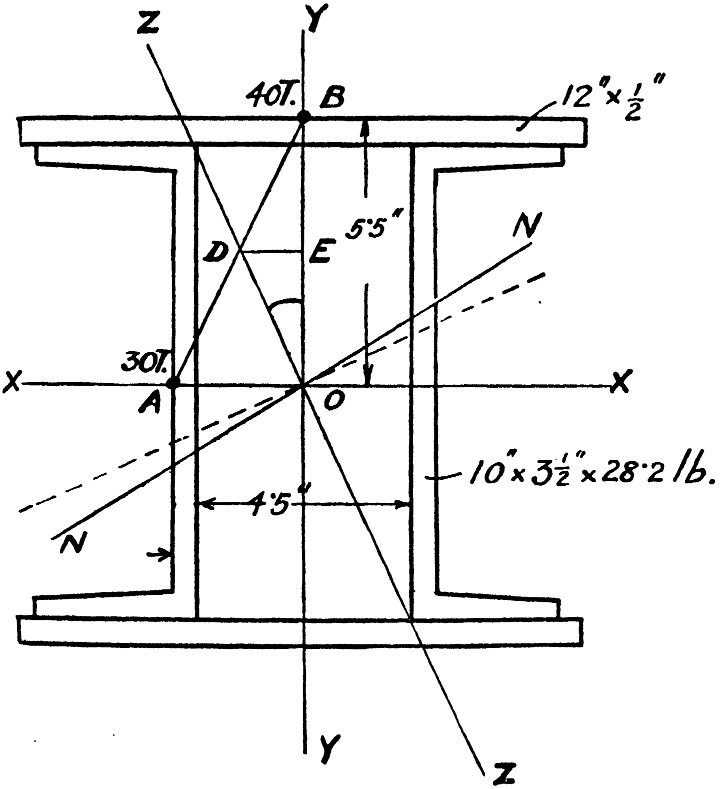


FIG. 120.

Calculate the stresses at any point for each bending moment separately about the corresponding neutral axes; then the total stress for the two bending moments will be the sum of the separate stresses.

CHAPTER IX

DEFLECTIONS OF BEAMS

We have found the relation which exists between the stresses in a beam and the bending moment; we now want to find the relation between the deflections and the bending moment.

Let C C′, Fig. 121, represent a *short* length of the centroid line of a beam, the original curvature of which was negligible, and which has become bent to a radius of curvature R. This radius R is that which agrees with the very short length C C′, and is not the same all along the beam. If the assumptions that we previously made with regard to the stresses in beams still hold, B F and A E are straight lines after bending, and they meet at O, the centre of curvature of C C′. Draw B′ F′ parallel to A E. Now consider the segments B B′ C′ and C C′ O.

$$\text{Since } \theta \text{ is very small } \frac{\text{B B}'}{\text{B}'\text{ C}'} = \frac{\text{C C}'}{\text{C O}}$$

$$\text{or } \frac{\text{B B}'}{\text{C C}'} = \frac{\text{B}'\text{ C}'}{\text{C O}} = \frac{d}{\text{R}} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

But A B′ represents the length of A B before bending occurs

$$\therefore \frac{\text{B B}'}{\text{A B}'} = \frac{\text{increase in length}}{\text{original length}} = \text{strain in A B}$$

$$\text{But A B}' = \text{C C}'$$

$\therefore$ We have $\frac{\text{B B}'}{\text{C C}'}$ = strain in A B $= \frac{f}{\text{E}}$, where f is the stress along A B.

∴ Putting this in equation (1) we have

$$\frac{f}{E} = \frac{d}{R}$$

$$\text{or } \frac{f}{d} = \frac{E}{R} \quad \text{.....................(2)}$$

But we have already shown that

$$M = \frac{fI}{d}$$

$$\text{or } \frac{f}{d} = \frac{M}{I}$$

∴ combining these results we have

$$\frac{f}{d} = \frac{M}{I} = \frac{E}{R} \quad \text{..................(3)}$$

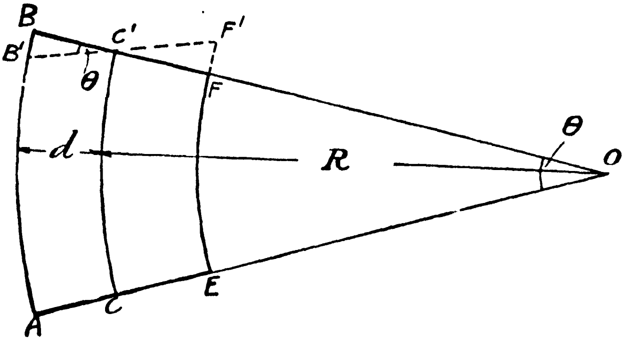


FIG. 121.

This is the complete relation between the stresses in beams, the bending moment, and the radius of curvature. In practice we do not so much want to know the radius of curvature at various points of a beam, but we require the deflection, and so we will next find the relation between radius of curvature and deflection, and then find the deflections for various kinds of loading.

Our investigation now divides itself into two parts according as we consider it from the graphical or the mathematical standpoint, and we will deal with it in this order.

INVESTIGATION FROM GRAPHICAL STANDPOINT*

Preliminary Note on Curvature.—Let A B (Fig. 122) represent any curve, and let P P_1 be points on it at a short distance *s* apart. Draw tangents P Q, P_1 Q_1 to meet any base line making angles θ and θ_1 with it, and draw lines perpendicular to the tangents, then the point of intersection of these perpendiculars is the centre of curvature of the short arc P P_1.

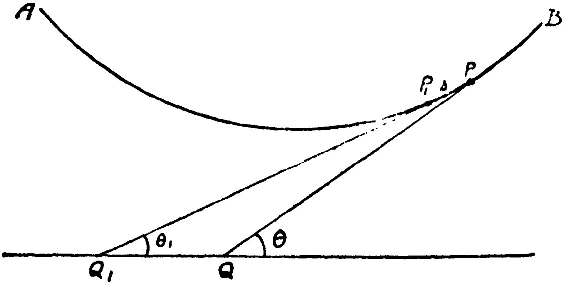


FIG. 122.

Then the angle subtended by P P_1 at the centre will be equal to $(\theta - \theta_1)$.

∴ if R is the radius of curvature $R \times (\theta - \theta_1) = s$.

$$\therefore R = \frac{s}{\theta - \theta_1}$$

$$\text{or } \frac{\theta - \theta_1}{s} = \frac{1}{R}$$

Then $\frac{1}{R}$ is called the *curvature at the given point*, or rather the curvature is the value which $\frac{\theta - \theta_1}{s}$ approaches as *s* gets smaller and smaller.

Mohr's Theorem.—Now imagine A B to be a cable loaded vertically in any manner, and let the load between

---

* The reader may take either the mathematical or the graphical reasoning. Each is complete in itself.

A the points P, P_1 be equal to w. Then it follows from the laws of graphic statics that the cable takes up the shape of the link polygon, for the load system on it, drawn with a polar distance equal to the horizontal pull in the cable.

Now let the tension in the cable at the points P, P_1 be T, T_1. Then the horizontal components of these tensions must be equal, since there is no horizontal force on the cable; let this horizontal component be H; the difference between the vertical components of the tensions must be equal to w, the load between the points.

$$\therefore \text{ We have } \quad H = T \cos \theta = T_1 \cos \theta_1$$

$$w = T \sin \theta - T_1 \sin \theta_1$$

$$\text{i. e. } w = \frac{H \sin \theta}{\cos \theta} - \frac{H \sin \theta_1}{\cos \theta_1}$$

$$= H (\tan \theta - \tan \theta_1)$$

Now if θ_1 and θ are small, as they will be when considering beams, we may say $\tan \theta_1 = \theta_1$ and $\tan \theta = \theta$

$$\therefore \text{ We have } \quad w = H (\theta - \theta_1)$$

$$\therefore \frac{w}{s} = \frac{H (\theta - \theta_1)}{s}$$

$$= \frac{H}{R}$$

But $\frac{w}{s}$ = load per unit length of the cable = say p.

$$\therefore p = \frac{H}{R}$$

$$\text{or } \frac{1}{R} = \frac{p}{H} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Now return to the case of the beam.

$$\text{From equation (3)} \quad \frac{1}{R} = \frac{M}{EI} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The quantity E × I depends solely on the shape and material of the beam, and is called the "flexural rigidity." Then if this flexural rigidity is constant throughout the span, by comparing statement A and equations (4) and (5) we see that: *A loaded beam takes up the same shape as an imaginary cable of the same span which is loaded with the bending moment curve on the beam, and subjected to a horizontal pull equal to the flexural rigidity* ($E\,I$).

This is **Mohr's Theorem,** and the deflected form of the beam is called the *elastic line* of the beam. We see, therefore, that to obtain the elastic line of a beam our procedure is as follows—

(1) Draw the bending moment curve for the beam.

(2) Divide this curve up into narrow vertical strips, and set down mid-ordinates on a vector line, and take a polar distance equal to the flexural rigidity (E I).

(3) Draw the link polygon for this vector polygon, and reduce it to a horizontal base, then this link polygon gives the elastic line to a scale which we shall determine later.

For the present we will assume that the section of the beam is uniform along its length, or rather that the flexural rigidity is constant. We shall see later how to proceed when such is not the case.

Standard Cases of Deflections.—In certain special cases we can calculate the maximum deflections by reasoning based on Mohr's Theorem, and we will deal with such cases now (Fig. 123).

(1) SIMPLY SUPPORTED BEAM WITH CENTRAL LOAD W.—Let A B represent a simply supported beam of span l with a central load W.

Then A D B is the B.M. diagram, the maximum ordinate being equal to $\frac{W\,l}{4}$. Let $A_1\,C_1\,B_1$ be the elastic line of the beam; then, according to Mohr's Theorem, the shape of this elastic line is the same as that of an imaginary cable of the same span loaded with the B.M. curve and subjected to a horizontal pull equal to the flexural rigidity.

Now consider the stability of one half of this cable. It is kept in equilibrium by three forces: the horizontal pull H at the point C_1; the resultant load P on half the cable; and the tension T at the point A_1.

Take moments about the point A_1, then we have

$$H \times \delta = P \times y$$

$$\therefore \delta = \frac{P \times y}{H}$$

In this case P = area of one-half of B.M. diagram

$$= \frac{1}{2} \cdot \frac{l}{2} \times \frac{Wl}{4} = \frac{Wl^2}{16}$$

y = distance of centroid of shaded triangle from A

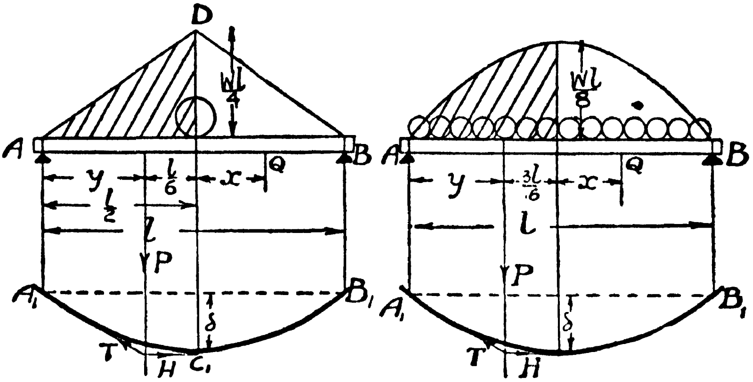


Fig. 123.—Deflections of simply supported Beams.

$$= \frac{l}{3}$$

$$H = EI$$

$$\therefore \delta = \frac{Wl^2}{16} \times \frac{l}{3.EI} = \frac{Wl^3}{48EI}$$

(2) Simply Supported Beam with Uniform Load.—Let AB represent a simply supported beam of span l, with a uniformly distributed load W.

The B.M. diagram is a parabola, the height being equal to $\frac{Wl}{8}$. Then considering the stability of half the imaginary cable, we have as before

$$\delta = \frac{P \times y}{H}$$

In this case P = area of one-half of B.M. diagram

$$= \frac{1}{2} \cdot \frac{2}{3} l \times \frac{W l}{8} = \frac{W l^2}{24}$$

$$y = \frac{5 l}{16}$$

$$H = E I$$

$$\therefore \delta = \frac{W l^2}{24} \cdot \frac{5 l}{16 E I} = \frac{5 W l^3}{384 E I}$$

(3) **Cantilever with an Isolated Load not at Free End.**—Let a cantilever of span L (Fig. 124) carry a load W at a point at distance l from the fixed end A.

Then the B.M. diagram is a triangle, A D being equal to $W l$, $A_1 B_1$ represents the elastic line of the beam and the imaginary cable. In this case we must imagine the load as acting upwards.

The cable is horizontal at A_1.

Take moments round B_1, then we have as before

$$H \times \delta = P \times y$$

$$\therefore \delta = \frac{P \times y}{H}$$

In this case P = area of B.M. curve A C D.

$$= \frac{W l . l}{2} = \frac{W l^2}{2}$$

$$y = L - \frac{l}{3}$$

$$H = E I$$

$$\therefore \delta = \frac{W l^2}{2 E I}\left(L - \frac{l}{3}\right)$$

In this case it should be noted that the portion of the beam beyond the load is straight.

(4) **Cantilever with an Isolated Load at Free End.**—This is the same as the previous case when $l = L$.

$$\therefore \delta = \frac{W L^2}{2 E I}\left(L - \frac{L}{3}\right)$$

$$= \frac{W L^3}{3 E I}$$

(5) **Cantilever with Uniform Load from Fixed End to a Point before the Free End.**—Let A B be a cantilever on span L, and let a load W be uniformly distributed from A to a point C, l being the length of A C.

Then as before

$$\delta = \frac{P y}{H}$$

In this case $\quad P = \text{area of B.M. curve A C D}$

$$= \frac{1}{3} \cdot \frac{W l}{2} \cdot l = \frac{W l^2}{6}$$

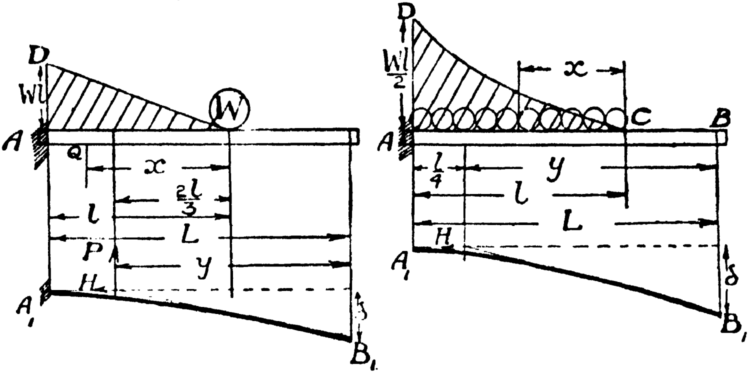


Fig. 124. —Deflections of Cantilevers.

$$y = L - \frac{l}{4}$$

$$H = E I$$

$$\therefore \delta = \frac{W l^2}{6 E I}\left(L - \frac{l}{4}\right)$$

(6) **Cantilever with Uniform Load over Whole Length.**—This is the same as the previous case when $l = L$.

$$\therefore \delta = \frac{W L^2}{6 E I}\left(L - \frac{L}{4}\right)$$

$$= \frac{W L^2}{6 E I} \cdot \frac{3 L}{4} = \frac{W L^3}{8 E I}$$

* (7) **Simply supported Beam with Isolated Load anywhere.**—The reasoning in this case is somewhat long, but should not otherwise present any great difficulties.

The first important point to notice is that the maximum deflection will not occur under the load, so that, as it is the maximum deflection that we nearly always require, it is of very little use to find the deflection directly under the load as is commonly done.

We have seen that the ordinate of the bending moment curve or link polygon of a beam is a maximum where the shear is zero, so that treating the B.M. curve as a load on the beam, the deflection will be a maximum where the shear due to this load is zero.

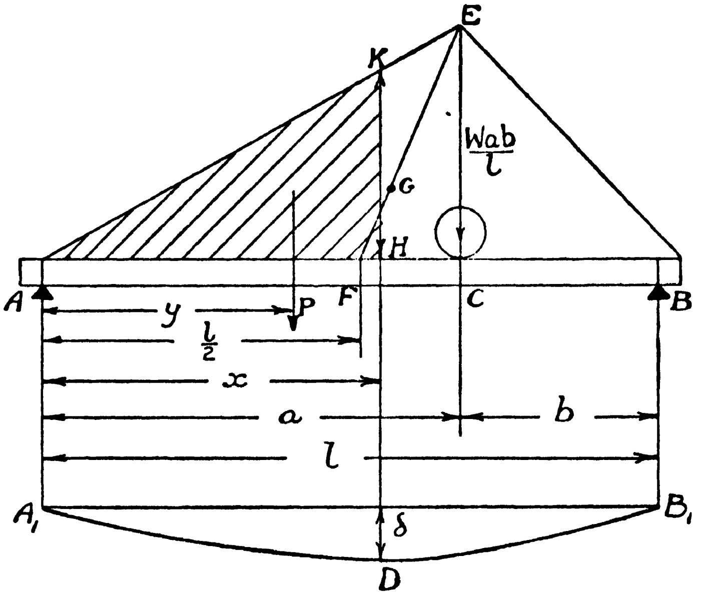


FIG. 125.

Let a load W be placed at a point C on a beam A B of span l, Fig. 125, C being at distances a, b from A and B. Then A E B is the B.M. diagram, C E being equal to $\frac{Wab}{l}$. The total load represented by this B.M. diagram treated as a load will be equal to the area of the Δ A E B $= \frac{Wab}{l} \times \frac{l}{2} = \frac{Wab}{2}$.

It acts at the centroid G of the Δ.

The vertical through this point G is at distance $\frac{2}{3}$ FC from C, F being the mid-point of the beam, so that the distance of this centroid from the end B is equal to

$$b + \frac{2}{3}\left(\frac{l}{2} - b\right) = \frac{l}{3} + \frac{b}{3} = \frac{(l+b)}{3}$$

∴ The reaction at A due to this imaginary load is equal to

$$\frac{\text{Total load}}{l} \times \frac{(l+b)}{3} = \frac{W a b}{2 l} \cdot \frac{(l+b)}{3}$$

Now let the deflection be a maximum at the point D at distance x from A.

Then the shear at this point is zero.

i.e. $$R_A - \frac{x}{2} \cdot KH = 0$$

i.e. $$\frac{W a b}{2 l}\left(\frac{l+b}{3}\right) - \frac{x}{2} \cdot \frac{W b x}{l} = 0$$

$$\therefore \frac{a(l+b)}{3} = x^2$$

or $$x = \sqrt{\frac{a(l+b)}{3}} \quad \ldots\ldots\ldots (1)$$

The maximum deflection δ is then obtained by considering the stability of the portion A_1 D of the imaginary cable.

Then we have as before $\delta = \frac{P \cdot y}{H}$

In this case P = area AKH

$$= \frac{W b x}{l} \cdot \frac{x}{2} = \frac{W b x^2}{2 l}$$

$$= \frac{W b \cdot a(l+b)}{6 l} = \frac{W a b(l+b)}{6 l}$$

$$y = \frac{2}{3} x$$

$$= \frac{2}{3}\sqrt{\frac{a(l+b)}{3}}$$

$$H = EI$$

$$\therefore \delta = \frac{W a b(l+b)}{6 l} \cdot \frac{2}{3}\sqrt{\frac{a(l+b)}{3}} \cdot \frac{1}{EI}$$

$$= \frac{W b}{3 E I l}\left\{\frac{a(l+b)}{3}\right\}^{\frac{3}{2}}$$

This can be put into somewhat simpler form for use by putting

$$a = \alpha\, l. \quad \text{Then } b = (1 - \alpha)\, l.$$

$$\text{Then } \delta = \frac{W(1-\alpha)}{3\,E\,I}\left\{\frac{\alpha\, l\,(2-\alpha)\, l}{3}\right\}^{\frac{3}{2}}$$

$$= \frac{W\, l^3}{3\,E\,I}(1-\alpha)\left\{\frac{2\,\alpha - \alpha^2}{3}\right\}^{\frac{3}{2}}$$

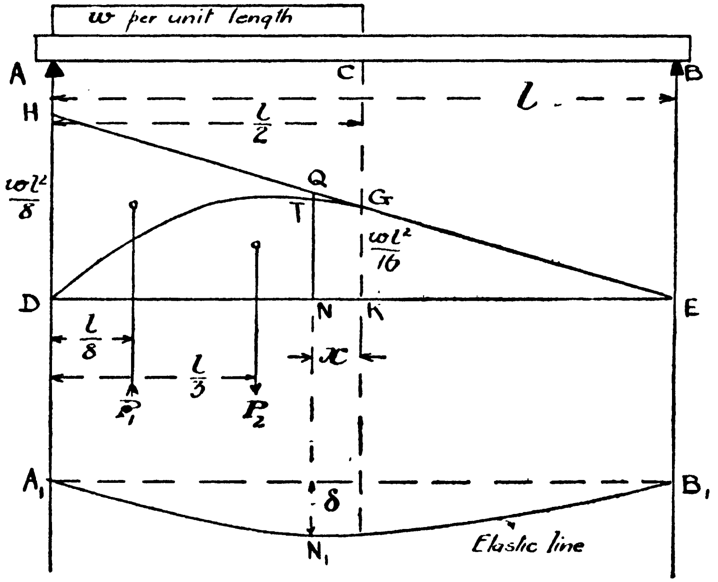


FIG. 126.—Deflections of Beams.

* (8) BEAM UNIFORMLY LOADED FROM ONE END TO THE CENTRE.

The B.M. diagram for this loading is given by the curve D T G E (Fig. 126), the curve D T G being a parabola tangential to the line E G.

We have first to find where this maximum deflection will occur. We do this by the rule that the Maximum Bending Moment in a beam occurs at the point where the shear is zero. We will treat the diagram D G E therefore as the load on the beam.

If E G be produced to H, the curve D T G will be a parabola tangential at G, and it is most convenient in the present problem to consider the B.M. diagram as made up of the difference between the triangle D H E and the parabolic segment D G H.

The first step is to find the imaginary reaction R_s at E. To do this we consider the area of the triangle as a force P_2 acting down its centre of gravity, which is at distance $\frac{l}{3}$ from D.

$$\text{Then } P_2 = \text{area of } \Delta\, \text{DHE} = \tfrac{1}{2}\, \text{DH} \,.\, \text{DE} = \frac{w\,l^3}{16}$$

The area of the parabolic segment will be considered as an upwardly acting force P_1 passing through its centre of gravity which will be at distance $\frac{l}{8}$ from D.

$$\text{Then } P_1 = \text{area DTGH} = \tfrac{1}{3}\, \text{HD} \,.\, \text{DK}$$

$$= \frac{w\,l^2}{3 \times 8} \cdot \frac{l}{2} = \frac{w\,l^3}{48}$$

To get the imaginary reaction R_s at E, take moments about D.

$$\text{Then } P_2 \cdot \frac{l}{3} - P_1 \frac{l}{8} = R_s \,.\, l$$

$$\therefore R_s = \frac{P_2}{3} - \frac{P_1}{8}$$

$$= \frac{w\,l^3}{48} - \frac{w\,l^3}{8 \times 48} = \frac{7\,w\,l^3}{384} \qquad \text{..........(1)}$$

Suppose that the maximum deflection occurs at a point N at distance x from the centre.

Then the imaginary shearing force S at this point $= 0$

Shear at N $= R_s$ − area N Q E + area Q T G

$$= \frac{7\,w\,l^3}{384} - \frac{w\,l\left(\frac{l}{2} + x\right)}{8} \cdot \frac{\left(\frac{l}{2} + x\right)}{2} + \frac{w\,x^2}{2} \cdot \frac{x}{3}$$

$$= \frac{7\,w\,l^3}{384} - \frac{w\,l}{16}\left(\frac{l^2}{4} + l\,x + x^2\right) + \frac{w\,x^3}{6}$$

$$= \frac{7\,w\,l^3}{384} - \frac{w\,l^3}{64} - \frac{w\,l^2\,x}{16} - \frac{w\,l\,x^2}{16} + \frac{w\,x^3}{6}$$

$$= \frac{w\,l^3}{384} + \frac{w\,x^3}{6} - \frac{w\,l^2\,x}{16} - \frac{w\,l\,x^2}{16}$$

If this = 0, we have on dividing through by $\frac{w}{384}$ and rearranging the terms,

$$64\,x^3 - 24\,x^2 l - 24\,x l^2 + l^3 = 0$$

$$\text{or}\quad 64\left(\frac{x}{l}\right)^3 - 24\left(\frac{x}{l}\right)^2 - 24\left(\frac{x}{l}\right) + 1 = 0 \quad \ldots\ldots(2)$$

This is a cubic equation that cannot be solved by direct methods.

We must proceed by trial as follows—

If $x = 0$, left-hand side, which we will call $y = +1$

If $x = \cdot 1l$ $y = \cdot 064 - \cdot 24 - 2 \cdot 4 + 1 = -1 \cdot 576$

If $x = \cdot 05l$ $y = \cdot 008 - \cdot 06 - 1 \cdot 2 + 1 = -\cdot 252$

If $x = \cdot 04l$ $y = \cdot 0041 - \cdot 038 - \cdot 96 + 1 = +\cdot 006$

If the values of y are plotted against x it will be found that $y = 0$ for $x = \cdot 0406l$ approximately, and for all practical purposes we may take $x = \cdot 04l$.

Having determined the point of maximum deflection, we have next to calculate the value of the deflection δ at this point.

We first find the imaginary Bending Moment M_1 at the point N

$$M_1 = R_2\left(\frac{l}{2} + x\right) + \text{moment of section QTG}$$

$$- \text{Moment of } \Delta \text{ QNE}$$

$$= \frac{7\,w\,l^3}{384} \times \cdot 54\,l + \frac{w \times (\cdot 04\,l)^2}{2} \times \frac{\cdot 04\,l}{3} \times \frac{\cdot 04\,l}{4}$$

$$- \frac{w\,l}{8} \times \cdot 54\,l \times \frac{\cdot 54\,l}{3} \times \frac{\cdot 54\,l}{2}$$

$$= w\,l^4\,\{\cdot 00984 + \cdot 0000001 + \cdot 00328\}$$

$$= \cdot 00656\ w\,l^4$$

$$\therefore\ \delta = \frac{M_1}{E\,I} = \frac{\cdot 00656\ w\,l^4}{E\,I} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

As an interesting comparison, let us suppose that the whole load were spread right over the span.

Then $\delta = \frac{5\,W\,l^3}{384\,E\,I}$, according to the well-known formula.

In this case $W = \frac{wl}{2}$

$$\therefore \delta = \frac{5\,wl^4}{768\,EI} = \frac{\cdot 00651\,wl^4}{EI} \quad\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

We see therefore that we shall only make a slight error—which is practically negligible considering the necessary deviations from theoretical conditions which occur in practice—if we treat for deflection purposes the present case as being the same as for the same load spread over the whole span, remembering that the maximum deflection occurs at ·54 l from the unloaded end.

Graphical Construction for any Loading.—Let A C B be the B.M. curve for any given load system. Divide the base into a convenient number of equal parts and let e be the length of each base segment. The number is such that each piece of the B.M. diagram is approximately a rectangle. Now set down the mid ordinates of each section diminished in the ratio $\frac{1}{n}$ on a vector line. These ordinates are diminished in order to keep the vector diagram of a workable size.

Now let the space scale be 1″ = x feet, and let the B.M. scale be 1″ = y foot tons. Then considering any section of the B.M. diagram, say 2, 3, the area of this section is e × mid ordinate. Therefore, on given scales, one inch in height of mid ordinate, since the area of each segment is proportional to the height of the mid ordinate, represents $e \times x \times y$ square ft. tons. Since each portion of the vector line is $\frac{1}{n}$ of the ordinates, the portion 2, 3 of the vector line represents the area of its corresponding section of the B.M. diagram to a scale 1″ = $n \times e \times x \times y$ square ft. tons. Now calculate the length of E I on this scale. This will be too large for practical use, so take a pole P at distance $\frac{EI}{r}$, where r is some convenient whole number. With this pole P, draw the link polygon A′ C′ B′, then this is the elastic line of the beam for the given loading, or, more strictly speaking, A′ C′ B′, when reduced to a horizontal base, would give the elastic line. The scale to which the deflections are to be read is then obtained as follows—

If the polar distance were taken equal to E I, the deflections would be to the space scale 1″ = x feet, but as the polar distance is $\frac{\text{E I}}{r}$, the deflections will be to a scale 1″ = $\frac{x}{r}$ feet. The following numerical example should clear up the difficulty as to scale—

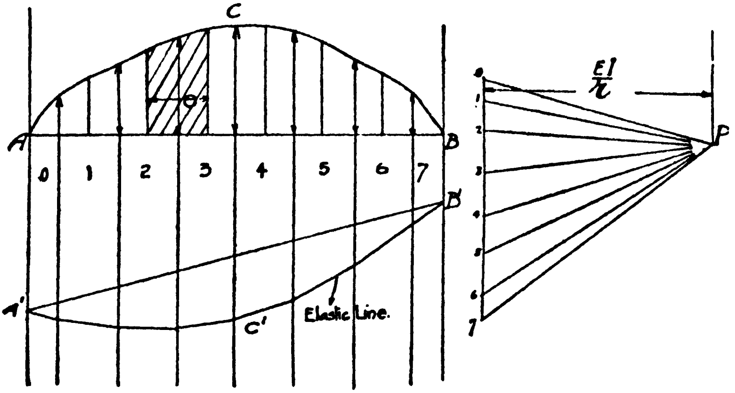


FIG. 127.—Graphical Construction for Deflections.

NUMERICAL EXAMPLE.—*A* 16″ × 6″ × 62 *lb. rolled steel joist of* 24 *ft. span carries a uniformly distributed load* (*including its own weight*) *of* 8 *tons, and also an isolated load of* 5 *tons, at a point* 6 *ft. from the left-hand support. Find the maximum deflection* (Fig. 128).

In this case E = 12,500 tons per sq. inch.

I = 725·7 inch units.

$$\therefore \text{E I} = \frac{12{,}500 \times 725{\cdot}7}{144} = 62{,}980 \text{ sq. ft. tons.}$$

First draw the B.M. diagrams for each of the loads, taking as linear scale, say 1″ = 4 ft., and for the B.M. scale, say 1″ = 20 ft. tons. Now divide the B.M. diagram into a convenient number of equal parts, say 12, and draw the mid ordinate of each part, treating these as force lines, then set these ordinates down a vector line, 0, 1, 2, etc. .. 12 to a reducèd scale, say one-fourth for convenience.

Then 1 in. down the vector line represents $\frac{4 \times 4 \times 20}{2} =$ 160 sq. ft. tons, because each base element is ½ in.

$$\therefore \text{E I on this scale} = \frac{62{,}980}{160} = 393{\cdot}7 \text{ ins.}$$

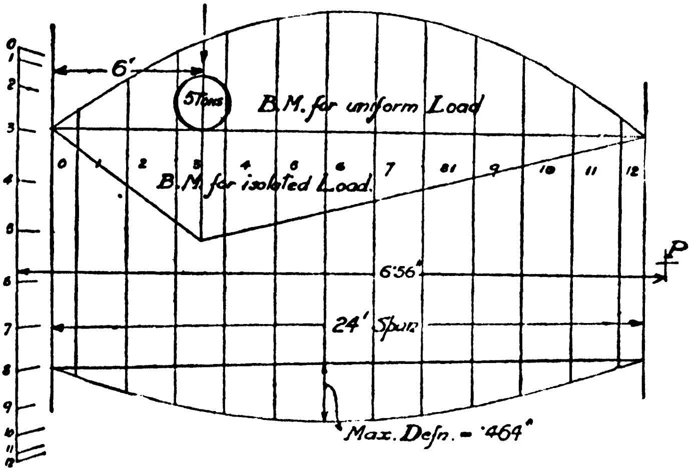


Fig. 128.—Example on Deflections.

This is obviously not convenient, so take $\frac{393{\cdot}7}{60} = 6{\cdot}56$ ins

Then 1 in. on the link polygon represents $\frac{48}{60}$ in. deflection.

The maximum ordinate of the link polygon will be found to be ·58 in.

$$\therefore \text{Maximum deflection} = {\cdot}58 \times {\cdot}8 = {\cdot}46 \text{ in.}$$

ALLOWANCE FOR DEVIATION OF CROSS SECTION.—The cases up to the present have all been on the assumption that the section is constant, or rather that the Moment of Inertia, I, is the same all along the span. If such is not the case, the deflection can be found accurately by first altering the B.M curve to make up for the variation in the section as follows—

Suppose A B C (Fig. 129) is the B.M. curve on any beam

A D B, and suppose that I_0 is the maximum moment of inertia or second moment of the section, this occurring at the point D. Then take any point along the beam at which B.M. is X Y and moment of inertia I_x and find $X Y_1$ so that $X Y_1 = \frac{X Y \times I_0}{I_x}$. Do this for a number of points along the span, and join up the points thus obtained, and we get the *corrected B.M. curve* from which the deflections can be found by the construction given above. The value I_0 is taken in obtaining the expression E I for this construction.

Deflections of Girders of Uniform Strength and Constant Depth.—If the cross section of a beam varies so that the maximum stresses are constant along the span,

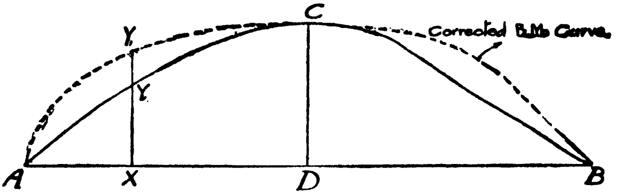


FIG. 129.

then the modulus of the section must vary in the same way as the B.M., and so the ratio $\frac{M}{Z}$ is constant. If the depth of the girder is also constant, then the ratio $\frac{M}{I}$ will also be constant.

The corrected B.M. diagram will in this case be a rectangle, and the deflection can be found by Mohr's theorem as follows—

As in the several previous cases we have

$$\delta = \frac{P \cdot y}{E I}$$

In this case P will be equal to $\frac{M \cdot L}{2}$ and $y = \frac{L}{4}$ since the curve is a rectangle.

$$\therefore \delta = \frac{M L^2}{8 E I}$$

In case of uniform loading $\mathrm{M} = \frac{\mathrm{W\,L}}{8}$

$$\therefore \delta = \frac{\mathrm{W\,L}^3}{64\,\mathrm{E\,I}}$$

In case of a central load $\mathrm{M} = \frac{\mathrm{W\,L}}{4}$

$$\therefore \delta = \frac{\mathrm{W\,L}^3}{32\,\mathrm{E\,I}}$$

Another simple proof of this relation will be found on p. 272.

Further numerical examples will be found at the conclusion of this chapter.

DEFLECTIONS FROM MATHEMATICAL STANDPOINT

From equation (3) $\frac{\mathrm{M}}{\mathrm{E\,I}} = \frac{1}{\mathrm{R}}$

Now when R is great, as it will be in this case, we have

$$\frac{1}{\mathrm{R}} = \frac{d^2 y}{d x^2}$$

$$\therefore \frac{d^2 y}{d x^2} = \frac{\mathrm{M}}{\mathrm{E\,I}}$$

$$\therefore \frac{d y}{d x} = \text{slope of beam} = \int \frac{\mathrm{M}\,d x}{\mathrm{E\,I}}$$

$$y = \text{deflection of beam} = \iint \frac{\mathrm{M}\,d x}{\mathrm{E\,I}}$$

Now consider the following standard cases (see Figs. 123, 124).

(1) **Simply Supported Beam with Central Load W.**—Consider a point Q at distance x from the centre of the beam.

$$\text{Then } \mathrm{M} = \frac{\mathrm{W}}{2}\left(\frac{l}{2} - x\right)$$

$$\therefore \iint \mathrm{M}\,d x = \iint \frac{\mathrm{W}}{2}\left(\frac{l}{2} - x\right)$$

$$\int \frac{\mathrm{W}}{2}\left(\frac{l}{2} - x\right) = \frac{\mathrm{W}\,l\,x}{4} - \frac{\mathrm{W}\,x^2}{4} + \mathrm{C}_1$$

$$\therefore \iint \frac{\mathrm{W}}{2}\left(\frac{l}{2} - x\right) = \int \frac{\mathrm{W}\,l\,x}{4} - \int \frac{\mathrm{W}\,x^2}{4} + \int \mathrm{C}_1$$

$$= \frac{\mathrm{W}\,l\,x^2}{8} - \frac{\mathrm{W}\,x^3}{12} + \mathrm{C}_1 x + \mathrm{C}_2$$

The slope is zero when $x = 0$ $\therefore c_1 = 0$, and the deflection is zero when $x = \pm \frac{l}{2}$

$$\therefore \frac{W l^3}{32} - \frac{W l^3}{96} + c_2 = 0$$

$$c_2 = \frac{- W l^3}{48}$$

Then maximum deflection occurs when $x = 0$

$$\text{Then } \delta = \frac{c_2}{E I} = - \frac{W l^3}{48 E I}\,^*$$

(2) **Simply Supported Beam with Uniform Load.**—Taking a point as before at distance x from the centre, we have

$$M = \frac{w l}{2}\left(\frac{l}{2} - x\right) - \frac{w}{2}\left(\frac{l}{2} - x\right)^2$$

$$= \frac{w}{2}\left(\frac{l}{2} - x\right)\left(l - \frac{l}{2} + x\right)$$

$$= \frac{w}{2}\left(\frac{l^2}{4} - x^2\right)$$

$$\int M\, d x = \frac{w l^2 x}{8} - \frac{w x^3}{6} + c_1$$

as before $c_1 = 0$

$$\therefore \iint M\, d x = \int \frac{w l^2 x}{8} - \int \frac{w x^3}{6}$$

$$= \frac{w l^2 x^2}{16} - \frac{w x^4}{24} + c_2$$

$$= 0 \text{ when } x = \frac{l}{2}$$

$$\therefore - c_2 = \frac{w l^4}{64} - \frac{w l^4}{384}$$

$$= w l^4 \left\{\frac{6 - 1}{384}\right\} = \frac{5 w l^4}{384}$$

Then the maximum deflection occurs when $x = 0$

$$\therefore \delta = \frac{c_2}{E I} = - \frac{5 w l^4}{384 E I} = - \frac{5 W l^3}{384 E I}$$

---

[*] The minus sign indicates only that the deflection is downward, and need not be employed in calculations.

(3) Cantilever with an Isolated Load not at Free End.—Take a point Q at distance x from load.

$$M = -Wx$$

$$\therefore \text{Slope} \times EI = \int M\,dx$$

$$= -\frac{Wx^2}{2} + C_1$$

When $x = l$, slope $= 0$

$$\therefore C_1 = \frac{Wl^2}{2}$$

$$\therefore EI \times \text{slope under load} = \frac{Wl^2}{2}.$$

$$\text{deflection} = \int\int \frac{M\,dx}{EI}$$

$$= \left(-\frac{Wx^3}{6} + \frac{Wl^2x}{2} + C_2\right) \times \frac{1}{EI}$$

When $x = l$, deflection $= 0$

$$\therefore C_2 = \frac{Wl^3}{6} - \frac{Wl^3}{2} = \frac{-Wl^3}{3}$$

$\therefore$ Deflection under load, where $x = 0$

$$= \frac{C_2}{EI} = \left(\frac{-Wl^3}{3}\right) \times \frac{1}{EI}$$

Deflection at free end

$$= \text{deflection under load} + \text{slope under load } (L - l)$$

$$= \left\{\frac{-Wl^3}{3} - \frac{Wl^2}{2}(L - l)\right\} \frac{1}{EI}$$

$$= -\frac{W}{EI}\left\{\frac{l^2L}{2} - \frac{l^3}{6}\right\}$$

$$= -\frac{Wl^2}{2EI}\left(L - \frac{l}{3}\right)$$

or neglecting the minus sign, which indicates only that the deflection is downward, we get

$$\text{Maximum deflection} = \delta = \frac{Wl^2}{2EI}\left(L - \frac{l}{3}\right)$$

(4) Cantilever with Isolated Load at Free End.—This is obtained by putting $l = L$ in the above case,

$$i.e.\ \delta = \frac{WL^3}{3EI}$$

(5) Cantilever with Uniform Load from Fixed End to a point before Free End.

$$\text{In this case } M = -\frac{w x^2}{2}$$

$$\therefore \text{Slope} \times EI = \int M \, dx$$

$$= -\frac{w x^3}{6} + c_1$$

$$\text{When } x = l, \text{ slope} = 0$$

$$\therefore c_1 = \frac{w l^3}{6}$$

$$\therefore EI \times \text{slope under load} = \frac{w l^3}{6}$$

$$EI \times \text{deflection} = \int\int M \, dx$$

$$= -\frac{w x^4}{24} + \frac{w l^3 x}{6} + c_2$$

$$\text{When } x = l, \text{ deflection} = 0$$

$$\therefore c_2 = \frac{w l^4}{24} - \frac{w l^4}{6} = \frac{-w l^4}{8}$$

$$\therefore EI \times \text{deflection under load, when } x = 0$$

$$= c_2 = \frac{-w l^4}{8}$$

$$\therefore EI \times \text{deflection at free end}$$

$$= \frac{-w l^4}{8} + \text{slope under load} \times EI \times (L - l)$$

$$= \frac{-w l^4}{8} + (L - l) \frac{(-w l^3)}{6}$$

$$= \frac{-w l^3}{2} \left\{ \frac{l}{4} + \frac{L}{3} - \frac{l}{3} \right\}$$

$$= \frac{-w l^2}{2} \left\{ \frac{L}{3} - \frac{l}{12} \right\}$$

$$= \frac{-w l^3}{6} \left(L - \frac{l}{4} \right)$$

$$= \frac{-W l^2}{6} \left(L - \frac{l}{4} \right)$$

$\therefore$ Neglecting $-$ sign we have

$$\delta = \frac{W l^2}{6 EI} \left(L - \frac{l}{4} \right)$$

(6) Cantilever with Uniform Load over Whole Length.—This is the same as the previous case when l = L.

$$\therefore \delta = \frac{W L^2}{6 E I}\left(L - \frac{L}{4}\right)$$

$$= \frac{W L^3}{8 E I}$$

(7) Simply Supported Beam with Isolated Load anywhere.—Let a load W be placed at a point C on a beam A B, Fig. 130, of span l, and let it be at distance $a\,l$ from the end A, the distance from the end B being $(1 - a)\,l$.

$$\text{Then } R_B = \frac{W a l}{l} = W a$$

$$R_A = \frac{W (1 - a) l}{l} = W (1 - a)$$

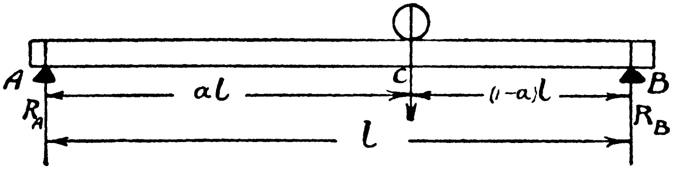


Fig. 130.

Consider a point at distance x from A between A and C.

$$\text{Then } M_x = R_A\, x = W (1 - a)\, x$$

$$\therefore E I \frac{d^2 y}{d x^2} = W (1 - a)\, x \quad \ldots\ldots(1)$$

$$\therefore E I \frac{d y}{d x} = \frac{W (1 - a)\, x^2}{2} + C_1 \quad \ldots\ldots(2)$$

$$E I y = \frac{W (1 - a)\, x^3}{6} + C_1 x + C_2 \quad \ldots\ldots(3)$$

Now consider a point at distance x_1 from A between C and B.

$$\text{Then } M_{x1} = R_A\, x_1 - W (x_1 - a\,l)$$

$$= W (1 - a)\, x_1 - W x_1 + W a l$$

$$= W a l - W a x_1$$

$$\therefore E I \frac{d^2 y}{d x_1^2} = W a l - W a x_1 \quad \ldots\ldots(4)$$

$$\therefore \mathrm{E\,I}\frac{d\,y}{d\,x_1} = \mathrm{W}\,a\,l\,x_1 - \mathrm{W}\,a\frac{x_1^2}{2} + c_3 \dots\dots\dots\dots(5)$$

$$\mathrm{E\,I}\,y = \frac{\mathrm{W}\,a\,l\,x_1^2}{2} - \frac{\mathrm{W}\,a\,x_1^3}{6} + c_3\,x_1 + c_4 \dots(6)$$

In equation (3) when $x = 0, y = 0$

$$\therefore c_2 = 0$$

In equation (6) when $x = l, y = 0$

$$\therefore \frac{\mathrm{W}\,a\,l^3}{2} - \frac{\mathrm{W}\,a\,l^3}{6} + c_3\,l + c_4 = 0.$$

$$\therefore c_4 = \frac{-\mathrm{W}\,a\,l^3}{3} - c_3\,l \dots\dots\dots\dots\dots(7)$$

These two equations representing the elastic line on either side of the load have the same slope and the same ordinate when

$$x = x_1 = a\,l$$

$\therefore$ putting in these values in equations (2) and (5) and equating we have

$$\frac{\mathrm{W}\,(1 - a)\,a^2\,l^2}{2} + c_1 = \mathrm{W}\,a^2\,l^2 - \frac{\mathrm{W}\,a^3\,l^2}{2} + c_3$$

$$c_1 = \frac{\mathrm{W}\,a^3\,l^2}{2} + c_3 \dots\dots\dots\dots(8)$$

Putting in value $x = x_1 = a\,l$ in equations (3) and (6) and equating we have

$$\frac{\mathrm{W}\,(1 - a)\,a^3\,l^3}{6} + c_1\,a\,l = \frac{\mathrm{W}\,a^3\,l^3}{2} - \frac{\mathrm{W}\,a^4\,l^3}{6} + c_3\,a\,l + c_4$$

$$c_1\,a\,l = \frac{\mathrm{W}\,a^3\,l^3}{3} + c_3\,a\,l - \frac{\mathrm{W}\,a\,l^3}{3} - c_3\,l$$

$$c_1\,a = \frac{\mathrm{W}\,a^3\,l^2}{3} - \frac{\mathrm{W}\,a\,l^2}{3} + c_3\,(a - 1)$$

$$= \frac{\mathrm{W}\,a^3\,l^2}{3} - \frac{\mathrm{W}\,a\,l^2}{3} + \left(c_1 - \frac{\mathrm{W}\,a^2\,l^2}{2}\right)(a - 1)$$

$$\therefore c_1 = \frac{\mathrm{W}\,a^3\,l^2}{3} - \frac{\mathrm{W}\,a\,l^2}{3} - \frac{\mathrm{W}\,a^3\,l^2}{2} + \frac{\mathrm{W}\,a^2\,l^2}{2}$$

$$= -\mathrm{W}\,a\,l^2\left(\frac{1}{3} + \frac{a^2}{6} - \frac{a}{2}\right)$$

$$= \frac{-\mathrm{W}\,a\,l^2}{6}(2 - 3\,a + a^2)$$

$$= \frac{-\mathrm{W}\,a\,l^2}{6}(1 - a)(2 - a) \dots\dots\dots(9)$$

$\therefore$ equation (3) becomes

$$\mathrm{E\,I}\,y = \frac{W(1-a)x^3}{6} - \frac{W a l^2 x}{6}(1-a)(2-a) \quad \ldots\ldots(10)$$

Assume $a > \frac{1}{2}$, then the maximum deflection occurs between A and C. y is a maximum when $\frac{dy}{dx} = 0$

$$i.e. \text{ when } \frac{W(1-a)\,3x^2}{6} - \frac{W a l^2}{6}(1-a)(2-a) = 0$$

$$i.e. \text{ when } x^2 = \frac{l^2 a(2-a)}{3}$$

$$\text{i. e. } x = l\sqrt{\frac{a(2-a)}{3}} \quad \ldots\ldots\ldots\ldots(11)$$

Putting this value in equation (10) we have

$$\mathrm{E\,I}\,\delta = \frac{W l^3(1-a)}{6}\cdot\left\{\frac{a(2-a}{3}\right\}^{\frac{3}{2}} - \frac{W a l^2 . l}{6}\left\{\frac{a(2-a)}{3}\right\}^{\frac{1}{2}} .(1-a)(2-a)$$

$$= W l^3(1-a)\left\{\frac{a(2-a)}{3}\right\}^{\frac{3}{2}}\left(\frac{1}{6}-\frac{3}{6}\right)$$

$$= -\frac{W l^3}{3}(1-a)\left\{\frac{a(2-a)}{3}\right\}^{\frac{3}{2}}$$

$$\text{or } \delta = -\frac{W l^3(1-a)}{3\,\mathrm{E\,I}}\left\{\frac{2a-a^2}{3}\right\}^{\frac{3}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

The deflection under the load is obtained by putting $x = a\,l$ in (10).

$$\text{Then } \mathrm{E\,I}\,y = \frac{W(1-a)a^3 l^3}{6} - \frac{W a^2 l^3}{6}(1-a)(2-a)$$

$$= \frac{W a^2 l^3(1-a)}{6}[a-2-a]$$

$$\therefore y = -\frac{W a^2 l^3(1-a)}{3\,\mathrm{E\,I}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

Deflection of Girder of Uniform Strength with Parallel Flanges.—If the section of a girder varies along its length so that the stress is constant, then $\frac{M}{Z}$ is constant, so that if the depth is also constant $\frac{M}{I}$ is also constant. Assuming also that E is also constant we have

$$\frac{1}{R} = \frac{M}{E\,I} = \text{constant.}$$

∴ Wherever $\frac{EI}{M}$ is constant, the beam bends to an arc of a circle.

Let Fig. 131 represent a beam bent to an arc of a circle. (N.B.—The beam is shown vertical instead of horizontal for convenience of figure.)

Then, if O is the centre of the circle and C D the deflection of the beam, we have from the property of the circle

$$CD(CO + OD) = AC^2$$

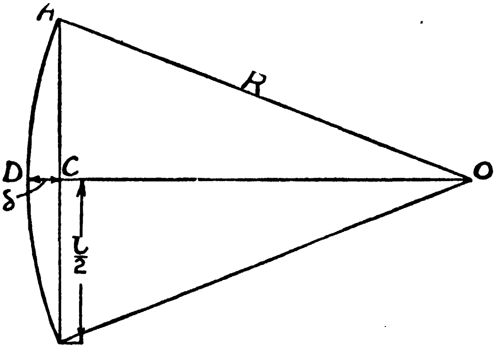


FIG. 131.

As C D is very small we may write

$$\delta . 2R = \left(\frac{l}{2}\right)^2 = \frac{l^2}{4}$$

$$\therefore \delta = \frac{l^2}{8R}$$

$$\text{Now } \frac{1}{R} = \frac{M}{EI}$$

$$\therefore \delta = \frac{Ml^2}{8EI}$$

This result agrees with that obtained on p. 264 by reasoning from Mohr's Theorem.

* **Resilience of Bending.**—The work done in bending a beam to a given stress may be obtained as follows—

The work done by a couple in moving through an angle is equal to the product of the moment of the couple into the angle turned through. Therefore, if a short portion of a beam subjected to a bending moment M is bent to a slope δi, the work done in bending is $\frac{M\,\delta i}{2}$, because M gradually increases from 0 to M.

$\therefore$ Total work done in bending over whole beam

$$= P = \int \frac{M\,d\,i}{2}$$

$$\text{Now } \frac{d\,i}{d\,x} = \frac{1}{R},\ \textit{i.e.}\ \frac{1}{R} = \text{rate of change of slope}$$

$$\therefore P = \int \frac{M}{2\,R}\,d\,x$$

$$\text{but } \frac{1}{R} = \frac{M}{E\,I}$$

$$\therefore P = \int \frac{M^2}{2\,E\,I}\,d\,x$$

If the B.M. is constant, and the section is rectangular, then

$$P = \frac{M^2}{2\,E\,I}\int_0^L d\,x = \frac{M^2\,L}{2\,E\,I}$$

$$\text{But } M^2 = \frac{f^2\,I^2}{d^2}$$

$$\therefore P = \frac{f^2\,I\,.\,L}{2\,.\,E\,.\,d^2}$$

$$\text{Now } I = \frac{b\,h^3}{12},\ d = \frac{h}{2}$$

$$\therefore P = \frac{f^2}{E}\cdot\frac{4 \times b\,h^3 \times L}{2 \times 12 \times h^2}$$

$$= \frac{f^2}{6\,E}\cdot V$$

Where V is the volume of the beam.

$$\therefore \text{Resilience} = \frac{P}{V} = \frac{f^2}{6\,E}$$

If the load is central and the section is rectangular.—Consider one-half of the beam, then x is the distance from the abutment.

$$M = \frac{W x}{2}$$

$$\therefore \frac{P}{2} = \int_0^{\frac{L}{2}} \frac{M^2 \, d x}{2 E I}$$

$$= \int_0^{\frac{L}{2}} \frac{W^2 x^2 \, d x}{8 E I}$$

$$= \frac{W^2 L^3}{192 E I}$$

$$P = \frac{W^2 L^3}{96 E I} = \frac{1}{2} W . \delta$$

$$\text{Now M at centre} = \frac{W L}{4} = \frac{f I}{d}$$

$$\therefore P = \frac{f^2 I^2 . L}{6 . E I . d^2}$$

$$= \frac{f^2}{6 E} . \frac{I L}{d^2}$$

$$\text{As before } I = \frac{b h^3}{12}, d = \frac{h}{2}$$

$$\therefore P = \frac{f^2}{6 E} . \frac{4 b h^3 . L}{12 h^2}$$

$$= \frac{f^2}{18 E} . V$$

$$\therefore \text{Resilience} = \frac{P}{V} = \frac{f^2}{18 E}$$

NUMERICAL EXAMPLES ON DEFLECTIONS, ETC.

(1) *A girder has a span of* 120 *feet, and has to support a uniformly distributed load of* $1\frac{1}{4}$ *tons per foot run. What depth must the girder have in the centre if the maximum deflection is not to exceed* $\frac{1}{1200}$ *of the span? The maximum stress in the flanges is not to exceed* $6\frac{1}{2}$ *tons per sq. in. and* E *is* 12,000 *tons per sq. in.* (*B.Sc. Lond.*)

This question is not quite clear, because if the depth is not the same throughout, we cannot calculate the deflection until we know the way it varies.

We will assume the depth constant—

$$\text{Now at centre } M = \frac{W\,l}{8} = \frac{1\tfrac{1}{4} \times 120 \times 120}{8} \text{ ft. tons}$$

$$= 27{,}000 \text{ in. tons.}$$

$$\therefore \text{ If maximum stress} = 6\tfrac{1}{2} \text{ tons per sq. in. since } f = \frac{M}{Z}$$

$$Z = \frac{27{,}000}{6\tfrac{1}{2}} \text{ in. units.}$$

$$\text{Now } \delta = \frac{5\,W\,L^3}{384\,E\,I}$$

$$\delta = \frac{L}{1{,}200} = \frac{1}{10} \text{ ft.}$$

$$\therefore \frac{12}{10} = \frac{5 \times 150 \times 120 \times 120 \times 120}{384 \times 12{,}000\,I} \times 1{,}728$$

$$\therefore I = 405{,}000 \text{ in. units.}$$

$$\text{Now } \frac{I}{Z} = \frac{D}{2} \text{ where } D = \text{depth.}$$

$$\therefore D = \frac{2\,I}{Z} = \frac{2 \times 405{,}000 \times 6{\cdot}5}{27{,}000} \text{ ins.}$$

$$= \frac{30 \times 6{\cdot}5}{12} = 16{\cdot}25 \text{ ft.}$$

This is a greater depth than would be usually adopted in practice for a solid web girder.

(2) *A cast-iron water pipe,* 10 *inches external diameter and* $\frac{1}{2}$ *inch thick, rests on supports* 40 *feet apart. Calculate the maximum stress in the outer fibre of the material when empty and when full of water, also the corresponding deflections.* (*A.M.I.C.E.*)

$$\text{In this case} \quad I = \frac{\pi\,(D^4 - d^4)}{64} = \frac{\pi\,(10^4 - 9^4)}{64}$$

$$= 168{\cdot}8 \text{ in. units.}$$

$$\therefore Z = \frac{I}{5} = \frac{168{\cdot}8}{5} = 33{\cdot}76 \text{ in. units.}$$

$$\text{Volume of pipe} = \frac{\pi}{4}\left(100 - 81\right) \times \frac{40}{144} = 4{\cdot}14 \text{ cub. feet.}$$

$$\text{Volume of water} = \frac{\pi}{4} \cdot \frac{81}{144} \times 40 = 17{\cdot}67 \text{ cub. feet.}$$

$$\therefore \text{Weight of pipe} = w = \frac{4{\cdot}14 \times 450}{2{,}240} = {\cdot}832 \text{ ton.}$$

$$\text{Weight of water} = w_1 = \frac{17{\cdot}67 \times 62{\cdot}5}{2{,}240} = {\cdot}492 \text{ ton.}$$

$$\therefore W = w + w_1 = 1{\cdot}324 \text{ tons (about).}$$

∴ Max. stress when empty

$$= \frac{M}{Z} = \frac{{\cdot}832 \times 40 \times 12}{8 \times 33{\cdot}76} = 1{\cdot}48 \text{ tons per sq. in.}$$

$$\text{Max. stress when full} = \frac{1{\cdot}48 \times 1{\cdot}324}{{\cdot}832} = 2{\cdot}35 \text{ tons per sq. in.}$$

Taking E as 8000 tons per sq. in.

$$\delta \text{ when empty} = \frac{5\,W\,L^3}{384\,E\,I}$$

$$= \frac{5 \times {\cdot}832 \times 40 \times 40 \times 40 \times 12 \times 12 \times 12}{384 \times 168{\cdot}8 \times 8000}$$

$$= {\cdot}89 \text{ inch.}$$

$$\delta \text{ when full} = \frac{{\cdot}89 \times 1{\cdot}324}{{\cdot}832} = 1{\cdot}41 \text{ inches.}$$

(3) *A pole made of mild steel tube,* 6 *inches diameter and* ½ *inch thick, is firmly fixed in the ground, the top being* 10 *feet above the ground level. A horizontal pull of* 2000 *lbs. is applied at a point* 6 *feet from the ground. Find the deflection at the top.* E = 13,500 *tons per square inch.* (*B.Sc. Lond.*)

In this case $I = \frac{\pi(D^4 - d^4)}{64} = \frac{\pi(6^4 - 5^4)}{64}$

$$= 32{\cdot}9 \text{ in. units.}$$

This is the same as Case (2).

$$\therefore \delta = \frac{W l^2}{2\,E\,I}\left(L - \frac{l}{3}\right)$$

In this case $l = 6$ ft. $L = 10$ ft.

$$W = 2000 \text{ lbs.} = \frac{2000}{2{,}240} \text{ ton.}$$

$$\therefore \delta = \frac{2000}{2{,}240} \times \frac{6 \times 6 \times 12 \times 12 \times 8 \times 12}{13{,}500 \times 32{\cdot}9 \times 2}$$

$$= {\cdot}5 \text{ inch, nearly.}$$

(4) *What is the least internal radius to which a bar of steel*

4 *inches wide by* $\frac{3}{8}$ *inch thick can be bent so that the maximum stress will not exceed* 5 *tons per square inch?* E = 13,000 *tons per square inch.* (*A.M.I.C.E.*)

The general formula for bending is

$$\frac{f}{d} = \frac{M}{I} = \frac{E}{R}$$

$$\therefore \frac{f}{d} = \frac{E}{R}$$

$$\text{or } R = \frac{d\,E}{f}$$

In this case d = distance from N.A. to extreme fibre,

$$= \frac{3}{16} \text{ in.}$$

$$\therefore R = \frac{3}{16} \times \frac{13{,}000}{5}$$

$$= 488 \text{ inches.}$$

$$= 40{\cdot}7 \text{ feet.}$$

It should be noted that the width of the bar is not necessary in this problem.

The result is the radius of the centre line.

(5) *A cast-iron beam has a rectangular cross section, the thickness being* 1 *inch and the depth of the section* 2 *inches. It is found that a load of* 10 *cwt. placed in the centre of a* 36-*inch span deflects this beam by* ·11 *inch. Through what height would a weight of* $\frac{1}{2}$ *cwt. have to fall on to the centre of the same span to produce a deflection of* ·30 *inch?* (*B.Sc. Lond.*)

It takes 10 cwt. to produce a deflection of ·11 inch.

∴ It would take $\frac{10 \times {\cdot}30}{{\cdot}11}$ to produce a deflection of ·30 inch.

Now the work done in deflecting a bar when loaded in the centre = $\frac{1}{2}$ W δ,

∴ Work done to produce ·30 inch deflection

$$= \frac{1}{2} \cdot \frac{10 \times {\cdot}30}{{\cdot}11} \times {\cdot}30 \text{ in. cwt.}$$

$$= {\cdot}341 \text{ ft. cwt.}$$

If h is the height from which the ½ cwt. falls, work done by it $= \frac{1}{2}\left(h + \frac{\cdot 30}{12}\right)$ ft. cwt., because we shall take h as the height above the unstrained position of the beam.

These two amounts of work must be the same,

$$\therefore \text{ We have } \quad \frac{1}{2}\left(h + \frac{\cdot 30}{12}\right) = \cdot 341$$

$$h = \cdot 682 - \frac{\cdot 30}{12} \text{ foot.}$$

$$= 8{\cdot}18 - \cdot 30 \text{ inches.}$$

$$= 7{\cdot}88 \text{ inches.}$$

CHAPTER X

COLUMNS, STANCHIONS AND STRUTS

The question of strength of columns of compression members is of very great importance, and has formed a field of discussion and investigation for many years. Interest in the subject has recently been aroused by the regrettable failure of the Quebec Bridge, and within the next few years many investigators will probably direct their energy towards giving us further information in this direction. Although the subject certainly presents difficulties, much of the confusion which is in the minds of many designers is undoubtedly due to insufficient grasp of the meaning of the various formulæ in use. We will endeavour to make this subject quite clear by approaching it in the following manner, which was suggested by the author in 1908.

In the design of a tie-bar we use a constant working stress, that is to say, the stress does not depend on the shape or the length of the tie; but in struts or compression members the working stress depends on the shape and the length and the manner in which the ends are fixed. The quantity which determines the working stress, and thus the strength of a pin-jointed strut, column, or stanchion is equal to

$$\frac{\text{Length of column}}{\textit{Least}\text{ radius of gyration about centroid}} = c$$

This quantity we will call the **Buckling Factor** of the strut.

For struts with ends fixed in other ways the buckling factor is obtained by dividing the *equivalent* length of the strut by the least radius of gyration. We will show later how the equivalent length is obtained.

Slenderness ratio.—Some writers use this term in place of buckling factor. The slenderness ratio

$$= \frac{\text{Length}}{\text{Least radius of gyration about centroid}}$$

but does not take into account the method of fixing the ends, and so is not the same as the buckling factor. Two struts of the same material and having the same buckling factor may carry the same stress, no matter how their ends are fixed: this is not the case for two columns with the same slenderness ratio.

The reason why a variable working stress has to be used is that struts fail by buckling and not by crushing, unless their length is extremely small. If for some reason the centre line of a strut is not quite straight or the load comes out of centre, there are bending stresses caused in the material, and the distortion due to these bending stresses tends to increase the eccentricity, and failure may ultimately occur due to this reason.

Strut Formulæ.—A large number of formulæ, some theoretical and some empirical, have been proposed for obtaining the working stress in compression in terms of the buckling factor of the strut and of the crushing strength of the material. Before these formulæ can logically be compared we must be careful to see that they are for the same crushing strength, and for the same manner of fixing the ends of the strut. We will consider the following :—

(*a*) **Euler's Formula.**—This formula is intended for long struts in which the direct stress is negligible compared with the buckling stress. It is usually given in the following form—

$$P = \frac{\pi^2 E I}{L^2}$$

where P = the breaking load (not the working load).
E = Young's modulus.
I = *least* moment of inertia.
L = length of pin-jointed strut.

We will now put it into more convenient use for practice as follows—

$$\therefore \frac{P}{A} = \text{breaking stress} = \frac{\pi^2 E A k^2}{A L^2}$$

$$= \frac{\pi^2 E}{\left(\frac{L}{k}\right)^2} = \frac{\pi^2 E}{c^2}$$

Adopting a factor of safety of 5, we get

$$\text{Working stress} = f_p = \frac{\text{breaking stress}}{5} = \frac{\pi^2 E}{5 c^2}$$

For mild steel, $E = 13{,}000$ tons per sq. in.

$$\therefore f^p = \frac{\pi^2 E}{5 c^2} = \frac{25{,}600}{c^2} \text{ tons per sq. in.}$$

For wrought iron, $E = 12{,}500$

$$\therefore f_p = \frac{24{,}600}{c^2} \quad \text{,,} \quad \text{,,}$$

$$\text{Similarily for cast iron } f_p = \frac{12{,}000}{c^2} \quad \text{,,} \quad \text{,,}$$

$$\text{For timber } f_p = \frac{1{,}600}{c^2} \quad \text{,,} \quad \text{,,}$$

Proof of Euler's Formula.—The proof of Euler's formula is found by many students to be somewhat difficult to follow, as it involves the solution of a differential equation. Suppose that a column in some way or other becomes deflected as shown in Fig. 132 (1). Then there are bending stresses induced in it, and the strut will exert a force P on the supports tending to straighten itself. Now, if the load on the strut is less than P, the strut will straighten, and so is safe; but if the load is greater than P, the strut will continue to deflect, and will ultimately break. When the load is equal to P, the strut is in unstable equilibrium, and so P is called the *critical, or buckling, or crippling load.*

Consider a point A on the strut.

The B.M. at A $= M_A = - P\,x$.*

Now, if R is the radius of curvature,

$$\frac{1}{R} = \frac{d^2 x}{d y^2} = \frac{M}{EI} = \frac{-Px}{EI}$$

$$\therefore \frac{d^2 x}{d y^2} = \frac{-P}{EI} . x = \text{say } - m^2 . x \; \ldots\ldots\ldots\ldots (1)$$

assuming that I is constant, or that the strut is of uniform section.

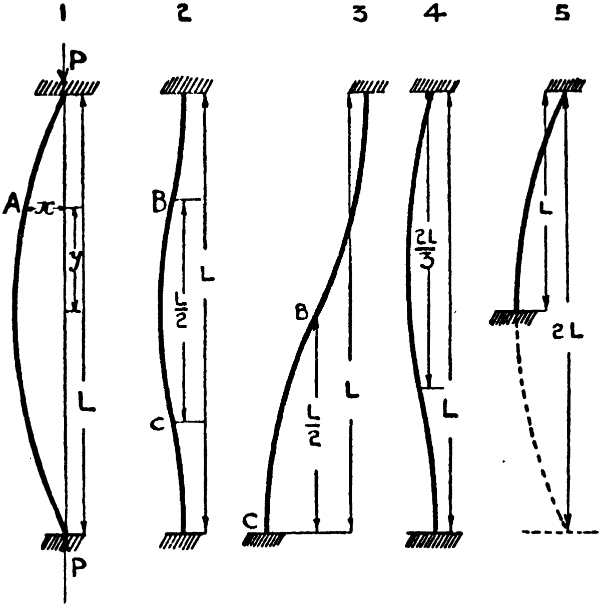


Fig. 132.—Methods of Fixing Ends of Columns.

The general solution of this differential equation is

$$x = A \cos m y + B \sin m y \; \ldots\ldots\ldots\ldots (2)$$

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[*] The negative sign occurs because we take anti-clockwise moments to the right as positive, and x in the figure is, on the usual convention, negative. If the figure be turned round so that the column is horizontal and has a downward deflection it will be seen that according to the rule given on p. 121, M_A is positive and x is negative. Px therefore is negative, and to make the moment positive we must write $M_A = - Px$. If we had drawn the column buckled in the other direction, M_A would have been negative and x positive, so that we still would have $M_A = - Px$.

where A and B are constants, which are obtained as follows—

When $y = \frac{-L}{2}$ and $\frac{+L}{2}$, $x = 0$

$$\therefore\ 0 = A\cos\frac{mL}{2} + B\sin\frac{mL}{2} \quad \dots\dots\dots\dots (3)$$

$$0 = A\cos\frac{-mL}{2} + B\sin\frac{-mL}{2} \quad \dots\dots\dots (4)$$

$$= A\cos\frac{mL}{2} - B\sin\frac{mL}{2}$$

$$\therefore\ B \text{ must} = 0$$

$$\therefore\ x = A\cos my \quad \dots\dots\dots\dots\dots (5)$$

When $y = 0$, x is finite, $\therefore$ A is not zero

$$\therefore \text{ if } A\cos\frac{mL}{2} = 0$$

$$\cos\frac{mL}{2} \text{ must} = 0$$

The general solution for this condition is that

$$\frac{mL}{2} = \frac{n\pi}{2}$$

$$\therefore\ m^2 = \frac{n^2\pi^2}{L^2}$$

$$\therefore\ \frac{P}{EI} = \frac{n^2\pi^2}{L^2}$$

$$P = \frac{n^2\pi^2 EI}{L^2} \quad \dots\dots\dots\dots\dots\dots (6)$$

The lowest value of P is given by $n = 1$, and as this is the most important for us, we write the result as

$$P = \frac{\pi^2 EI}{L^2} \quad \dots\dots\dots\dots\dots\dots\dots (7)$$

It should be noted that P is independent of the quantity x, so that the force necessary to keep the strut deflected at large radius of curvature is the same as that to keep it at a small radius, and so if the load is the least amount greater than P the strut will go on deflecting, and so break.

Use of Euler's Formula.—It must be remembered that in this formula we have not taken into account the direct compression stress on the strut. If the safe stress given by Euler's formula is greater than the safe compressive stress for very short lengths of the material, then obviously we should not use Euler's result. Thus, if Euler for mild steel gives f_p greater than 6 tons per sq. in. we should use 6 tons per sq. in.

Another way of using it is as follows—

$$\text{Safe load} = \frac{\pi^2 \text{E I}}{5 l^2}$$

$$= \text{P} = \frac{2 \text{E I}}{l^2}$$

$$\therefore \text{I required} = \frac{\text{P} l^2}{2 \text{E}}$$

$$\text{A required} = \frac{\text{P}}{f_c}$$

We have thus the least area and moment of inertia that the section must have and can so choose a suitable section from tables.

Method of Fixing Ends—Equivalent Length of Strut.—In the above working we have considered the ends as pin-jointed. If the ends are fixed in any other way we must take as the length of the strut the length of the equivalent pin-jointed strut; this we will call the *equivalent length of the strut.*

Now consider the following methods of fixing the ends (see Fig. 132).

(1) PIN JOINTS AT EACH END.—This is the standard case.

(2) BOTH ENDS FIXED IN POSITION AND DIRECTION.—In this case the buckled form is as shown in the figure, and B C is the equivalent length, *i. e.* a pin-jointed strut of length B C is as strong as the fixed strut.

$$\therefore \text{in this case equivalent length of strut} = \frac{\text{L}}{2}$$

$$\text{Buckling factor} = c = \frac{\text{L}}{2 k}$$

(3) BOTH ENDS FIXED IN DIRECTION ONLY.—The buckled form in this case is as shown in the figure. On comparing

with Case 1, it will be seen that the portion BC is equivalent to one-half the strut in Case 1, and so in this case, since

$$BC = \frac{L}{2}$$

$$\text{equivalent length of strut} = L$$

$$\therefore \text{Buckling factor} = c = \frac{L}{k}$$

(4) ONE END FIXED IN DIRECTION AND POSITION, OTHER END PIN-JOINTED.—It will be clear from the figure that in this case

$$\text{equivalent length of strut} = \frac{2L}{3}$$

$$\therefore \text{Buckling factor} = c = \frac{2L}{3k}$$

(5) ONE END FIXED IN DIRECTION AND POSITION, OTHER END FREE.—In this case

$$\text{equivalent length of strut} = 2L$$

$$\therefore \text{Buckling factor} = c = \frac{2L}{k}$$

SUMMARY OF VALUES OF BUCKLING FACTORS

	Case 1.	Case 2.	Case 3.	Case 4.	Case 5.
Buckling factor $= c$	$\frac{L}{k}$	$\frac{L}{2k}$	$\frac{L}{k}$	$\frac{2L}{3k}$	$\frac{2L}{k}$

These values should be used in Euler's and the other formulæ involving the buckling factor.

(*b*) **Rankine's Formula.**—This formula is sometimes called the Gordon-Rankine formula, and is of the form

$$f_p = \frac{f_c}{1 + a\left(\frac{L}{k}\right)^2}$$

$$= \frac{f_c}{1 + a \cdot c^2}$$

Where

f_c = safe compressive stress for very short lengths of the material

a = a constant depending on the material

c = buckling factor of the strut

f_p = working stress per sq. in. for the strut.

The following values of a may be taken according to different authorities—

Mild steel $a = \frac{1}{9000}$ to $\frac{1}{6000}$, $f_c = 6$ tons per sq. in.

Wrought iron $a = \frac{1}{9000}$ to $\frac{1}{8000}$, $f_c = 4$ „ „ „

Cast iron $a = \frac{1}{2500}$ to $\frac{1}{1800}$, $f_c = 7$ „ „ „

Timber $a = \frac{1}{2000}$, $f_c = {\cdot}5$ „ „ „

In each case we prefer to use the higher value of the constant a

There is a very large amount of variation in the values of the constants as given by various authorities, and in comparing the above with those given by others, the reader should be careful to compare the *safe stresses* given with the above figures with the safe stresses given by others, because the value of f_c also varies in the various forms of the formula and thus, although the constants may be different, the resulting safe stress may be nearly the same. Care must also be taken to see whether pin-jointed or fixed ends are taken as the standard case.

CONSTRUCTION OF RANKINE'S FORMULA.—Rankine's formula may be looked upon as a corrected form of Euler's.

If c is very small, *i. e.* if the strut is very short, the term $a\,c^2$ is negligible, and so we get $f_p = f_c$

This is, of course, the result which we ought to obtain.

If c is great, *i. e.* if the strut is very long, the term $a\,c^2$ will be so great that 1 may be neglected in comparison with it, and so we get

$$f_p = \frac{f_c}{a\,c^2}.$$

This will give the same result as Euler if

$$\frac{f_c}{a} = \frac{\pi^2 E}{5}$$

$$i.e. \text{ if } \frac{1}{a} = \frac{\pi^2 E}{5 f_c} = \frac{25,600}{6} = 4,267$$

Although some writers state that constants obtained in this manner agree with experimental results, the constants are not usually calculated theoretically in this way, but are obtained from experiments.

It is believed that the figures recommended above will agree well with the best practice.

It is interesting to note that in one form of Rankine's formula, giving the breaking or crippling stress, viz.

$$\frac{P}{A} = \frac{f}{1 + a\left(\frac{L}{k}\right)^2}$$

is the stress at the elastic limit.

In an earlier chapter we pointed out the desirability of obtaining the working stresses from elastic limit, *i. e.* basing the factor of safety on the elastic limit.

An interesting and important paper by Mr. C. P. Buchanan, in *Engineering News*, December 26, 1907—published after the Quebec Bridge disaster—gives the results of tests on full-size built-up columns such as are actually used in bridge practice. The tests extend over a period of fourteen years, and show that even for the short columns the buckling or crippling stress is not more than 90 per cent. of the tensile yield point (see p. 4).

We thus see that in columns as actually used in practice, the buckling stress is certainly not more than the elastic limit stress, and so the only reasonable factor of safety is that based on the elastic limit.

(*c*) **Straight Line Formula.**—These empirical formulæ are used principally in America, and give very good approximations for rough working. They are of the form

$$\begin{aligned} f_p &= f_e\left(1 - e \, . \, \frac{L}{k}\right) \\ &= f_e(1 - e \, . \, c) \end{aligned}$$

Where f_p and f_c are as before

e = a constant depending on the material.

The following values of e may be taken—

For mild steel $e = \cdot 0053$
„ wrought iron $e = \cdot 0053$
„ cast iron $e = \cdot 008$
„ timber $e = \cdot 0083$

As in Rankine's formula the values of constants vary considerably according to different authorities.

(*d*) **Johnson's Parabolic Formula.**—This is also an empirical formula devised to agree with Euler for long lengths, and to agree with the ordinary compression strength for short lengths. It is of the form

$$f_p = f_c\left\{1 - g\left(\frac{L}{k}\right)^2\right\}$$
$$= f_c\,(1 - g\,.\,c^2)$$

g is a constant of such value as to make the curve of f_p plotted against c tangential to Euler, and the curve is used up to the point where it meets the Euler curve.

The following values may be taken for g—

For mild steel $g = \cdot 000057$
„ wrought iron $g = \cdot 000039$
„ cast iron $g = \cdot 00016$

(*e*) **Gordon's Formula.**—This formula is often confused with Rankine's, and was used largely for some time, but it is now quickly going out of use in favour of the Rankine formula. This is probably due to the fact that designers are now more used to making calculations involving the radius of gyration, a quantity which practical men have usually looked upon with suspicion. Now that tables are published giving k for most sections, it is as easy to use as the diameter d.

Gordon's formula is of the form

$$f_p = \frac{f_c}{1 + j\,.\left(\frac{L}{d}\right)^2}$$

Where f_c, f_p, and L have their usual meaning.

j is constant depending on the material *and on the shape of the section.*

d is the least diameter or breadth of the section.

The objection to this formula as compared with Rankine's lies in the fact that one has to use different constants for different shapes of section for the same material. Otherwise it is very similar to Rankine's.

The following values for j may be taken, f_c being the same as in Rankine—

Shape of Section.	j			
	Mild Steel.	Wrought Iron.	Cast Iron.	Timber.
Solid circle	$\frac{1}{370}$	$\frac{1}{500}$	$\frac{1}{110}$	$\frac{1}{125}$
Hollow circle	$\frac{1}{600}$	$\frac{1}{800}$	$\frac{1}{180}$	$\frac{1}{200}$
L, T, H, etc.	$\frac{1}{300}$	$\frac{1}{400}$	$\frac{1}{90}$	$\frac{1}{100}$
Built-up sections . . .	$\frac{1}{400}$	$\frac{1}{550}$	—	—
Rectangle (solid) . . .	$\frac{1}{500}$	$\frac{1}{700}$	$\frac{1}{120}$	$\frac{1}{160}$

(f) **Fidler's Formula.**—The reader is referred to Fidler's *Bridge Construction* for a very complete analysis of the strut problem.

The formula which Mr. Fidler obtains gives the breaking stress, and is

$$\text{Minimum breaking stress} = \frac{f + \mathrm{R} + \sqrt{(f + \mathrm{R})^2 - 2\,m\,f\,\mathrm{R}}}{m}$$

Where f = ultimate pure compressive strength of material

R = Euler's breaking stress = $\frac{\pi^2 \mathrm{E}}{c^2}$

m = a constant of average value 1·2.

The following values of f_p, the safe stress in tons per sq. in. for struts, are suggested by Fidler and are used by some authorities—

$\frac{L}{k}$	Mild Steel.		Wrought Iron.		Cast Iron.	
	Pin ends.	Fixed ends.	Pin ends.	Fixed ends.	Pin ends.	Fixed ends.
20	5·20	5·29	3·92	3·99	8·07	8·65
40	4·76	5·09	3·64	3·89	5·68	7·56
60	4·02	4·83	3·17	3·73	3·35	6·10
80	3·15	4·45	2·60	3·48	1·96	4·68
100	2·40	4·00	2·03	3·17	1·29	3·35
120	1·83	3·46	1·57	2·82	·93	2·37
140	1·42	2·96	1·24	2·48	·70	1·78
160	1·13	2·51	·98	2·14	·56	1·40
180	·91	2·13	·80	1·84	·43	1·14

Lilly's Formula.—Professor Lilly of Dublin has devised formulæ for columns which allow for secondary flexure or wrinkling in the column,* and may be regarded as a modification or correction of Rankine's formula.

It is of the form

$$f_p = \frac{f_c}{1 + m\,.\,\frac{k}{t} + ac^2}$$

Where m is a constant depending on the shape of section

t is the thickness of the metal in the section

a is $\frac{5\,f_c}{\pi^2\,\text{E}}$. (Compare p. 287.)

$m = \frac{5\,\text{N}\,f_c}{\text{E}}$, the values of N being given for some sections in Fig. 132*a*.

Use of Strut Formulæ.—Fig. 133 shows curves of f_p for mild steel for various values of the buckling factor according to the first four formulæ. It is advisable to draw such a curve to a good scale, choosing one of the formulæ—say

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* See *Engineering*, January 10, 1908, or a more complete paper and bibliography in *Proc. Am. Soc. C. E.* Vol. LXXVI (1913); also *The Design of Plate Girders and Columns* (Chapman & Hall, Ltd.).

Rankine—with $a = \frac{1}{6000}$; such curve can then be used whenever the value of f_p is required.

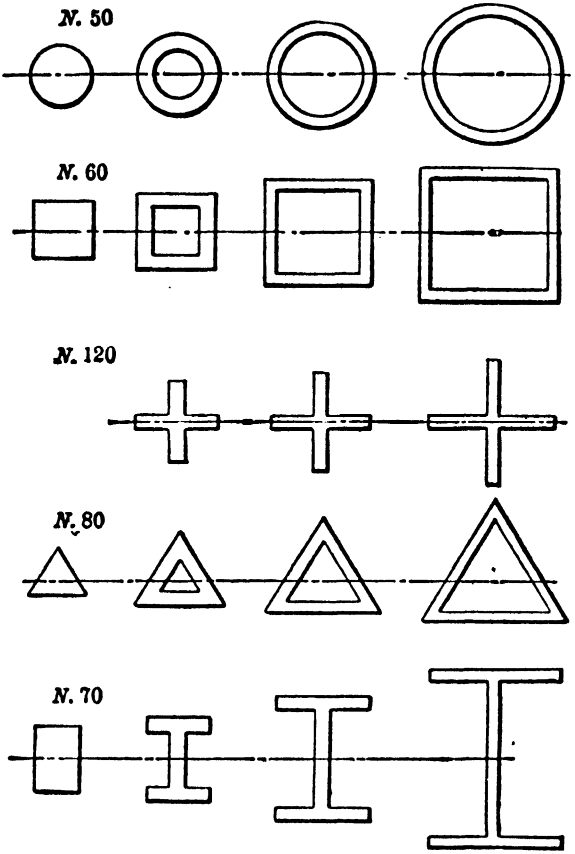

Fig. 132*a*.—Lilly's Column Formula.

It will be remembered that f_p gives the safe stress per sq. in. for struts *with central loads*. If the loads are eccentric we must proceed as described later.

Then if **A** = area of section of strut,

$$\text{Safe load} = P_s = f_p \, . \, A.$$

If, as often occurs in practice, we are given the load but have not designed the section, so that we do not know the buckling factor, we can often get a rough idea by taking a trial value of f_p equal to about $\frac{2}{3}$ f_c, *i. e.* 4 tons per sq. in. for steel, and finding the area requisite for this stress. This will give us an idea of the area required, and we can choose a section with roughly this area, and see by finding its buckling factor what is the safe load on it.

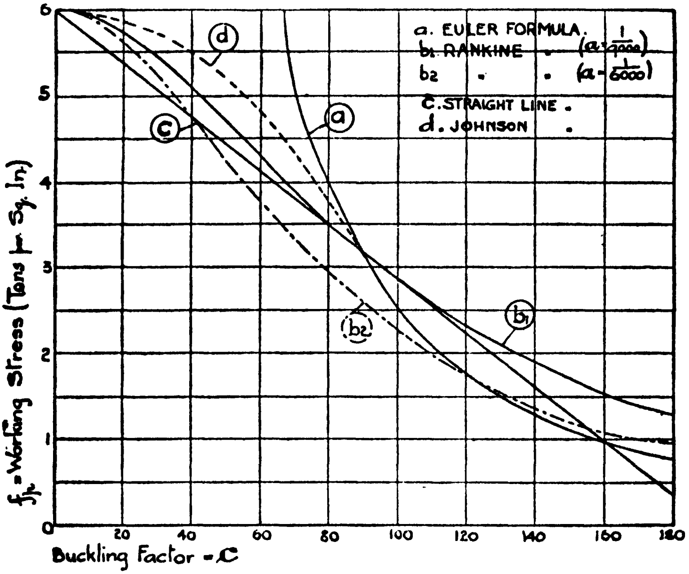

FIG. 133.—Curves for various Column or Strut Formulæ. (Mild Steel)

Many of the leading constructional steelwork firms publish tables of safe loads on various struts. Having previously checked one or two to see that these firms work with similar formulæ, we can choose a suitable section for our case, and then apply our formula and see if such section is satisfactory.

REINFORCED CONCRETE COLUMNS

Short Columns Centrally Loaded.—We have shown on p. 38 that the safe load in a column in which buckling is negligible (the length being less than 15 times the least diameter, and the notation being modified) is given by

$$P = c\,(A_c + m\,A_s)$$

Cross-binding of Reinforcement.—In addition to the longitudinal reinforcement, some force of binding is necessary to keep the bars at the requisite distance apart. This is due to the following reason—

Suppose that a reinforced column with bars A B, C D, Fig. 133*a*, be compressed; then, quite apart from any buckling of the whole column, the column will bulge out somewhat as shown, and the reinforcing bars will buckle because the value of $\frac{L}{k}$ or the buckling factor for them will be large. If we bind the reinforcing bars together, as shown diagrammatically, so that they cannot buckle, the column will not bulge to anything like the same extent, and so will be considerably strengthened. From a large number of experiments M. Considère found that the best results are obtained when spiral coils are placed round the reinforcing bars at distances apart equal to $\frac{1}{7}$ to $\frac{1}{10}$ of the diameter of the coil.

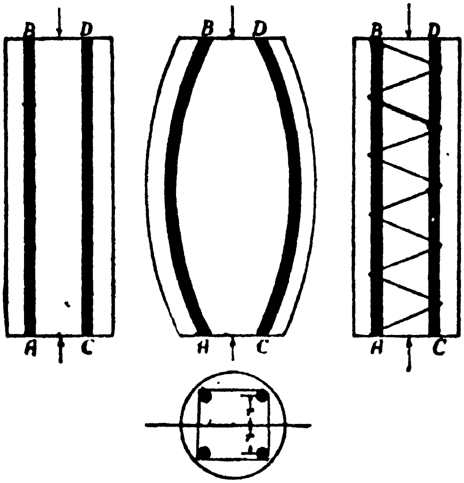

FIG. 133*a*.

M. Considère suggested the following allowance for the coils in the strength of the column—

Let A_h be the equivalent area of longitudinal reinforcement of the spiral coils $\left(i.e.\ A_h = \dfrac{\text{volume of metal in coils}}{\text{length of column}}\right)$

Then safe load $= c\ (A_c + m\,A_s + 2{\cdot}4\,m\,A_h)$

Long Columns Centrally Loaded.—Some authorities use Euler's formula applied to the homogeneous section, viz.—

$$\text{Safe stress} = f_p = \frac{\pi^2 E}{5\,c^2}$$

c being the buckling factor. In obtaining c the radius of gyration of the equivalent homogeneous section (see p. 183) is used,

$$\text{i. e. } k = \sqrt{\frac{I_e}{A}}$$

where $A = A_c + m\,A_s$.

I_e = equivalent second moment

$= I' + (m - 1)\,A_s\,r^2$ for section shown in Fig. 133*a*.

I' being the moment of the section apart from the reinforcement,

$$I_e = \frac{\pi\,D^4}{64} + (m - 1)\ A_s\ r^2 \text{ for circle}$$

$$= \frac{b\,h^3}{12} + (m - 1)\ A_s\ r^2 \text{ for rectangle}$$

Then safe load $= f_p \times (A_c + m\,A_s)$

Rankine's formula can also be used in the form

$$f_p = \frac{500}{1 + \dfrac{c^2}{8000}}$$

Braced Columns, Struts, and Stanchions.—Struts are often formed of rolled sections such as beams and channels braced together by diagonal bracing or plates. The strut that failed in the Quebec Bridge was a braced strut, and the report of the Commission states that there is not yet sufficient information for the design of such struts for very heavy loads.* For ordinary comparatively light work, however,

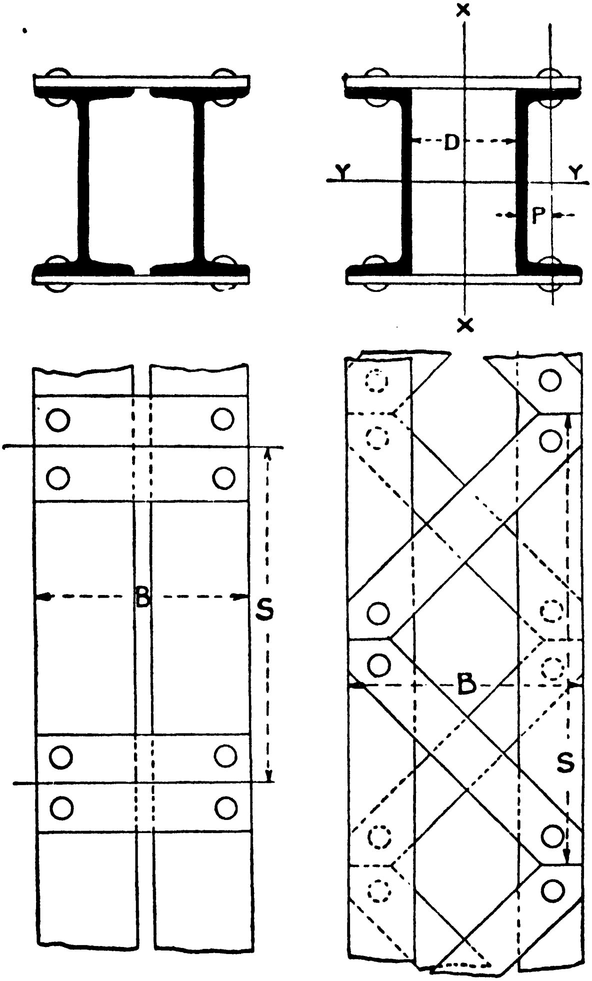

FIG. 134.—Columns with Open Webs.

braced struts such as shown in Fig. 134 are commonly used but the diagonal bracing should preferably have one rivet passing through the two diagonal bows as in the top of Fig. 135 instead of two rivets as shown. The unbraced length of one of the beams or channels must be such that the load per sq. in. on them is not more than the safe stress for them considered as struts. We can get an idea of the maximum unbraced length as follows—

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* See *Illinois University Bulletin*, No. 44, G. Talbot and Moore, for an experimental investigation of the subject.

Let c = buckling factor of whole strut
,, k_1 = least radius of gyration of one channel or beam
,, P = total load carried by strut
,, 2 A = total area of strut
,, S = maximum unbraced length of channel or beam.

Then, using Euler's Formula, $\frac{P}{2A} = \frac{\pi^2 E}{5c^2} = \frac{B}{c^2}$

Each channel or beam carries ½ load

$$\therefore \frac{\frac{1}{2}P}{A} = \text{stress} = \frac{\pi^2 E k_1^2}{5 S^2} = \frac{B k_1^2}{S^2}$$

$$\therefore \frac{B}{c^2} = \frac{B k_1^2}{S^2}$$

$$\therefore S = k_1 c$$

Or, since $c = \frac{\text{Equivalent length of strut}}{\text{Least radius of gyration of whole strut}} = \frac{L}{k}$

$$\frac{S}{L} = \frac{k_1}{k}$$

$$\therefore \frac{L}{S} = \text{least number of panels} = \frac{k}{k_1}$$

$$= \frac{\text{Least radius of gyration of whole strut}}{\text{Least radius of gyration of channel or beam}}$$

As a rule a spacing of 2 to 3 times the breadth B or 30° to 45° inclination of the diagonals will be found to be satisfactory, and in practice would be adopted, unless the calculation required them to be less.

The strength of the strut in this case is calculated as if the section consisted of the two channels or beams held at the requisite distance apart. See worked Example No. 4.

Relative Values of Different Bracings.—Professor H. F.

Moore, of Illinois University,* has given the results shown in Fig. 135 of the " flexual efficiency " of various forms of bracing. This flexual efficiency is the ratio of the calculated fibre stress to that obtained by measuring the strain, the calculation being made on the assumption that the braced section behaves as an integral one. The tests were made by ordinary cross bending and not as columns, but the comparative results may be taken as representing the relative values of the different kinds of bracing for column purposes. Particular attention is directed to the great advantage that single diagonal bracing with single rivets (the third from the top) has over that with the separate rivets (bottom) in which heavy secondary stresses occur.

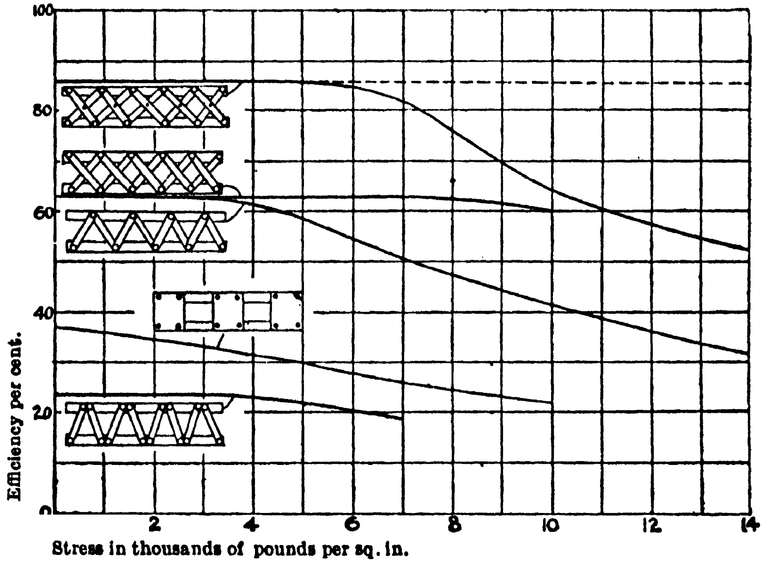

FIG. 135.—Efficiency of Bracing.

Least Radius of Gyration.—The least radius of gyration will be about or at right angles to an axis of symmetry if there be one, so that in this case we need only calculate *k* for the axis of symmetry and at right angles to it. If there is no such axis we should proceed as indicated on p. 172.

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* See a paper by Professor A. H. Basquin, *Proc. Western Society of Engineers*, 1914, on " The Design of Columns." This is one of the best papers which have been published on the subject.

Examples on Struts, etc., with Central Loads.—The following numerical examples should make the question of the design of struts, etc., clear.

(1) *A* 10″ × 6″ × 42 *Standard* **I** *beam of mild steel is used as a stanchion, the length being* 16 *ft. and one end being fixed and one end pin-jointed. Find the safe load for it to carry.*

From the table of standard sections we see

$$A = 12{\cdot}35 \text{ sq. ins.}$$

$$\text{Least } k = 1{\cdot}36$$

$$\therefore \text{ Buckling factor} = c = \frac{\text{equivalent length}}{1{\cdot}36} = \frac{2\,L}{3\,k}$$

$$= \frac{2 \times 16 \times 12}{3 \times 1{\cdot}36} = 94{\cdot}2 \text{ about}$$

$$\therefore \text{ Safe stress} = f_p = \frac{6}{1 + \frac{94{\cdot}2^2}{6000}} \text{ using Rankine's formula}$$

$$= \frac{6}{1 + 1{\cdot}47} = 2{\cdot}43 \text{ tons per sq. in.}$$

$$\therefore \text{ Safe load} = 12{\cdot}35 \times 2{\cdot}43 = 30 \text{ tons.}$$

(2) *A solid cast-iron column,* 6 *inches in diameter and* 15 *feet long, is fixed at the lower end and carries a load at its free upper end. Calculate the load the column will safely carry, assuming a reasonable factor of safety.* (*B.Sc. Lond.*)

$$\text{In this case } k = \frac{D}{4} = 1{\cdot}5''$$

$$\text{Equivalent length} = 2\,L = 30$$

$$\therefore c = \frac{\text{equivalent length}}{k} = \frac{30 \times 12}{1{\cdot}5}$$

$$= 240$$

$$\therefore \text{ Safe stress per sq. in.} = f_p = \frac{7}{1 + \frac{240 \times 240}{1{,}800}}$$

$$= \frac{7}{1 + 32}$$

$$= {\cdot}212 \text{ ton per sq. in.}$$

$$\therefore \text{ Safe load} = \cdot 212 \times \frac{\pi \times 36}{4} = 6 \text{ tons.}$$

$$\text{According to Euler } f_p = \frac{\pi^2 E}{5\, c^2} = \frac{12{,}000}{c^2}$$

$$= \frac{12{,}000}{240 \times 240} = \cdot 208$$

$$\therefore \text{ Safe load} = \frac{\cdot 208 \times \pi \times 36}{4} = 5{\cdot}88 \text{ tons.}$$

(3) *A steel rolled joist is used as a strut with built-in ends, the length of the strut being* 15 *feet. Find, from the data given below, the cross section of the joist, if it has to support a compressive load of* 40 *tons with a factor of safety of* 4.

(*a*) *The total depth of the cross section of the joist is twice the width of the flanges, and the thickness of metal is to be* $\frac{1}{8}$ *of the width of the flanges.*

(*b*) *The crushing strength of a short strut of this quality of steel is* 24 *tons per square inch.*

(*c*) *The constant in Rankine formula is* $\frac{1}{36{,}000}$. (*B.Sc. Lond.*)

In this problem we must first find the breaking stress from the formula. In this case we do not use the equivalent length of the strut because the constant is given for fixed ends.

$$\text{Breaking stress} = \frac{24}{1 + \frac{1}{36{,}000}\left(\frac{L}{k}\right)^2}$$

$$\therefore \text{ Safe stress} = \frac{\text{breaking stress}}{4} = \frac{6}{1 + \frac{1}{36{,}000}\left(\frac{L}{k}\right)^2}$$

Now let A = area of section

and let B = breadth of flange

then 2 B = depth of beam

$\frac{B}{8}$ = thickness of metal.

$$\text{Then} \quad A = \frac{2\,B \times B}{8} + \frac{B}{8}\left(2\,B - \frac{2\,B}{8}\right)$$

$$= \frac{B^2}{4} + \frac{7\,B^2}{32} = \frac{15\,B^2}{32} = \cdot 4687\,B^2$$

The least radius of gyration will be about an axis perpendicular to the flanges.

Then
$$I = \frac{B}{4} \cdot \frac{B^3}{12} + \frac{7\,B}{4 \times 12} \cdot \left(\frac{B}{8}\right)^3$$
$$= \frac{B^4}{48} + \frac{7\,B^4}{12 \times 2{,}048} = \cdot 02111\ B^4$$
$$\therefore\ k^2 = \frac{I}{A} = \frac{\cdot 02111\ B^4}{\cdot 4687\ B^2} = \cdot 045\ B^2$$
$$\therefore\ \text{Safe stress} = \frac{40}{A} = \frac{6}{1 + \dfrac{15 \times 12 \times 15 \times 12}{36{,}000 \times \cdot 045\ B^2}}$$
$$\therefore\ \frac{40}{\cdot 4687\ B^2} = \frac{6}{1 + \dfrac{9}{\cdot 45\ B^2}}$$
$$\therefore\ \left(1 + \frac{9}{\cdot 45\ B^2}\right) = \frac{6}{40} \cdot \cdot 4687\ B^2$$
$$= \cdot 15 \times \cdot 4687\ B^2$$
$$\therefore\ B^4(\cdot 15 \times \cdot 4687 \times \cdot 45) - \cdot 45\ B^2 - 9 = 0$$
$$3{\cdot}16\ B^4 - 45\ B^2 - 900 = 0$$

The solution of this quadratic gives
$$B^2 = 25{\cdot}3 \text{ nearly}$$
$$\text{say } B = 5$$

$\therefore$ Adopt a joist 10″ × 5″ with metal $\frac{5}{8}$″ thick.

We could work this problem roughly by the given rule, as follows—

$$\text{Take } f_p = \frac{2}{3} \times 6 = 4$$
$$\therefore\ A = \frac{40}{4} = 10 \text{ sq. ins.}$$
$$\therefore\ \frac{15\ B^2}{32} = 10$$
$$B^2 = \frac{10 \times 32}{15} = \frac{64}{3}$$
$$B = \frac{8}{\sqrt{3}} = 4{\cdot}62, \text{ say } 5''$$

(4) *A steel column in a bridge-truss has pin-jointed ends and is* 26 *feet long. It consists of two standard* 10″ × 3½″ × 28·21 *lb.*

channels placed 4½ inches apart. Find a safe load for the section. (See Fig. 134.)

On looking up the tables, we see that for a 10″ × 3½″ × 28·21 lb. channel,

$$A = 8{\cdot}296$$

$$k_{\text{max.}} = 3{\cdot}77$$

$$k_{\text{min.}} = {\cdot}994$$

$$\text{Dist. of C.G. from edge} = P = {\cdot}933$$

Then for whole strut

$$k_{YY} = 3{\cdot}77$$

$$k_{XX}^2 = \left(\frac{D}{2} + p\right)^2 + k_{\text{min.}}^2$$

$$= 3{\cdot}183^2 + {\cdot}994^2$$

$$\therefore k_{XX} = 3{\cdot}33$$

$$\therefore c = \frac{\text{Length}}{\text{Least radius of gyration}} = \frac{26 \times 12}{3{\cdot}33}$$

$$= 93{\cdot}6$$

$$\therefore f_p = \frac{6}{1 + \dfrac{93{\cdot}6 \times 93{\cdot}6}{6000}} = \frac{6}{2{\cdot}46}$$

$$= 2{\cdot}44$$

$$\therefore \text{Safe load} = 2{\cdot}44 \times \text{area}$$

$$= 2{\cdot}44 \times 2 \times 8{\cdot}296$$

$$= 40{\cdot}4 \text{ say } \underline{40 \text{ tons}}.$$

STRUTS WITH ECCENTRIC LOADING

Simple Approximate Method.—If the thrust in the strut is out of the centre, *i.e.* where there is bending moment as well as direct thrust on the strut, we cannot use the same rules for design as in the ordinary case.

In such case we may obtain approximate results by proceeding as follows—

Let the load P be at distance *e* from the centroid of the cross section, then $M = P \cdot e$ (Fig. 136).

CASE 1. VERY SHORT STRUTS.—If the length is less than

10 times the least diameter of the strut, the stresses are obtained as shown on p. 237.

$$\text{i.e. } f_c = \frac{P}{A} + \frac{M}{Z_c}$$

$$f_t = \frac{M}{Z_t} - \frac{P}{A}$$

In this case $$f_c = \frac{P}{A} + \frac{Px}{Z_c}$$

$$= \frac{P}{A} + \frac{P.e.d_c}{A k^2}$$

$$= \frac{P}{A}\left(1 + \frac{e d_c}{k^2}\right)$$

$$\therefore \frac{P}{A} = \frac{f_c}{1 + \frac{e d_c}{k^2}}$$

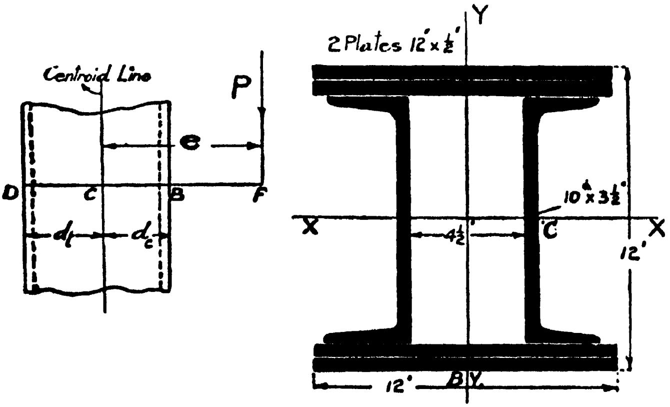

Fig. 136.—Columns with Eccentric Loads.

This gives the safe load P for a compressive stress f_c. This case is fully dealt with in Chap. VIII.

Case 2. Struts Longer than 10 Diameters.—In this case we must make some allowance for buckling tendencies, and we may proceed as follows—

As in the previous case we have

$$\text{Combined compressive stress} = \frac{P}{A}\left(1 + \frac{e d_c}{k^2}\right)$$

Now in this case this compressive stress should not be more than the safe stress per sq. in. obtained by considering the buckling formulæ,

$$i.e.\ \frac{P}{A}\left(1 + \frac{e\,d_c}{k^2}\right) = f_p$$

$$\frac{P}{A} = \frac{f_p}{1 + \frac{e\,d_c}{k^2}}$$

$$i.e.\ \text{Safe eccentric load on strut} = \frac{\text{Safe central load on strut}}{\left(1 + \frac{e\,d_c}{k^2}\right)}$$

where e = eccentricity of load

d_c = distance from centroid to edge of section nearest load

k = radius of gyration about axis perpendicular to the plane containing the centroid and the load.

This formula may be put into a form which is sometimes more useful as follows—

Let P_1 be the central load, which is equivalent to the eccentric load P.

$$\text{Then } P_1 = P\left(1 + \frac{e\,d_c}{k^2}\right)$$

In this formula e should be taken as the *effective eccentricity*, *i. e.* $\left(\frac{M}{P}\right.$ where M is the Bending Moment on the column), the value of e shown on the drawings is only true when the column is free at the top. For other cases see articles by the author in the *Architect's and Builder's Journal*, March 3 and 31, 1915.

Then $\frac{P_1}{P}$ may be called the *eccentricity factor* for the strut.

In using this formula it should be noted that it is worked on the assumption that the buckling will take place in the plane of the figure, and so the value of k for the strut in this direction should be used in finding the safe central load.

If the safe eccentric load according to this formula comes more than the safe central load for the least value of k (this can of course only occur when the least value of k is about the axis D B), the lower value should be used.

Stanchions with Web and Flange Connections.—The loads on stanchions are often communicated from girders connected by cleats, etc., to the web or flange of the stanchion. If such connections come on one side only, or if the loads communicated from the two sides are not equal, the load will not be central, and allowance for the eccentricity should be made.

NUMERICAL EXAMPLE.—*A mild steel stanchion* 30 *feet long and with ends fixed has the section shown in Fig.* 136. *Find the safe central load and also the safe loads communicated at the points* B *and* C.

$$\text{In this case } A = 40{\cdot}59 \text{ sq. ins.}$$
$$k_{XX} = 4{\cdot}87 \quad ,, \quad ,,$$
$$k_{YY} = 3{\cdot}41 \quad ,, \quad ,,$$

$$\therefore \text{Buckling factor} = c = \frac{L}{2k} = \frac{30 \times 12}{2 \times 3{\cdot}41} = 52{\cdot}8$$

$$\therefore f_p = \frac{6}{1 + \frac{52{\cdot}8 \times 52{\cdot}8}{6000}} = \frac{6}{1{\cdot}464} = 4{\cdot}10$$

$$\therefore \text{Safe central load} = 40{\cdot}59 \times 4{\cdot}10 = 166 \text{ tons nearly.}$$

Load at C.—

$$e = 2{\cdot}25 + {\cdot}475 = 2{\cdot}725$$
$$\therefore d_c = 6''$$
$$\therefore \frac{e\,d_c}{k^2} = \frac{2{\cdot}725 \times 6}{3{\cdot}41^2} = 1{\cdot}41$$

$$\therefore \text{Safe eccentric load at C} = \frac{166}{1 + 1{\cdot}41} = 69 \text{ tons nearly.}$$

Load at B.—We must now first calculate f_p as if k_{XX} were minimum radius of gyration, i. e. $c = \frac{30 \times 12}{2 \times 4{\cdot}87} = 36{\cdot}9$.

$$\therefore f_p = \frac{6}{1 + \frac{36{\cdot}9 \times 36{\cdot}9}{6000}} = 4{\cdot}89$$
$$e = 6''$$
$$d_c = 6''$$
$$\therefore \frac{e\,d_c}{k^2} = \frac{6 \times 6}{4{\cdot}87^2} = 1{\cdot}52$$

$$\therefore \text{ Safe eccentric load at B} = \frac{4{\cdot}89 \times 40{\cdot}59}{1 + 1{\cdot}52}$$

$$= \frac{4{\cdot}89 \times 40{\cdot}59}{2{\cdot}52}$$

$$= 77{\cdot}7 \text{ tons nearly.}$$

In this case the eccentricity factors for C and B are 2·41 and $\frac{166}{77{\cdot}7} = 2{\cdot}14$ respectively.

A rough rule sometimes adopted is to use 2½ and 1½ as eccentricity factors for flange and web connections respectively, but such rule is not reliable. It is more nearly true for **I** beams used as stanchions than for built-up sections.

Alternative Approximate Method.—Where the bending stress is large compared with the direct stress it seems reasonable to allow that instead of the previous treatment we shall subtract the bending stress from the value of f_c used in the strut formula.

The compressive bending stress for an effective eccentricity $e\left(e = \frac{M}{P}\right)$ is $\frac{P \, . \, e d_c}{A k^2}$

$\therefore$ using the Rankine formula we shall have

$$f_p = \frac{P}{A} = \frac{f_c - \frac{P e d_c}{A k^2}}{1 + ac^2}$$

$$i.e. \ \frac{P}{A}\left(1 + a c^2 + \frac{e d_c}{k^2}\right) = f_c$$

$$\frac{P}{A} = \frac{f_c}{\left(1 + a c^2 + \frac{e d_c}{k^2}\right)}$$

Cast-iron Struts Eccentrically Loaded.—In dealing with cast-iron struts with eccentric loads it must be remembered that they will probably fail by tension.

The safe load P from the tension standpoint

$$= \frac{f_t A}{\left(\frac{e d_t}{k^2} - 1\right)}$$

when f_t is the safe tensile stress, and this should be compared with the safe load from the compression standpoint, and the lower value adopted.

***Modified Euler Theory for Eccentric Loading.**— In this case we have, Fig. 137,

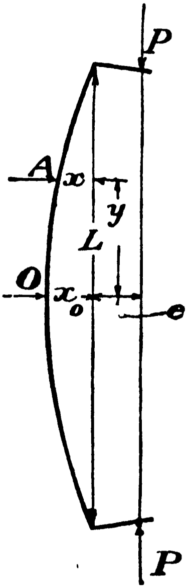

FIG. 137.—Eccentric Loading of Columns.

$$P(e + x) = M = -EI\frac{d^2x}{dy^2}$$

$$\therefore \frac{d^2x}{dy^2} = -\frac{P}{EI}(x + e)$$

or putting
$$m = \sqrt{\frac{P}{EI}}$$

$$\frac{d^2x}{dy^2} = -m^2(x + e) \ldots\ldots\ldots\ldots\ldots(1)^*$$

The general solution of this is

$$(x + e) = A\cos my + B\sin my$$

---

* Cf. p. 282, equation (1).

Since $x = 0$ for $y = \pm \frac{L}{2}$, $B = 0$ as before

$$\therefore (x + e) = A \cos m y \quad \ldots\ldots\ldots\ldots\ldots (2)$$

$$\therefore \text{ when } x = 0, e = A \cos \frac{m L}{2}$$

$$\therefore A = e \sec \frac{m L}{2}$$

$$\therefore x = e \sec \frac{m L}{2} . \cos m y - e$$

$$= e\left(\sec \frac{m L}{2} . \cos m y - 1\right) \quad \ldots\ldots\ldots (3)$$

At point o where $y = 0$ eccentricity $= x_o + e = e_1$

$$= e\left(\sec \frac{m L}{2} - 1\right) + e = e \sec \frac{m L}{2}$$

$$= e \sec \left(\frac{L}{2}\sqrt{\frac{P}{E I}}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$= e \sec \left(\frac{L}{2 k}\sqrt{\frac{f_c}{E}}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

where $\qquad f_c = \frac{P}{A}$

$$\therefore \text{ stress at o } = \frac{P}{A}\left(1 + \frac{e_1 d_c}{k^2}\right)$$

$$= \left(\text{putting } \frac{L}{k} = c\right) f_c \left(1 + \frac{e d_c}{k^2} \sec \frac{c}{2}\sqrt{\frac{f_c}{E}}\right) \quad \ldots\ldots\ldots\ldots (6)$$

Now let $\theta = \frac{c}{2}\sqrt{\frac{f_c}{E}}$ and call it the Eulerian angle.

Then stress at o $= f_c\left(1 + \frac{e d_c}{k^2} \sec \theta\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$

Values of sec θ are given in the table on p. 308, taken from Professor Basquin's paper * previously referred to.

$$\therefore \text{ Safe eccentric load } = \frac{\text{Safe central load}}{1 + \frac{e d_c}{k^2} \sec \theta}$$

This can only be used by trial if the load is not given.

---

* *Proc. Western Society of Engineers*, 1914.

Average Unit Stress Thousand Pounds per Square Inch	Buckling Factor c.									
	50	60	70	80	90	100	110	120	130	140
5	1·05	1·08	1·11	1·15	1·20	1·25	1·32	1·40	1·50	1·62
6	1·07	1·10	1·14	1·18	1·24	1·31	1·40	1·51	1·63	1·82
7	1·08	1·11	1·16	1·22	1·29	1·38	1·50	1·64	1·83	2·08
8	1·09	1·14	1·19	1·26	1·35	1·46	1·60	1·79	2·05	2·41
9	1·10	1·15	1·22	1·30	1·41	1·54	1·72	1·97	2·32	2·85
10	1·12	1·17	1·25	1·34	1·47	1·63	1·86	2·18	2·65	3·46
11	1·13	1·19	1·28	1·39	1·54	1·74	2·02	2·44	3·12	4·38
12	1·14	1·21	1·31	1·43	1·61	1·85	2·22	2·56	3·74	5·88
13	1·15	1·23	1·34	1·49	1·69	1·98	2·42	3·16	4·63	8·69
14	1·17	1·25	1·37	1·54	1·77	2·12	2·68	3·69	6·02	16·4
15	1·18	1·28	1·41	1·60	1·87	2·29	2·99	4·40	8·46	172
16	1·19	1·30	1·45	1·66	1·97	2·47	3·38	5·43	14·4	
17	1·21	1·32	1·49	1·72	2·09	2·69	3·87	7·04	39·0	
18	1·22	1·35	1·53	1·79	2·21	2·95	4·51	9·86		
19	1·24	1·37	1·57	1·87	2·36	3·25	5·39	16·4		
20	1·25	1·40	1·62	1·95	2·51	3·62	6·65	47·4		
21	1·27	1·43	1·66	2·04	2·69	4·07	8·63			
22	1·28	1·45	1·71	2·13	2·90	4·65	12·3			
23	1·30	1·48	1·77	2·24	3·13	5·40	20·9			
24	1·31	1·51	1·82	2·35	3·40	6·41	65·9			
25	1·33	1·54	1·88	2·47	3·73	7·87				
26	1·35	1·57	1·94	2·61	4·10	10·1				
27	1·37	1·61	2·00	2·76	4·57	13·8				
28	1·38	1·64	2·08	2·93	5·15	22·9				
29	1·40	1·68	2·15	3·11	5·84	57·3				
30	1·42	1·72	2·23	3·32	6·79					
31	1·44	1·75	2·32	3·56	8·04					
32	1·46	1·79	2·41	3·83	9·86					
33	1·48	1·84	2·51	4·14	12·7					
34	1·50	1·88	2·61	4·50	17·3					
35	1·52	1·92	2·73	4·93	28·7					
36	1·54	1·97	2·83	5·43	68·7					
37	1·56	2·02	2·98	6·02						
38	1·59	2·07	3·13	6·80						
39	1·61	2·13	3·29	7·73						
40	1·63	2·18	3·46	9·07						
41	1·66	2·24	3·66	10·8						
42	1·68	2·31	3·87	13·3						
43	1·71	2·37	4·11	17·2						
44	1·74	2·44	4·38	24·6						
45	1·76	2·51	4·68	43·0						
46	1·79	2·59	5·03	172						
47	1·82	2·67	5·42							
48	1·85	2·76	5·88							
49	1·88	2·85	6·42							
50	1·91	2·95	7·06							

Table of Eulerian Secants

$$\sec\left(\frac{c}{2}\cdot\sqrt{\frac{f_c}{E}}\right)$$

$E = 30{,}000{,}000.$

Multipliers of $\frac{e\,d_c}{k^2}$
in Expression for
Maximum Stress.

Numerical Examples.—(1) *Take the same case as dealt with on p.* 304 *for the load at* B.

We found load = 77·7 tons = $\frac{77{\cdot}7 \times 2{,}240}{40{\cdot}59}$ lbs. per sq. in.

= 4,300 lbs. per sq. in. nearly

$\therefore$ $\sec\theta = 1{\cdot}03$ about

The effect of this upon the result is negligible.

(2) *Find the stress produced in the column of question* (4), p. 300, *if the load is* 1″ *out of centre, in the weak direction.*

$$\text{Here } \frac{P}{A} = \frac{40 \times 2{,}240}{2 \times 8{\cdot}296} = 5{,}400 \text{ lbs. per sq. in.}$$

$$c = 93{\cdot}6 \;\therefore\; \sec\theta = 1{\cdot}24 \text{ approx. (from table)}$$

$$\therefore \text{stress} = 5{,}400\left(1 + \frac{1 \times 1{\cdot}24 \times 5{\cdot}75}{3{\cdot}33^2}\right)$$

$$= 5{,}400\,(1{\cdot}64)$$

$$= 8{,}900 \text{ lbs. per sq. in. nearly}$$

$$= 4{\cdot}0 \text{ tons per sq. in.}$$

The approximate method would have given

$$\text{stress} = 5{,}400\left(1 + \frac{1 \times 5{\cdot}75}{3{\cdot}33^2}\right) = 10{,}800 \times 1{\cdot}52$$

$$= 8{,}200 \text{ lbs. per sq. in. nearly}$$

$$= 3{\cdot}66 \text{ tons per sq. in.}$$

Johnson's Formula for Eccentric Loading.—This formula, due to Professor Johnson, is obtained by adding the additional eccentricity due to the deflection and is

$$\text{maximum stress in column} = \frac{P}{A} + \frac{P\,e\,d_c}{I - \frac{P\,L^2}{10\,E}} \;\cdots\cdots (8)$$

$$= \frac{P}{A} + \frac{P\,e\,d_c}{A\,k^2 - \frac{P\,L^2\,k^2\,A}{10\,E\,k^2\,A}}$$

$$= \frac{P}{A}\left\{1 + \frac{e\,d_c}{k^2\left(1 - \frac{P\,L^2}{10\,E\,A\,k^2}\right)}\right\}$$

$$= f_c\left\{1 + \frac{e\,d_c}{k^2\left(1 - \frac{f_c\,c^2}{10\,E}\right)}\right\} \;\cdots\cdots\cdots\cdots\cdots (9)$$

A somewhat more correct but similar formula can be obtained by regarding the bending moment as uniform; this gives a deflection (see p. 264) $= \delta = \frac{M L^2}{8 E I} = \frac{P e L^2}{8 E A k^2}$

$$= \frac{f_c e c^2}{8 E}$$

$\therefore$ effective eccentricity $= \delta + e = e\left(1 + \frac{f_c c^2}{8 E}\right)$

$$\therefore \text{ bending stress } = \frac{P(\delta + e) d_e}{I}$$

$$= \frac{P d_e e}{A k^2}\left(1 + \frac{f_c c^2}{8 E}\right)$$

$$= \frac{P d_c e}{A k^2\left(1 - \frac{f_c c^2}{8 E}\right)} \text{ (approx.)}$$

$$= \frac{f_c d_c e}{k^2\left(1 - \frac{f_c c^2}{8 E}\right)}$$

$\therefore$ Total stress $=$ direct stress $+$ bending stress

$$= f_c + \frac{f_c d_c e}{k^2\left(1 - \frac{f_c c^2}{8 E}\right)}$$

$$= f_c \left\{1 + \frac{e d_e}{k^2\left(1 - \frac{f_c c^2}{8 E}\right)}\right\} \quad \text{..........(10)}$$

Professor Morley * obtained the same result by the expansion

$$\sec \theta = 1 + \frac{\theta^2}{2!} + \frac{5 \theta^4}{4!} + \frac{61 \theta^6}{6!} + \ldots$$

Taking first the two terms as an approximation

$$\sec \frac{c}{2}\sqrt{\frac{f_c}{E}} = 1 + \frac{c^2 f_c}{8 E}$$

$\therefore$ Equation (7) becomes

$$\text{Total stress } = f_c \left\{1 + \frac{e d_c}{k^2}\left(1 + \frac{c^2 f_c}{8 E}\right)\right\}$$

$$= f_c \left\{1 + \frac{e d_e}{k^2\left(1 - \frac{f_c c^2}{8 E}\right)}\right\} \text{ approx.}$$

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* *Theory of Structures* (Longmans).

CHAPTER XI

TORSION AND TWISTING OF SHAFTS

We have seen that in a beam the bending moment is resisted by a complex series of stresses in tension and compression which vary in intensity at different points in the depth of the beam. In the case of shafts we have twisting moment in place of the bending moment and the stresses are pure shear stresses which vary in intensity at different distances from the centre of the shaft.

Stresses in a Shaft Coupling.—As an introduction to the subject consider the case of the stresses in the bolts of the flange coupling shown in Fig. 138.

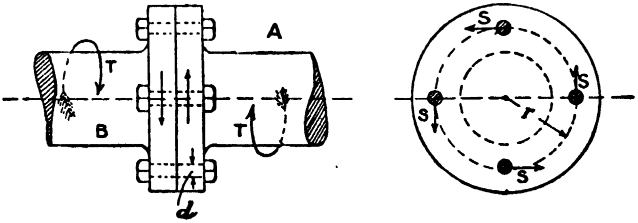


FIG. 138.—Stresses in Coupling Bolts.

Suppose that a twisting moment or torque T is being transmitted through the coupling from the shaft A to shaft B. The shaft B has a resistance to its motion which produces a reverse torque numerically equal to T and the effect of these opposite torques upon the coupling is a tendency to shear the bolts. Suppose that the bolts are at the same distance from the centre and are so small that the shear stress over them may be regarded as constant and that they are equal in area and equally stressed.

Then S = shearing force on each bolt

$$= s \frac{\pi d^2}{4}$$

∴ Taking moments about the axis of the shaft we have

$$\text{Resisting Torque} = \frac{4 \,.\, s \pi d^2}{4} \times r$$

$$\therefore \text{ we have } \mathrm{T} = \frac{4 s \,.\, \pi d^2 \,.\, r}{4} \quad \ldots\ldots\ldots\ldots (1)$$

or if n is the number of bolts

$$\mathrm{T} = \frac{n s \pi d^2 \,.\, r}{4} \quad \ldots\ldots\ldots\ldots (2)$$

If the shaft is transmitting a horse-power, H.P., and is rotating at N revolutions per minute, the work done per revolution is $2 \pi \mathrm{T}$ so that the work done per minute is $2 \pi \mathrm{N} \mathrm{T}$

we have $$\mathrm{T} = \frac{\text{H.P.} \times 33{,}000}{2 \pi \mathrm{N}} \text{ ft. lbs.}$$

$$= \frac{\text{H.P.} \times 33{,}000 \times 12}{2 \pi \mathrm{N}} \text{ in. lbs.} \ldots\ldots\ldots (3)$$

In calculations upon the strength of shafting it is always desirable to work in in. lbs., and taking the unit of 1000 lbs. as a "*kip*" we can work in in. kips to save writing a number of 0's and thus running the risk of error in dealing with large numbers.

In the case of the shaft coupling that we have considered it should be pointed out that we have made a great assumption in regarding the bolts as being equally stressed, because if, say, three of them are loose fits and the fourth is a good fit, the fourth one will carry all the load; the same point holds with ordinary riveted joints. In practice, however, a number of bolts are always used and each is always regarded as carrying its proportion of the load, and the best way to meet the difficulty seems to be to have the workmanship as good as possible.

General Case of Torque on Groups of Bolts or Rivets.—Suppose that we have any number of bolts or rivets of different areas and at different radii from the axis o about which a twisting action may be considered as taking place; in Fig. 139 we have shown three such bolts. Then if we imagine a slight rotational movement of one part of the joint or coupling about the point o relatively to the other it will be seen that the movements of the centres of the bolts will be proportional to their radii r_1, r_2, r_3, etc., and therefore the strain and consequently the stress on any bolt is proportional to its distance from the axis.

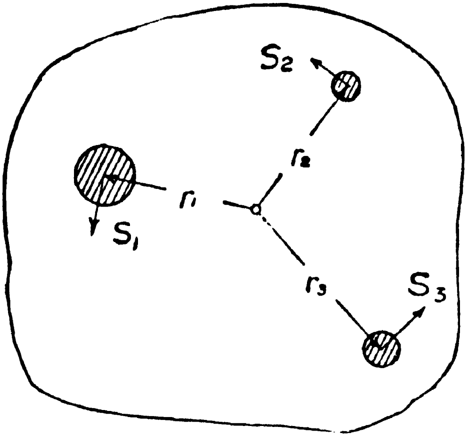


FIG. 139.

Let s_u be the shear stress at unit distance from the axis and let A_1, A_2, A_3, etc., be the areas of the various bolts.

We then have, if s_1, s_2, and s_3, etc., are the shear stresses on the various bolts and S_1, S_2, S_3, etc., the forces

$$s_1 = s_u r_1$$
$$s_2 = s_u r_2$$
$$s_3 = s_u r_3$$

∴ we have

$$\begin{aligned}\text{Total Torque} = \mathrm{T} &= \mathrm{S}_1 r_1 + \mathrm{S}_2 r_2 + \mathrm{S}_3 r_3 + \dots \\ &= s_1 \mathrm{A}_1 r_1 + s_2 \mathrm{A}_2 r_2 + s_3 \mathrm{A}_3 r_3 + \dots \\ &= s_u \mathrm{A}_1 r_1^2 + s_u \mathrm{A}_2 r_2^2 + s_u \mathrm{A}_3 r_3^2 + \dots \\ &= s_u \{\mathrm{A}_1 r_1^2 + \mathrm{A}_2 r_2^3 + \mathrm{A}_3 r_3^2 + \dots\} \\ &= s_u \Sigma \mathrm{A}_1 r_1^2 \quad \dots\dots\dots\dots (4)\end{aligned}$$

But $\Sigma \mathrm{A}_1 r_1^2$ is the polar moment of inertia of the group of bolts, etc. $= \mathrm{I}_r$

$$\therefore s_u = \frac{\mathrm{T}}{\mathrm{I}_r} \quad \dots\dots\dots\dots (5)$$

NUMERICAL EXAMPLES ON COUPLINGS AND JOINTS.—(1) *Find the diameter of bolts necessary in a coupling which transmits* 120 H.P. *at* 75 *revolutions per minute. The diameter of the circle of the bolt centres is* $10\frac{1}{2}$ *inches* (i. e. r = 5·25 *inches*) *and the coupling has* 6 *bolts. The stress allowed is* $2\frac{1}{2}$ *tons per sq. in.*

In this case from equation (3)

$$\begin{aligned}\mathrm{T} &= \frac{120 \times 33{,}000 \times 12}{2\pi \times 75} \text{ in. lbs.} \\ &= 100{,}000 \text{ in. lbs. nearly.}\end{aligned}$$

Also from equation (2)

$$\begin{aligned}\mathrm{T} &= \frac{6 \times 2{\cdot}5 \times \pi d^2}{4} \times 5{\cdot}25 \text{ in. tons} \\ &= \frac{6 \times 2{\cdot}5 \times \pi \times d^2 \times 2{,}240}{4} \times 5{\cdot}25 \text{ in. lbs.} \\ &= 139{,}000\, d^2 \text{ in. lbs. nearly.}\end{aligned}$$

$$\therefore d^2 = \frac{100{,}000}{139{,}000}$$

$$d = \sqrt{\frac{100{,}000}{139{,}000}} = {\cdot}85 \text{ in. nearly}$$

∴ Adopt bolts $\frac{7}{8}''$ in diameter.

(2) EXAMPLE OF CLEAT.—*We will now take the case shown in Fig.* 140 *of the cleat given in the Handbook of Messrs. Dorman, Long & Co., Ltd., for a* 16 *in. by* 6 *in. standard* **I** *beam with a minimum span of* 18 *ft., the rivets being of* $\frac{3}{4}$ *in. diameter.*

The safe uniformly distributed load given for this span and beam is 25 tons, so that the reaction at each end will be $\frac{25}{2} = 12{\cdot}5$ tons, and half of this will be carried by each angle, or the load P will be 6·25 tons.

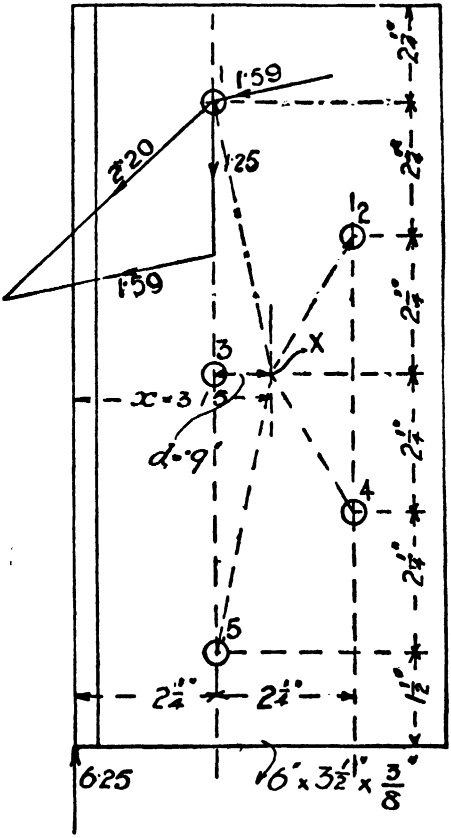


FIG. 140.

First find the position of the centre of gravity of the rivets. It is clearly on the horizontal line through the rivet 3, and its distance from the line 1, 3, 5 is obtained by moments thus—

$$5\,d = 2 \times 2\tfrac{1}{4}$$

$$\text{i.e.}\quad d = \frac{4{\cdot}5}{5} = {\cdot}9 \text{ in.}$$

Then we tabulate the dimensions as follows—

No. of Rivet.	r	r^2
1	4·58	21·06
2	2·62	6·88
3	·90	·81
4	2·62	6·88
5	4·58	21·06
		$\Sigma r^2 = 56{\cdot}69$

$$\therefore s = \frac{\text{P}x}{\Sigma r^2} = \frac{6{\cdot}25 \times 3{\cdot}15}{56{\cdot}69}$$
$$= {\cdot}348 \text{ ton.}$$

The moment load will be a maximum on rivets 1 and 5 because they are farthest from x, and will be equal to

$$\text{T}_5 = {\cdot}348 \times 4{\cdot}58 = 1{\cdot}59 \text{ tons.}$$

The direct load W on these rivets $= \frac{6{\cdot}25}{5} = 1{\cdot}25$ tons.

Therefore resultant load $= \text{R}_5 = 2{\cdot}20$ tons. [See Fig. 140.]

Now bearing area of a ¾-in. rivet in a ⅜-in. plate

$$= \frac{3}{4} \times \frac{3}{8} = \frac{9}{32} \text{ sq. in.}$$

Bearing stress on rivet $= \frac{2{\cdot}20 \times 32}{9} = 7{\cdot}82$ tons per sq. in.

Area of a ¾-in. rivet in section $= \frac{\pi}{4} \times \left(\frac{3}{4}\right)^2 = {\cdot}442$

$\therefore$ Shear stress on rivet $= \frac{2{\cdot}20}{{\cdot}442} = 4{\cdot}98$ tons per sq. in.

The above calculation shows that the rivets are stressed just about up to what is commonly taken as a safe working stress for rivets in shear, viz. 5 tons per sq. in. The importance of allowing for the eccentricity of the stress will be clear from this example, because the resultant maximum stress on the rivets comes to nearly twice the value which would have been found if the eccentricity had not been taken into account.

Torsion of a Circular Shaft.—Suppose that a circular shaft of length l and diameter D is subjected to a twisting moment or torque T (Fig. 141). To preserve the equilibrium of the shaft, equal and opposite torques must act at the two ends and each normal section of the shaft will be subjected to strains which will allow a slight twisting motion of such section without causing it to bend or warp out of its plane.

We will therefore make the following assumptions in developing our theory of torsion—

(1) That plane normal sections of the shaft remain plane after twisting.

(2) That stress is proportional to strain, *i. e.* all the stresses are within the elastic limit.

A line A B on the circumference of the shaft initially parallel to the axis becomes bent to the form A B′ as a result of the application of the torque, the end A being regarded for convenience as fixed. Then B B′ may be called the *arc of torsion* and it subtends at the centre O an angle θ called the *angle of torsion*. This angle of torsion may be regarded as a *torsional deflection*.

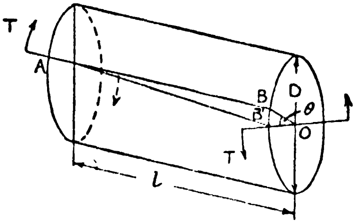


Fig. 141.—Torsion.

If we imagine the shaft divided up into a number of very small equal slices it follows that since each slice is exactly like every other slice the angle of twist in each slice will be equal, and if we regard each slice for convenience as of unit length we have

$$\text{Angle of twist per unit length of shaft} = \frac{\theta}{l}$$

Now consider the slice contained between sections X X and Y Y, Fig. 142, the thickness x being regarded as very small, and consider a square $a\,b\,c\,d$ of side x at distance r from the centre. In our figure bc is appreciably curved, but that is only because we cannot draw the figure clearly without making x of appreciable size.

The result of the twisting action is to make $abcd$ take up the form $ab'c'd$, this being the typical form (cf. Fig. 1) indicating pure shear strain, and the unital shear strain β is given by the relation

$$\beta = \frac{bb'}{x}$$

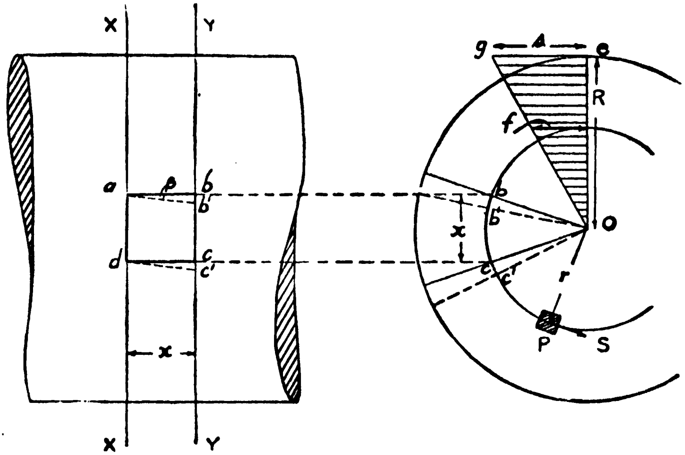


FIG. 142.—Torsion of Shafts.

$$\begin{aligned}\text{Shear stress} &= \text{shear modulus} \times \text{unital shear strain}\\ &= \beta G\\ &= \frac{bb'}{x}.G \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)\end{aligned}$$

$$\begin{aligned}\text{Now the angle } bob' &= \text{angle of twist in length } x\\ &= \text{angle of twist per unit length} \times x\\ &= \frac{\theta x}{l}\end{aligned}$$

$$\text{and } bb' = \text{arc} = \text{radius} \times \text{angle} = \frac{r\theta x}{l}$$

$$\begin{aligned}\therefore \text{ In (1) shear stress} &= \frac{r\theta x . G}{xl}\\ &= \frac{r\theta G}{l} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)\end{aligned}$$

This gives the important result that: *The shear stress at any point in a shaft is proportional to the distance of that point from the centre.*

The shear stress is therefore the same at all points on circles concentric with the shaft and the variation of shear stress is indicated by the triangle O $e\,g$, the stress at the extreme fibre being s and that at any other radius r being equal to

$$f = \frac{s\,r}{\mathrm{R}}$$

Now consider a very small element of area a at a point P in the section at distance r from the centre, the area being so small that the stress over it is constant.

Then from (2) stress on element $= \dfrac{r\,\theta\,\mathrm{G}}{l}$

$\therefore$ Force on element = S = stress × area

$$= \frac{r\,\theta\,\mathrm{G}}{l} \,.\, a$$

$\therefore$ Moment about O of force on element

$$= \mathrm{S} \times r$$

$$= \frac{r\,\theta\,\mathrm{G}}{l} \,.\, a \,.\, r$$

$$= \frac{\theta\,\mathrm{G}}{l} \,.\, a\,r^2$$

$\therefore$ Total moment about the axis of all the forces on the section

= Sum of separate moments

$$= \sum \frac{\theta\,\mathrm{G}}{l} \,.\, a\,r^2$$

$$= \frac{\theta\,\mathrm{G}}{l} \sum a\,r^2 \text{ because } \theta, \mathrm{G}, \text{ and } l \text{ are constant}$$

$$= \frac{\theta\,\mathrm{G}}{l} \times \text{Polar moment of Inertia of section}$$

$$= \frac{\theta\,\mathrm{G}\,\mathrm{I}_p}{l}$$

But the total moment about the axis of all the forces on the section must be equal to the twisting moment, so that we get

$$\mathrm{T} = \frac{\theta\,\mathrm{G}\,\mathrm{I}_p}{l} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

From (2) we get $\frac{\text{stress}}{r} = \frac{\theta G}{l}$

and from (3) we get $\frac{T}{I_p} = \frac{\theta G}{l}$

By combining these results we obtain the following complete relation for torsion which should be compared with the corresponding relation on p. 115 for the bending of beams.

$$\frac{\text{stress}}{r} = \frac{T}{I_p} = \frac{\theta G}{l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

In practical calculations we are usually concerned with the maximum shear stress s

∴ we write $\frac{s}{R} = \frac{T}{I_p}$

$$\therefore s = \frac{TR}{I_p} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

By analogy with the method of dealing with the section modulus of a beam we call $\frac{I_p}{R}$ the *polar modulus* Z_p.

We thus get $s = \frac{T}{Z_p}$

$$\text{or } T = f_s Z_p \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Cases in which the Formulæ are Applicable.—These formulæ are based upon the assumption that plane sections remain plane after twisting and this is true only for circular sections (solid and hollow); for shafts of other section the approximate formulæ given on p. 333 may be used.

SOLID CIRCULAR SECTION.—This is of course by far the most common case of shafting which occurs, and in this case

$$I_p = \frac{\pi D^4}{32}$$

$$\therefore Z_p = \frac{\pi D^4}{32} \div \frac{D}{2} = \frac{\pi D^3}{16}$$

$$\therefore T = s\frac{\pi D^3}{16} = \cdot 196\, s\, D^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Hollow Circular Section.—In this case, Fig. 143 (*a*),

$$I_p = \frac{\pi}{32}(D_1^4 - D_2^4)$$

$$\therefore Z_p = \frac{\pi(D_1^4 - D_2^4)}{16 D_1}$$

$$i.e.\ T = \frac{s\pi(D_1^4 - D_2^4)}{16 D_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

When the metal is very thin, Fig. 143 (*b*), we have

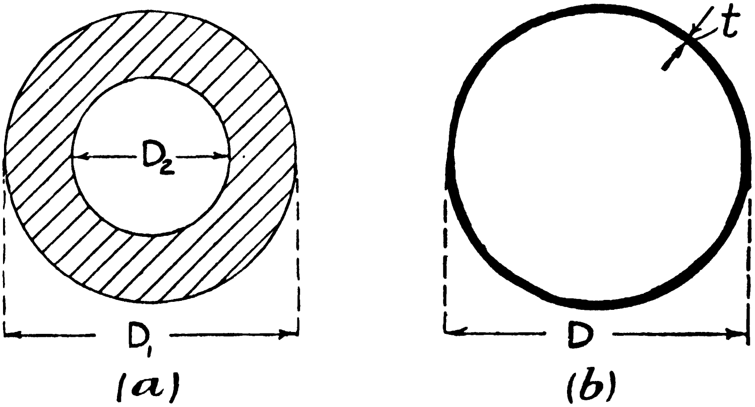


Fig. 143.

$I_p = \frac{\pi}{32}\{D^4 - (D-2t)^4\}$ and if t is so small that squares and higher powers of $\frac{t}{D}$ may be neglected we may write

$$(D-2t)^4 = D^4 - 8D^3 t$$

$$\therefore I_p = \frac{\pi D^3 t}{4}$$

$$i.e.\ Z_p = \frac{I_p}{\frac{D}{2}} = \frac{\pi D^2 t}{2}$$

$$\therefore T = \frac{s\pi D^2 t}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Alternative Derivation of Formula for Solid Shaft.—The formulæ (7) can also be derived as follows, and although we recommend the previous method as the more satisfactory we find by experience that some students find the alternative method more easy to follow. It is similar to the method which we have already used for beams in some cases (see pp. 220–222). Consider a very small sector A O B, Fig. 144, of the circle, so small that we may consider A O B as being practically a very narrow triangle. Set up A D, B C, to represent the maximum shear stress *s* and complete the pyramid A B C D O as shown. Then if this pyramid be considered as divided up into a number of slices as indicated, the volume of each slice = area of piece of sector × stress on it = load on each piece of sector; the volume of pyramid therefore represents the whole load acting on the sector, and to find the moment about the point O of the force on the sector we regard the volume of the pyramid as acting at the centre of gravity of the pyramid which is at distance $\frac{3R}{4}$ from O.

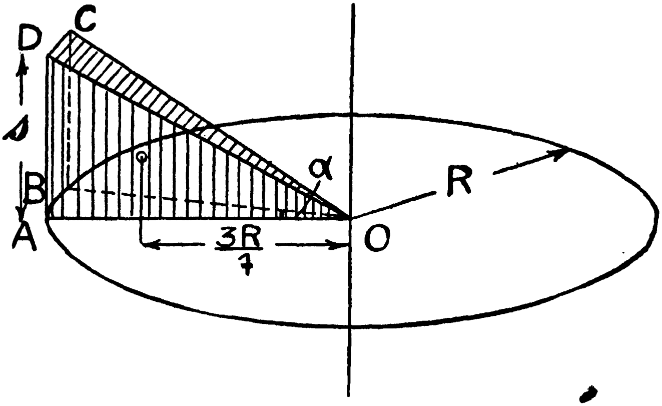


FIG. 144.

Now the volume of a pyramid = ⅓ area of base × height

$$= \frac{1}{3} . AB . AD \times OA$$

$$= \frac{1}{3} R\alpha . s . R = \frac{1}{3} s \alpha R^2$$

$$\therefore \text{ Moment about o} = \text{Volume} \times \frac{3\,R}{4}$$

$$= \frac{1}{3}\, s\, a\, R^2 \times \frac{3\,R}{4}$$

$$= \frac{s\, a\, R^3}{4}$$

Now in the whole section there will be $\frac{2\,\pi}{a}$ of these sectors, because the whole circumference subtends an angle $2\,\pi$ at the centre.

$\therefore$ Total moment about o of all the forces on the section

$$= \frac{s\, a\, R^3}{4} \times \frac{2\,\pi}{a}$$

$$= \frac{s\,\pi\, R^3}{2} = \frac{s\,\pi\, D^3}{16}$$

But this total moment must be equal to the torque T

$$\therefore T = \frac{s\,\pi\, D^3}{16}$$

This agrees with our previous result (equation (7)).

Horse-Power Transmitted by Shafting.

We have seen on p. 312 that

$$T = \frac{\text{H.P.} \times 33{,}000 \times 12}{2\,\pi\, N} \text{ in. lbs.}$$

$$= \frac{\text{H.P.} \times 33{,}000 \times 12}{2\,\pi\, N \times 2{,}240} \text{ in. tons.}$$

$\therefore$ putting this into equation (7) we have

$$\frac{s\,\pi\, D^3}{16} = \frac{\text{H.P.} \times 33{,}000 \times 12}{2\,\pi\, N}$$

$$\therefore D^3 = \frac{\text{H.P.} \times 33{,}000 \times 12 \times 16}{2\,\pi^2\, N\, s}$$

$$\text{This gives } D = 5{\cdot}23 \sqrt[3]{\frac{\text{H.P.}}{N\, s}} \text{ for } s \text{ in tons per in.}^2$$

$$= 68{\cdot}4 \sqrt[3]{\frac{\text{H.P.}}{N\, s}} \text{ for } s \text{ in lbs. per in.}^2$$

In using this formula it should be remembered that N is to be in revolutions per minute, the resulting diameter being in inches.

Taking s = 7,500 lbs. per sq. in. this gives

$$D = 3{\cdot}5\sqrt[3]{\frac{\text{H.P.}}{N}}$$

This is a very convenient formula for use. In practice s is often taken rather less than 7,500 lbs. per sq. in., the table given on p. 335 being often used; this is based upon s = 6,800 and is arranged to give round numbers for the 10 in. shaft.

Calculation of Angle of Twist.

From equation (3)

$$\theta = \frac{T\,l}{G\,I_p}$$

This, of course, will be in radians.

For solid shafts this gives

$$\theta = \frac{32\,T\,l}{G\,\pi\,D^4} \text{ radians} \quad \ldots\ldots\ldots\ldots\ldots (10)$$

$$= \frac{583\ Tl}{G\,D^4} \text{ degrees} \quad \ldots\ldots\ldots\ldots\ldots (11)$$

Comparison between Solid and Hollow Shafts.—Suppose that we have two shafts—one solid and of diameter D, and the other hollow and of external diameter D and internal diameter $\frac{D}{2}$.

Then I_p for solid shaft $= \frac{\pi\,D^4}{32}$

Then I_p for hollow shaft $= \frac{\pi}{32}\left\{D^4 - \left(\frac{D}{2}\right)^4\right\}$

$$= \frac{\pi}{32} \cdot \frac{15\,D^4}{16}$$

For the solid shaft we have

$$T_1 = \frac{s \times \pi\,D^3}{16}$$

and for the hollow shaft we have

$$T_2 = \frac{s \times 15\,D^4}{16} \times \frac{\pi}{32} \div \frac{D}{2}$$

$$= s \times \frac{15}{16} \cdot \frac{\pi\,D^3}{16}$$

$$\therefore \frac{T_2}{T_1} = \frac{15}{16}$$

The hollow shaft will therefore transmit $\frac{15}{16}$ of the torque of the solid shaft and therefore $\frac{15}{16}$ of the horse-power.

The area of the hollow shaft will be $\frac{\pi}{4}\left(D^2 - \frac{D^2}{4}\right) = \frac{3\pi D^2}{4}$, and of the solid shaft it will be $\frac{\pi D^2}{4}$; so that the area, and therefore the weight per given length of the hollow shaft, is $\frac{3}{4}$ of the corresponding value for the solid shaft.

Summing up our results, therefore, we may say that "the hollow shaft has a weight of $\frac{3}{4}$ that of the solid shaft and transmits $\frac{15}{16}$ of the horse-power; so that weight for weight the hollow shaft will be $\frac{15}{16} \times \frac{4}{3}$, *i. e.* $1\frac{1}{4}$ times as efficient as the solid shaft; the angle of torsion or torsional deflection will, however, be greater for the hollow shaft." This illustration brings out the fact that we can easily see from a consideration of the stress diagram that the material at the centre of a shaft is not used so effectively as that at the outside. This result agrees with that which we found for beams (p. 201).

NUMERICAL EXAMPLES ON CIRCULAR SHAFTING.—(1) *Find diameter of wrought-iron shaft to transmit* 90 *H.P. at* 130 *revolutions per minute if the working stress is to be* 5000 *lbs. per sq. in.*

In this case H.P. = 90, N = 130, and s = 5000

$\therefore$ Working from the general formulæ, which are much easier to remember than the particular one, we have

$$T = \frac{s \times \pi D^3}{16} = \frac{5000\,\pi D^3}{16}$$

$$\text{also } T = \frac{90 \times 33{,}000 \times 12}{2\pi \times 130}$$

$$\therefore D^3 = \frac{16 \times 90 \times 33{,}000 \times 12}{5000\,\pi \times 2\pi \times 130}$$

$$= 44\cdot45 \text{ ins.}^3$$

$$\therefore D = \sqrt[3]{44\cdot45}$$

$$= 3\cdot54 \text{ ins.}$$

Adopt $3\frac{1}{2}$ ins. diameter.

(2) *A steel shaft 4 in. in diameter is running at 130 revolutions per minute and is found to have a twist or "spring" of 9 degrees measured upon a length of 30 feet. What horse-power is being transmitted, taking G = 12×10^6 lbs. per sq. in., and what is the maximum stress in the shaft?*

Our general formula is

$$\frac{s}{R} = \frac{T}{I_p} = \frac{G\theta}{l}$$

In our case we are given the following—

$$R = 2; \quad I_p = \frac{\pi D^4}{32} = \frac{\pi \times 4^4}{32} = 8\pi$$

$$\theta = 9^\circ = \frac{9 \times \pi}{180} \text{ radians} = \frac{\pi}{20}$$

$$l = 30 \text{ ft.} = 360 \text{ ins.}; \ G = 12 \times 10^6 \text{ lbs. per sq. in.}$$

$$\therefore T = \frac{G\theta I_p}{l} = \frac{12 \times 10^6 \times \pi}{360 \times 20} \times 8\pi = \frac{2\pi^2 \times 10^5}{15} \text{ in. lbs.}$$

$$\text{but } T = \frac{\text{H.P.} \times 33{,}000 \times 12}{2\pi N}$$

$$\therefore \text{H.P.} = \frac{2\pi \times N \times 2\pi^2 \times 10^5}{33{,}000 \times 12 \times 15}$$

$$= \frac{4\pi^3 \times 130 \times 10^2}{33 \times 12 \times 15}$$

$$= \underline{271}$$

To test whether the stress is within safe limits we write

$$s = \frac{G\theta R}{l}$$

$$= \frac{12 \times 10^6}{360} \times \frac{\pi}{20} \times 2$$

$$= \underline{10{,}500 \text{ lbs. per sq. in. nearly.}}$$

(3) *What diameter of hollow shaft would you use to transmit 5000 H.P. at 60 revolutions per minute if the maximum torque is $1\frac{1}{4}$ times the mean and the safe shear stress is 7,500 lbs. per sq. in.? Take the internal diameter as half the external.*

$$\text{In this case mean torque} = \frac{\text{H.P.} \times 33{,}000 \times 12}{2\,\pi\,\text{N}}\ \text{in. lbs.}$$

$$= \frac{5000 \times 33{,}000 \times 12}{2\,\pi \times 60}\ \text{in. lbs.}$$

$$\text{Max. torque} = \text{T} = \frac{1{\cdot}5 \times 5000 \times 33{,}000 \times 12}{2\,\pi \times 60}\ \text{in. lbs.}$$

$$\text{But T} = \frac{s\,\pi\left\{\text{D}^4 - \left(\frac{\text{D}}{2}\right)^4\right\}}{16\,\text{D}} = \frac{7{,}500\,\pi}{16} \cdot \frac{15\,\text{D}^3}{16}$$

$$\therefore \text{D}^3 = \frac{1{\cdot}5 \times 5000 \times 33{,}000 \times 12 \times 256}{2\,\pi \times 60 \times 7{,}500\,\pi \times 15}$$

This gives D = 17·9 inches.

Adopt 18 inches external diameter.

*** Combined Bending and Torsion.**—In a large number of cases in practice shafts are subjected to bending as well as torsional stresses; a common example occurs in the case of a crank shaft and also in the bending stresses caused by the weight of the shafts themselves or by pulleys carried between the points of support.

Fig. 145 illustrates the action in an overhung crank shaft. The driving force P is applied at the point A at the crank pin and causes a twisting moment equal to P . A B = P r about the axis B O. In the vertical plane we have the couple formed of P at A and P at B in an opposite direction; to preserve equilibrium at B an equal and opposite force P must act which in the horizontal plane of B C combines with the reactionary force P at O. B O may therefore be regarded as a cantilever subjected to a bending moment equal to P × B C = P l. It is usual to assume the cantilever as extending to the centre of the bearing for the purpose of calculating the bending moment; it is difficult to obtain a much more accurate result until we know the distribution of the pressures upon the bearing. Our procedure for the calculation of the combined stresses is as follows—

First find the maximum bending moment M and the twisting moment T acting at any point along the length of the shaft and calculate the corresponding maximum tensile, compressive and shear forces contributed by the bending moment and twisting moment respectively. If the shear stress contributed from the consideration of the shaft as a beam is at all appreciable we should add this stress to the maximum shear stress given by the torsion. Let f and s be the maximum bending and shear stresses, then, as explained on p. 44, we have three alternative formulæ to apply, one of which gives the resultant shear stress and the other two the resultant tensile stress.

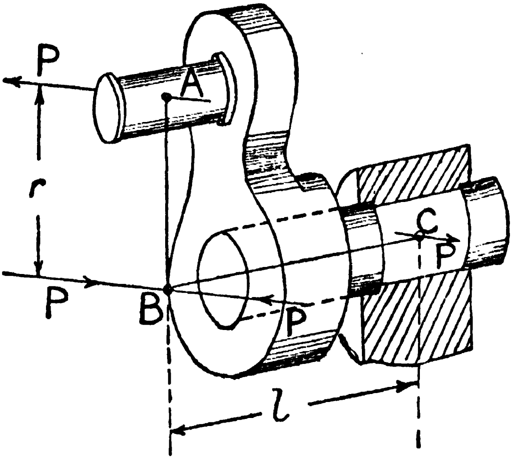


Fig. 145.

They are

(1) Rankine's formula—

$$\text{Equivalent tensile stress} = \frac{f}{2}\left(1 + \sqrt{1 + \frac{4s^2}{f^2}}\right)$$

(2) St. Venant's formula—

$$\text{Equivalent tensile stress} = \frac{f}{2}\left(\frac{3}{4} + \frac{5}{4}\sqrt{1 + \frac{4s^2}{f^2}}\right)$$

(3) Guest's formula—

$$\text{Equivalent shear stress} = \frac{f}{2}\sqrt{1 + \frac{4s^2}{f^2}}$$

$$= s\sqrt{1 + \frac{f^2}{4s^2}}$$

Equivalent Bending and Twisting Moments.—It is common to express these formulæ in terms of equivalent bending or twisting moments.

Now the polar modulus of a solid or hollow circular section is twice the ordinary or bending modulus of the same section.

For the solid circle $Z_p = \frac{\pi D^3}{16}$ and $Z = \frac{\pi D^3}{32}$

∴ we shall have

$$f = \frac{M}{Z}, \ s = \frac{T}{Z_p} = \frac{T}{2Z}$$

$$\therefore \frac{s}{f} = \frac{T}{2M}$$

∴ taking Rankine's formula we have, if M_E is the equivalent B.M.

$$\text{Equivalent tensile stress} = \frac{M_E}{Z} = \frac{M}{2Z}\left(1 + \sqrt{1 + \frac{T^2}{M^2}}\right)$$

$$\therefore M_E = \frac{M}{2}\left(1 + \sqrt{1 + \frac{T^2}{M^2}}\right) \quad \ldots\ldots(4a)$$

$$\text{or} = \frac{1}{2}\left(M + \sqrt{M^2 + T^2}\right) \quad \ldots\ldots(4b)$$

St. Venant's formula would give

$$M_E = \frac{M}{2}\left(\frac{3}{4} + \frac{5}{4}\sqrt{1 + \frac{T^2}{M^2}}\right) \quad \ldots\ldots(5a)$$

$$\text{or} = \frac{3M}{8} + \frac{5}{8}\sqrt{M^2 + T^2} \quad \ldots\ldots(5b)$$

Guest's formula will give

$$\text{Equivalent shear stress} = \frac{T_E}{Z_p} = \frac{M}{2Z}\sqrt{1 + \frac{T^2}{M^2}}$$

$$= \frac{M}{Z_p}\sqrt{1 + \frac{T^2}{M^2}}$$

$$\therefore T_E = M\sqrt{1 + \frac{T^2}{M^2}} \quad \ldots\ldots\ldots\ldots(6a)$$

$$\text{or} = \sqrt{M^2 + T^2} \quad \ldots\ldots\ldots\ldots\ldots(6b)$$

For ductile materials, such as steel, formulæ (6) are recommended for use in design, the safe shear stress being used for determining the necessary value of the polar modulus Z_p, to carry the twisting moment.

For brittle materials, such as cast iron, formulæ (4) or (5) should be used, the St. Venant formula being recommended as the more reliable of the two; the safe tensile stress should be taken in this case.

There has been considerable confusion with these formulæ because Rankine's formula is often given as an equivalent twisting moment and has been compared with Guest's formula for the same working stress; the point to keep in mind is that if Rankine's formula is used the safe tensile stress and bending modulus should be taken, but if Guest's formula is adopted the safe shear stress and polar modulus should be used.

NUMERICAL EXAMPLE.—*What must be the diameter of a solid shaft to transmit a twisting moment of* 160 *ft. tons and a bending moment of* 40 *ft. tons, the tensile stress being limited to* 4 *tons per sq. in.? What diameter would you use if the shear stress is limited to* 3 *tons per sq. in.?*

Let D be the diameter of the shaft.

Then
$$s = \frac{T}{\frac{\pi D^3}{16}} = \frac{16\,T}{\pi\,D^3}$$

$$f = \frac{M}{\frac{\pi D^3}{32}} = \frac{32\,M}{\pi\,D^3}$$

$$\therefore \frac{3}{4} + \frac{5}{4}\sqrt{1 + \frac{4s^2}{f^2}} = \frac{3}{4} + \frac{5}{4}\sqrt{1 + \frac{4 \times (16\,T)^2}{(32\,M)^2}}$$

$$= \frac{3}{4} + \frac{5}{4}\sqrt{1 + \left(\frac{T}{M}\right)^2}$$

$$= \frac{3}{4} + \frac{5}{4}\sqrt{1 + \left(\frac{160}{40}\right)^2}$$

$$= \cdot75 + \frac{5}{4}\sqrt{17} = 5{\cdot}9$$

$$\therefore \text{ Max. stress} = 4 = \frac{5{\cdot}9 f}{2} = \frac{5{\cdot}9}{2} \times \frac{32 \times 40 \times 12}{\pi D^3}$$

$$D^3 = \frac{5{\cdot}9 \times 32 \times 40 \times 12}{8\pi} = 3{,}600 \text{ nearly},$$

$$D = 15{\cdot}3 \text{ inches about.}$$

$$\text{Equivalent shear stress} = s\left(\sqrt{1 + \left(\frac{40}{160}\right)^2}\right) = \frac{16 T_E}{\pi D^3}$$

$$= \frac{16\sqrt{40^2 + 160^2}}{\pi D^3} = 3 = \frac{16 \times 40 \times 12 \times 4{\cdot}12}{\pi D^3}$$

$$D^3 = \frac{16 \times 40 \times 12 \times 4{\cdot}12}{3\pi}$$

$$D = 15 \text{ inches about.}$$

Shafts with Axial Pull or Thrust.—If in addition to the torsional stresses (and bending stresses if they occur) there exists an axial pull (P) or thrust Q we add $\frac{P}{A}$ or $\frac{Q}{A}$ to the bending stress to get the value of f to use in the formulæ. In the case of end thrust acting, when the direct stress is taken as a criterion, the resultant direct stress should not exceed the safe stress upon the shaft considered as a column, and the design should be treated similarly to that of a column with an eccentric load (p. 301). In the case of steel shafts, which are most common, we suggest that a double test should be applied; the shaft should be designed to carry the equivalent direct stress as a compressive stress on a column and also by the Guest formula, the diameter chosen being the greater of the two results.

Torsional Resilience.—As we have already explained in connection with resilience in bending, the work done by a couple is equal to the product of the couple into the angle turned through.

If, therefore, the angle turned through is θ, the work done by the couple which increases gradually from 0 to T is $\frac{T\theta}{2}$.

$$\frac{s}{R} = \frac{\theta G}{l} = \frac{T}{I},$$

$$\therefore \text{ Work stored in shaft} = \frac{T\theta}{2} = \frac{I_p . s}{2R} \cdot \frac{sl}{GR} = \frac{I_p s^2 l}{D . G \frac{D}{2}}$$

$$= \frac{2 I_p . s^2 l}{G D^2} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

For a solid shaft $\frac{2 I_p l}{D^2} = \frac{\pi D^2 l}{16}$

the volume of the shaft $= \frac{\pi D^2}{4} . l = V$

$$\therefore \frac{2 I_p l}{D^2} = \frac{V}{4}$$

i.e. Work stored in solid shaft $= \frac{s^2}{4G} \cdot V$

$\therefore$ Resilience = work stored in unit volume $= \frac{s^2}{4G} \ldots . (2)$

If we take $G = \frac{2}{5} E$ (cf. p. 12) this gives

$$\text{Resilience} = \frac{5s^2}{8E} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

For a hollow shaft of external diameter D and internal diameter D_1

$$\frac{2 I_p l}{D^2} = \frac{\pi (D^4 - D_1^4) . l}{16 D^2} = \frac{\pi (D^2 + D_1^2)(D^2 - D_1^2) l}{16 D^2}$$

and $V = \frac{\pi (D^2 - D_1^2) l}{4}$

$$\therefore \frac{2 I_p l}{D^2} = \left(\frac{D^2 + D_1^2}{D^2}\right) \cdot \frac{V}{4}$$

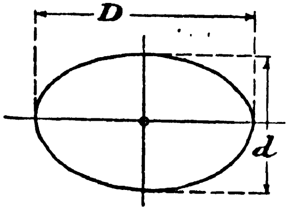
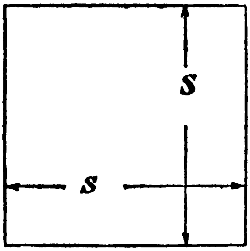
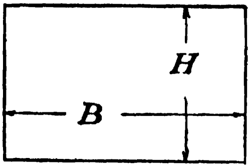
$$= \frac{V}{4}\left\{1 + \left(\frac{D_1}{D}\right)^2\right\}$$

$$\therefore \text{ Resilience} = \frac{s^2}{4G}\left\{1 + \left(\frac{D_1}{D}\right)^2\right\} \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

In the limiting case of a very thin tube $\frac{D_1}{D}$ approaches 1 and we have

$$\text{Resilience} = \frac{s^2}{2G} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Torsion of Non-Circular Shafts.—As we have already indicated on p. 320, the ordinary theory of torsion is true only for circular sections because in other sections the sections originally plane become bent out of the plane upon twisting. The following cases have been worked out fully by St. Venant, who has given the following approximate formulæ—

Section.	Relation of Maximum Stress to Twisting Moment.	Angles of Torsion in Degrees.
	$T = s . \frac{\pi D d^2}{16}$	$\theta = \frac{292 T l (d^2 + D^2)}{G D^3 d^3}$
	$T = s . \cdot 208 S^3$	$\theta = \frac{410 T l}{G S^4}$
	$T = \frac{s B H^2}{3 + 1{\cdot}8 \frac{H}{B}}$	$\theta = \frac{205 T l (B^2 + H^2)}{G B^3 H^3}$
Any symmetrical section not containing re-entrant angles.	$T = \frac{s A^4}{40 I_p . y}$	$\theta = \frac{40 I_p T l}{A^4 G}$
	Where A = area of section I_p = polar moment of Inertia y = distance of farthest edge from centre of shaft	

Numerical Example.—*A square steel shaft is required for transmitting power to a* 30-*ton overhead travelling crane. The load is lifted at a rate of* 40 *ft. per minute. Taking the mechanical* *efficiency of the crane gearing as* 35 %, *calculate the necessary size of shaft to run at* 160 *revolutions per minute. The twist must not exceed* 1° *in a length equal to* 30 *times the side of the square. Take* $G = 13 \times 10^6$ *lbs. per sq. in.*

Work per minute to lift weight $= 30 \times 40$ ft. tons.

If η of crane $= 35$ %.

Work to be supplied per minute $= \dfrac{30 \times 40 \times 100}{35}$

∴ If revolutions per minute $= 160$

$$\text{Torque} = \frac{30 \times 40 \times 100}{35 \times 2\pi \times 160} \text{ ft. tons} = T$$

$$= 3 \cdot 42 \text{ ft. tons.}$$

$$\therefore 1 = \frac{410 \times 3 \cdot 42 \times 2{,}240 \times 12 \times 30 S}{13 \times 10^6 . S^4}$$

$$S^3 = \frac{410 \times 3 \cdot 42 \times 2{,}240 \times 12 \times 30}{13 \times 10^6}$$

$$S = 4\tfrac{1}{2} \text{ inches (say).}$$

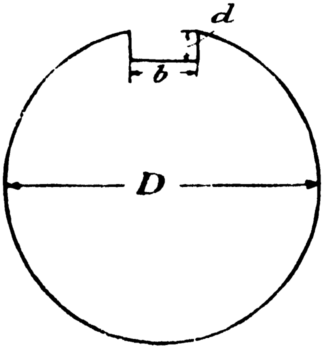


FIG. 140.

Effect of Keyways upon the Torsion of a Circular Shaft.—Professor H. F. Moore of Illinois University has given the following results of experimental investigations upon shafts with keyway grooves cut in them.

$$\frac{\text{Strength of cut shaft}}{\text{Strength of uncut shaft}} = 1 - \frac{\cdot 2 b}{D} - \frac{1 \cdot 1 d}{D}$$

$$\frac{\text{Angle of torsion of cut shaft}}{\text{Angle of torsion of uncut shaft}} = 1 + \frac{\cdot 4 b}{D} + \frac{\cdot 7 d}{D} .$$

HORSE-POWER TRANSMITTED BY STEEL SHAFTING

Revolutions per minute.	Diameter of Shaft in Inches.																		
	1½	1¾	2	2¼	2½	2¾	3	3½	4	4½	5	5½	6	6½	7	7½	8	9	10
50	3·3	5·3	8·0	10·9	15·6	20·8	27	43	64	91	125	166	216	275	343	422	512	729	1000
60	4·0	6·4	9·6	13·1	18·8	25·0	32	51	77	109	150	200	259	330	412	506	614	875	1200
70	4·7	7·5	11·2	15·2	21·9	29·1	38	60	89	128	175	233	302	385	480	591	717	1021	1400
80	5·4	8·5	12·8	17·4	25·0	33·3	43	69	102	146	200	266	346	439	549	675	819	1166	1600
90	6·0	9·6	14·4	19·6	28·1	37·4	49	77	115	164	225	299	389	494	617	759	922	1312	1800
100	6·7	10·7	16·0	21·8	31·2	41·6	54	86	128	182	250	333	432	549	686	844	1024	1458	2000
110	7·4	11·8	17·6	23·9	34·4	45·8	59	94	141	200	275	366	475	604	755	928	1126	1604	2200
120	8·1	12·9	19·2	26·1	37·5	49·9	65	103	154	219	300	399	518	659	823	1012	1229	1750	2400
130	8·7	13·9	20·8	28·3	40·6	54·1	70	111	166	237	325	433	562	714	892	1097	1331	1895	2600
140	9·4	15·0	22·4	30·5	43·8	58·2	76	120	179	255	350	466	605	769	960	1181	1434	2041	2800
150	10·1	16·1	24·0	32·6	46·9	62·4	81	129	192	273	375	499	648	824	1029	1265	1536	2187	3000
160	10·8	17·1	25·6	34·8	50·0	66·5	86	137	205	292	400	532	691	879	1097	1350	1638	2333	3200
170	11·5	18·2	27·2	37·0	53·1	70·7	92	146	218	310	425	566	734	934	1166	1434	1741	2479	3400
180	12·2	19·3	28·8	39·2	56·3	74·9	97	154	230	328	450	599	778	989	1235	1519	1843	2624	3600
190	12·8	20·4	30·4	41·3	59·4	79·0	103	163	243	346	475	632	821	1044	1303	1603	1945	2770	3800
200	13·5	21·4	32·0	43·5	62·5	83·2	108	172	256	365	500	665	864	1099	1372	1687	2048	2916	4000
225	15·2	24·1	36·0	49·0	70·3	93·6	122	193	288	410	563	749	972	1236	1543	1898	2304	3280	4500
250	16·9	26·8	40·0	54·4	78·1	104·0	135	214	320	456	625	832	1080	1373	1715	2109	2560	3645	5000
275	18·6	29·5	44·0	59·8	85·9	114·4	149	236	352	501	688	915	1188	1510	1886	2320	2816	4009	5500
300	20·3	32·2	48·0	65·3	93·7	124·8	162	257	384	547	750	998	1296	1648	2058	2531	3072	4374	6000
325	21·9	34·8	52·0	70·7	101·6	135·2	176	279	416	592	813	1081	1404	1785	2229	2742	3328	4739	6500
350	23·6	37·5	56·0	76·2	109·4	145·6	189	300	448	638	875	1165	1512	1922	2401	2953	3584	5103	7000
375	25·3	40·2	60·0	81·6	117·2	156·0	203	322	480	683	938	1248	1620	2060	2572	3164	3840	5468	7500
400	27·0	42·9	64·0	87·0	125·0	166·4	216	343	512	729	1000	1331	1728	2197	2744	3374	4096	5832	8000
425	28·7	45·6	68·0	92·5	132·8	176·8	230	364	544	775	1063	1414	1836	2334	2915	3585	4352	6197	8500
450	30·4	48·2	72·0	97·9	140·6	187·2	243	386	576	820	1125	1497	1944	2472	3087	3796	4608	6562	9000
475	32·1	50·9	76·0	103·4	148·4	197·6	257	407	603	866	1188	1580	2052	2609	3258	4007	4864	6926	9500
500	33·7	53·6	80·0	108·8	156·2	208·0	270	429	640	911	1250	1684	2160	2746	3430	4218	5120	7290	10000

For Power of Wrought-iron Shafts take 70 per cent. of above Values.

CHAPTER XII

SPRINGS

SPRINGS may be regarded as devices for storing up energy in the form of resilience and are used either as a storage for energy, as in clocks, phonographs, etc., or else are used in order to absorb excess energy which would otherwise do damage, buffer and carriage springs being very common examples.

The best form of spring will be that which will absorb the greatest amount of energy for a given stress, and in the case of springs placed upon trains it should be remembered that the kinetic energy of the spring itself will add to the energy that has to be absorbed. Failure to appreciate this fact has led many people to suggest that railway collisions could be prevented by the use of heavy springs placed in front, whereas the weight of spring necessary to do this would be so great that its advantage would be lost.

Since all elastic bodies have resilience, all forms such as ties, struts, beams and shafts are strictly springs, but in ordinary form they are too stiff to be of use. The principal forms of springs may be divided into torsion springs and bending springs.

We can consider the relative values of tensile, bending and torsion springs by comparing the relative resiliences. For a beam of uniform strength loaded at the centre such as occurs in a leaf spring we have

$$\text{Resilience} = \frac{f^2}{6\,E}$$

For tension we have

$$\text{Resilience} = \frac{f^2}{2\,E}$$

For torsion we have

$$\text{Resilience} = \frac{s^2}{4\,G} = \frac{5\,s^2}{8\,E}$$

Taking $s = \frac{4}{5}\,f$ this would give

$$\text{Resilience} = \frac{2\,f^2}{5\,E}$$

or, on the Guest theory, if $s = \frac{f}{2}$ we shall have

$$\text{Resilience} = \frac{5\,f^2}{32\,E}$$

The pure tension spring, therefore, which is practically never used in practice, is the most economical, and the relative economy of the torsion and bending springs depends upon the view taken as to the relative safe stresses for the two cases.

Time of Vibration of a Spring.—The vibration of a spring follows the laws of simple harmonic motion, so that the following general formula will enable the time of vibration to be obtained.

Time of complete vibration

$$= t = 2\,\pi\sqrt{\frac{\text{Weight of spring}}{g \times \text{Force to cause unit deflection}}} \quad \ldots(1)$$

$\therefore$ number of complete vibrations per second $= \frac{1}{t}$

TORSION SPRINGS

Close-Coiled Helical Springs.—If a helical spring is so closely coiled that each turn is practically a plane, the stresses upon the material will be almost pure torsion. The twisting moment, Fig. 147, will be W R and the spring will be equivalent to a shaft of diameter d and of length l equal to the total length of wire in the spring, *i. e.* if n are the number of turns, $l = 2\,\pi\,R\,n$ (approx.); the torque applied to this equivalent shaft will be W R as indicated.

By our general torsion formula we have

$$\frac{s}{\frac{d}{2}} = \frac{T}{I_p} = \frac{G\,\theta}{l}$$

$$\therefore\ \theta = \frac{l\,T}{G\,I_p} = \frac{l\,W\,R}{\frac{G\,\pi\,d^4}{32}}$$

$$= \frac{32\,l\,W\,R}{G\,\pi\,d^4} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

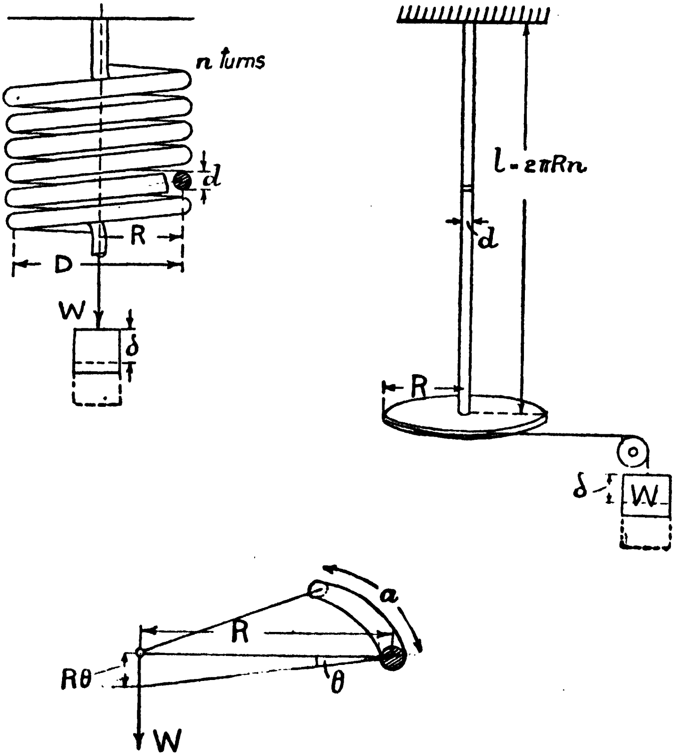


Fig. 147.—Close-coiled Helical Springs.

Now consider a very short length *a* of the spring and suppose that *θ* is the angle of torsion of this short length; then due to this angle *θ* the weight W will go down by an amount R *θ* as indicated in Fig. 147. This is true for every short length and the separate deflections due to each piece add together giving a total deflection

$$\delta = R\,\theta = \frac{32\,W\,R^2\,l}{G\,\pi\,d^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3a)$$

$$= \frac{8\,W\,D^2\,l}{G\,\pi\,d^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3b)$$

If n is the number of turns we have $l = 2\,\pi\,R\,n$

$$\therefore\ \delta = \frac{64\,\pi\,R\,n\,W\,R^2}{G\,\pi\,d^4}$$

$$= \frac{64\,R^3\,n}{G\,d^4}\,.\,W \quad \ldots\ldots\ldots\ldots\ldots\ldots (4a)$$

$$= \frac{8\,D^3\,n}{G\,d^4}\,.\,W \quad \ldots\ldots\ldots\ldots\ldots\ldots (4b)$$

∴ the load W to cause a deflection δ is given by

$$W = \frac{G\,d^4}{64\,R^3\,n}\,.\,\delta \quad \ldots\ldots\ldots\ldots\ldots\ldots (5a)$$

$$= \frac{G\,d^4}{8\,D^3\,n}\,.\,\delta \quad \ldots\ldots\ldots\ldots\ldots\ldots (5b)$$

We might have obtained our result as follows from the consideration of the torsional resilience (p. 331)—

$$\text{Torsional resilience} = \frac{s^2}{4\,G}$$

$$\therefore\ \text{Work absorbed} = \frac{s^2}{4\,G} \times \text{volume} = \text{Work done by weight}$$

$$= \frac{W\,\delta}{2}$$

$$s = \frac{16\,W\,R}{\pi\,d^3}\,;\ V = \frac{\pi\,d^2\,l}{4}$$

$$\therefore\ \frac{W\,\delta}{2} = \frac{16^2\,W^2\,R^2\,.\,\pi\,d^2\,l}{\pi^2\,d^6\quad 4\,G\,.\,4}$$

$$= \frac{16\,W^2\,R^2\,l}{\pi\,d^4\,G}$$

$$\delta = \frac{32\,W\,R^2\,l}{\pi\,d^4\,G} \quad \text{as before.}$$

Time of Vibration.—Putting $\delta = 1$ in equation (5) we get the force to cause unit deflection; this can then be put into equation (1) to find the time of swing, remembering that if all the other dimensions are in inch units, g (the gravity acceleration) must be also reduced to inch units.

NUMERICAL EXAMPLES ON CLOSELY COILED SPRINGS.—(1) *A closely wound helical spring is formed of* 30 *coils of* ¼ *in. round wire, the mean diameter of the coil being* 4 *ins. What axial load will produce a shear stress of* 9 *tons per sq. in., and if* G *is* 4,700 *tons per sq. in., what will be the extension of the spring under the load?*

$$T = W \times 2 = \frac{s \times \pi d^3}{16}$$

$$\therefore\ 2\,W = \frac{9\,\pi \times (\tfrac{1}{4})^3}{16}$$

$$W = \frac{9\,\pi}{32 \times 64} \text{ tons}$$

$$= \frac{9\,\pi \times 2{,}240}{32 \times 64} \text{ lbs.}$$

$$= \underline{30{\cdot}9 \text{ lbs.}}$$

$$\delta = \frac{64\,W\,R^3\,n}{G\,d^4}$$

$$= \frac{64 \times 30{\cdot}9 \times 30 \times 8 \times 256}{4{,}700 \times 2{,}240} = \underline{11{\cdot}57 \text{ ins.}}$$

Before this extension occurred the spring would have ceased to be closely wound if the load were such as to stretch the spring, and the spring would have been fully closed if the load were such as to compress the spring.

(2) *If a closely wound helical spring made of wire* ¼ *in. in diameter has* 10 *coils, each* 4 *ins. mean diameter, find the frequency of the free vibrations when it carries a load of* 15 *lbs (taking* $G = 12 \times 10^6$ *lbs. per sq. in.).*

$$t = 2\,\pi\sqrt{\frac{\text{Weight}}{g \times \text{Force to cause unit displacement}}}$$

Putting $\delta = 1$ in equation (5*b*) we have

$$\text{Force to cause unit displacement} = \frac{G\,d^4}{8\,D^3\,n}$$

$$= \frac{(\tfrac{1}{4})^4 \times 12 \times 10^6}{8 \times 4^3 \times 10} = 9{\cdot}156 \text{ lbs.}$$

$$\therefore\ t = 2\,\pi\sqrt{\frac{15}{32 \times 12 \times 9{\cdot}156}} \text{ seconds}$$

$$\therefore\ \text{Frequency} = \frac{1}{t} = \frac{1}{2\pi}\sqrt{\frac{32 \times 12 \times 9{\cdot}156}{15}}$$

$$= \underline{2{\cdot}43 \text{ per second.}}$$

It should be noted that in the above calculation we have neglected the weight of the spring itself, so that the result can only be regarded as approximate.

Weight of Springs.—A cubic inch of steel weighs about ·284 lb.

$$\begin{aligned} \therefore \text{Weight of spring} &= \cdot 284 \times \text{volume} \\ &= \cdot 284 \times \frac{\pi d^2 l}{4} \\ &= \cdot 284 \times \frac{\pi d^2 \pi D n}{4} \\ &= \cdot 7 d^2 D n \text{ very nearly.} \end{aligned}$$

Safe Loads on Circular Springs.—The following are the highest safe torsion stresses upon steel spring wire found by experiments by Mr. Wilson Hartnell—

Diameter of Wire.	Safe Stress in lbs. per sq. inch.	Safe Load on Spring (lbs.)
$\frac{1}{4}$	70,000	$\frac{429}{D}$
$\frac{3}{8}$	60,000	$\frac{1{,}240}{D}$
$\frac{1}{2}$	50,000	$\frac{2{,}450}{D}$

D = mean diameter of spring in inches.

$$\text{From the formula } T = \cdot 196\, d^3 s = \frac{W D}{2}$$

we have $W = \frac{2 \times \cdot 196\, s\, d^3}{D}$ from which the values in the third column of the table are obtained.

Restriction on Use of Formulæ.—When the springs are used so that when stressed they are shorter than when unloaded (*i. e.* if the load were to act upwards in Fig. 147), it should be remembered that the spring may become shut before the safe load has been reached; this does not diminish its strength, but impairs its value as a spring.

Alignment Charts for Close-coiled Circular Springs.

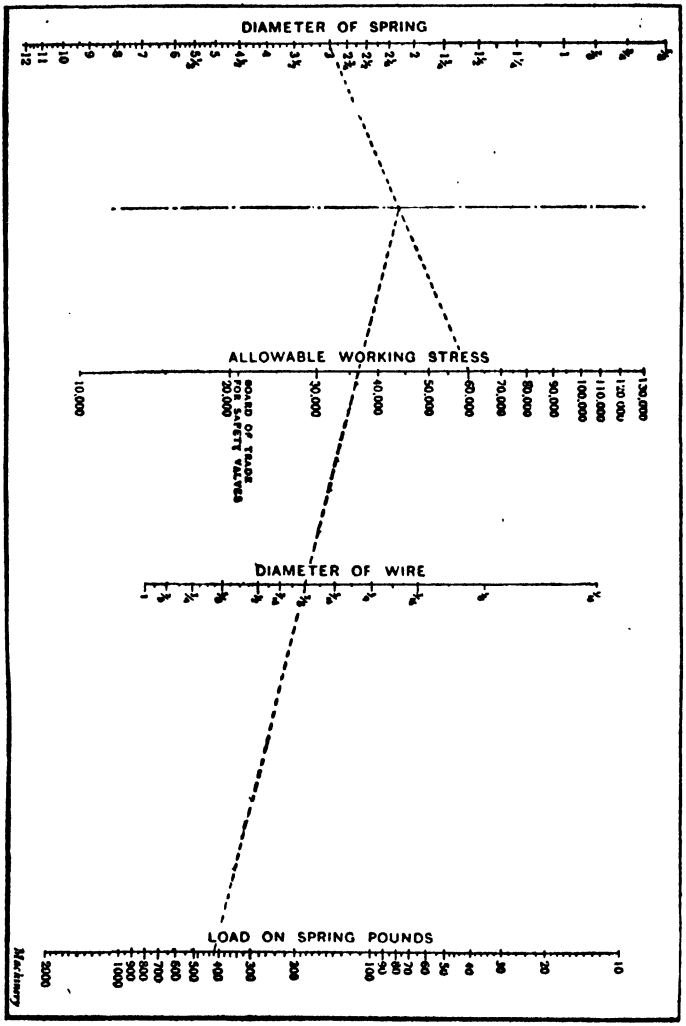


Chart for Strength of Round Wire Helical Springs

FIG. 148.

—Figs. 148, 149 show alignment charts * for the strength and

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* For an explanation of the principle of these charts see a booklet by the author on *Alignment Charts*, published by Messrs. Chapman and Hall, Ltd. [Price 1/3 net.]

deflection of round wire helical springs, taken from an article by Mr. F. Fitchett in *Machinery* for January 14, 1915.

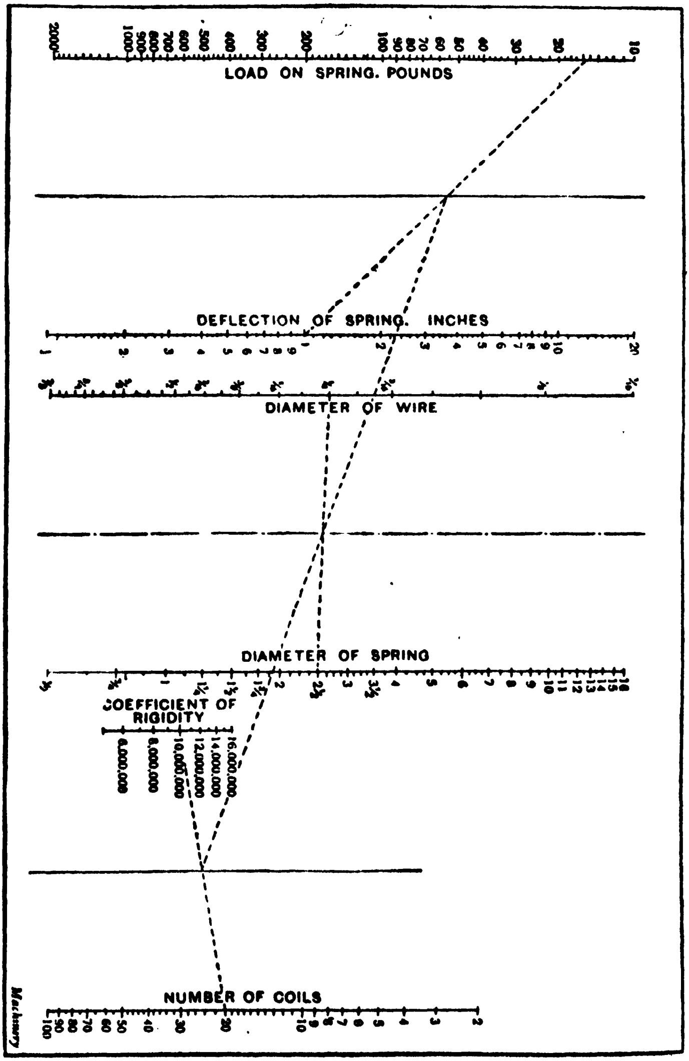

Chart for Deflection of Round Wire Helical Springs

Fig. 149.

To use the charts we proceed as follows: Suppose, for instance, mean diameter of coil = 3 in., diameter wire = $\frac{3}{8}$ in., safe stress = 60,000 lbs. per sq. in. On Fig. 148 connect "diameter of spring" to "safe working stress" and note the intersection on the dotted axis. Connect this point of axis through "diameter of wire" on to "safe load," which will give the result required, viz. 415 lbs. approximately. The chart can be used similarly to find any one of the four quantities if the other three are given.

To use the deflection chart of Fig. 149 take, for example, a load of 16 lbs., $\frac{1}{4}$ in. diameter wire, $2\frac{1}{2}$ in. diameter of spring consisting of 20 coils, using $G = 10{,}000{,}000$ lbs. per sq. in. Connect "number of coils" to "coefficient of rigidity" and note intersection on the right-hand intersection or "support line." Then join "diameter of wire" to "diameter of spring" and note intersection on centre support. Join these two intersections to meet the left-hand support and connect the point thus obtained to "load on spring" and produce to the "deflection" which gives 1 inch approx.

Springs of Square Section Wire.—If our spring is like that shown in Fig. 147 with the exception that the wire is square instead of round in section, the length of each side being S, we can proceed as follows—

From p. 333
$$\theta = \frac{410\,T\,l}{G\,S^4} \text{ (degrees)}$$
$$= \frac{7{\cdot}11\,T\,l}{G\,S^4} \text{ (radians)}$$
$$\text{and } T = s\,{\cdot}208\,S^3$$

Now
$$T = W\,R = \frac{W\,D}{2}$$
$$\therefore W = \frac{s \times {\cdot}208\,S^3}{\frac{D}{2}} = \frac{{\cdot}416\,s\,S^3}{D} \quad \ldots\ldots\ldots(1)$$
$$\delta = R\,\theta = \frac{7{\cdot}11\,R \times W\,R\,l}{G\,S^4}$$
$$= \frac{7{\cdot}11\,W\,R^2\,l}{G\,S^4}$$
$$= \frac{1{\cdot}78\,W\,D^2\,l}{G\,S^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

putting $l = \pi D . n$

$$\delta = \frac{1{\cdot}78\,\pi\,W\,D^3\,n}{G\,S^4}$$

$$= \frac{5{\cdot}6\,W\,D^3\,n}{G\,S^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Taking the same safe stresses as for round wire we get from formula (1) the following formulæ for the safe loads on square wire springs—

Side of Square (in.)	Safe Load in lbs.
$\frac{1}{4}$	$\frac{455}{D}$
$\frac{3}{8}$	$\frac{1316}{D}$
$\frac{1}{2}$	$\frac{2660}{D}$

COMPARISON OF SQUARE SECTION AND CIRCULAR SECTION SPRINGS.—The volume of the square spring $= S^2 l = V$

$$\therefore \text{ Work absorbed } = \frac{T\,\theta}{2}$$

$$= \frac{{\cdot}208\,s\,S^3}{2} \cdot \frac{7{\cdot}11\,T\,l}{G\,S^4}$$

$$= \frac{{\cdot}208\,s\,S^3}{2} \times \frac{7{\cdot}11 \times {\cdot}208\,s\,S^3\,l}{G\,S^4}$$

$$= {\cdot}154\,\frac{s^2\,S^2\,l}{G}$$

$$= \frac{{\cdot}154\,s^2}{G} . V$$

$$\therefore \text{ Resilience} = \frac{{\cdot}154\,s^2}{G}$$

This is less than for a solid circular spring, the ratio of circular to square resilience being $\frac{{\cdot}25}{{\cdot}154} = 1{\cdot}62$. Also for a spring of given diameter the load carried is greater for a circular section than for a square section of the same area.

For the circular section we have

$$T_c = \frac{s \times \pi d^3}{16} = \frac{s\,d}{4} \times \text{area}$$

$$\text{for square } T_s = \cdot 208\, s\, S^3 = \cdot 208\, s\, S \times \text{area}$$

$$\therefore \frac{T_c}{T_s} = \frac{d}{4 \times \cdot 208\, S}$$

If $\frac{\pi d^2}{4} = S^2$, *i.e.* if the areas are equal as suggested

$$d = \sqrt{\frac{4}{\pi}}\,.\,S = 1{\cdot}128\,S$$

$$\therefore \frac{T_c}{T_s} = \frac{1{\cdot}128}{4 \times \cdot 208} = 1{\cdot}36.$$

Weight for weight, therefore, a closely coiled circular section helical spring of given diameter is 1·36 *times as strong and will absorb* 1·62 *times as much energy as one of square section of the same diameter.*

Open-Coiled Helical Springs.—In the case of open-coiled helical springs the stress is principally a torsioned one, so that we will deal with it here although there is also bending stress.

Referring to Fig. 150, let the centre line of the spring at any point be inclined at an angle a to the horizontal; then the normal section plane x x of the wire will be at an angle a to the vertical load W. The load W has a moment W R about the line centre o of the section and this moment has a component o a = W R cos a which is a moment in the plane x x and causes a twisting action, and a component a b = W R sin a which is a moment normal to the plane x x and causes a bending action.

$$\left.\begin{aligned} \therefore T &= W R \cos a \\ M &= W R \sin a \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

It is not altogether easy to follow this resolution into a twisting and a bending moment at first, partly because it is not very easy to give a very clear diagram with the forces acting in different planes, but a little consideration of the problem will probably remove the difficulty.

The angle of torsion in the plane x x will be given by

$$\theta = \frac{T\,l}{G\,I_p}$$

$\therefore$ Work done against torsional stress $= \frac{T\,\theta}{2} = \frac{T^2\,l}{2\,G\,I_p}$

$$= \frac{W^2\,R^2\cos^2\alpha\,l}{2\,G\,I_p} \quad \text{..................(2)}$$

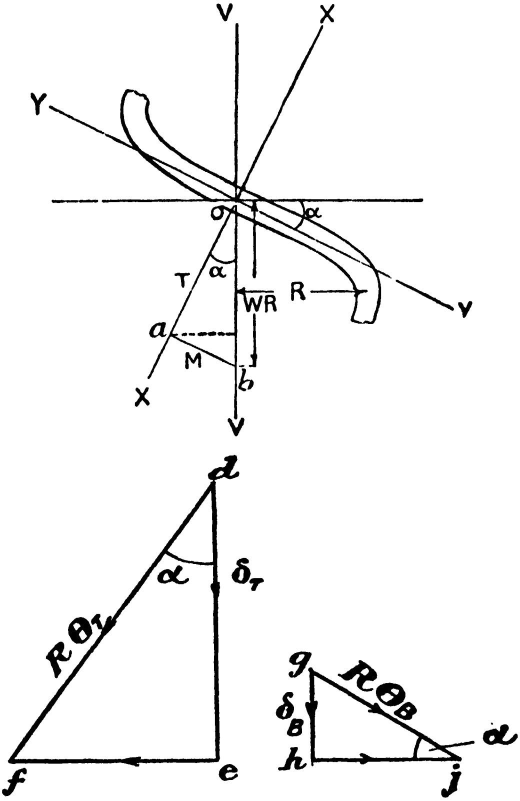

FIG. 150.—Open-coiled Springs.

The bending moment is constant, so that (see p. 273)

Work done against bending stress $= \frac{M^2\,l}{2\,E\,I}$

$$= \frac{W^2\,R^2\sin^2\alpha\,l}{2\,E\,I} \quad \text{..................(3)}$$

$\therefore$ Total work done against stress

$$= \frac{W^2 R^2 l}{2}\left(\frac{\cos^2 a}{G I_p} + \frac{\sin^2 a}{E I}\right) \quad \ldots\ldots\ldots\ldots (4)$$

But total work done against stress = Work done by weight $= \frac{W \delta}{2}$.

$$\therefore \delta = W R^2 l\left(\frac{\cos^2 a}{G I_p} + \frac{\sin^2 a}{E I}\right) \quad \ldots\ldots\ldots\ldots (5)$$

Taking a solid circular section we have $2 I = I_p = \frac{\pi d^4}{32}$

$$\therefore \delta = \frac{32 W R^2 l}{\pi d^4}\left(\frac{\cos^2 a}{G} + \frac{2 \sin^2 a}{E}\right)$$

$$= \frac{32 W R^2 l}{G \pi d^4}\left(\cos^2 a + \frac{2 \sin^2 a}{\frac{E}{G}}\right)$$

$$\left(\text{and taking } \frac{E}{G} = 2{\cdot}5\right)$$

$$= \frac{32 W R^2 l}{G \pi d^4}(\cos^2 a + {\cdot}8 \sin^2 a) \quad \ldots\ldots\ldots\ldots (6)$$

$$= \frac{8 W D^2 l}{G \pi d^4}(\cos^2 a + {\cdot}8 \sin^2 a)$$

$$= \frac{8 W D^2 l}{G \pi d^4}(1 - {\cdot}2 \sin^2 a) \quad \ldots\ldots\ldots\ldots (7)$$

The movement due to twisting per unit length of the coil will be equal to R θ_T, where θ_T is the angle of torsion, and takes place in the plane X X. This is equivalent to a movement *d e* vertically and a horizontal movement *e f*.

$$\therefore \delta_T = R\, \theta_T \cos a \quad \ldots\ldots\ldots\ldots (8)$$

$$e f = R\, \theta_T \sin a \quad \ldots\ldots\ldots\ldots (9)$$

This movement *e f* tends to increase the number of the turns of wire.

The bending deflection upon a unit length will be equal to R θ_B, where θ_B is the angular change in the centre line of the beam. This movement takes place in the plane Y Y and will tend to unwind the coil; it has a vertical component *g h* which is the part of the deflection contributed by the bend ing and a horizontal component hj which is an unwinding tendency and opposes the winding-up strain ef.

$$\therefore\ \delta_B = R\,\theta_B \,.\sin\alpha$$

$$hj = -R\,\theta_B\cos\alpha$$

$$\text{Now } \theta_T = \frac{T}{G\,I_p} \text{ per unit length}$$

$\theta_B = \dfrac{M}{E\,I}$ per unit length because the bending moment is constant.*

$\therefore$ per unit length

$$\text{deflection} = \delta_T + \delta_B = \frac{R\,T\cos\alpha}{G\,I_p} + \frac{R\,M\sin\alpha}{E\,I}$$

$$= \frac{W\,R^2\cos^2\alpha}{G\,I_p} + \frac{W\,R^2\sin^2\alpha}{E\,I}$$

$$= W\,R^2\left(\frac{\cos^2\alpha}{G\,I_p} + \frac{\sin^2\alpha}{E\,I}\right)$$

The deflection will be the same for each unit of length—

$$\therefore\ \delta = l\,(\delta_T + \delta_B) = W\,R^2\,l\left(\frac{\cos^2\alpha}{G\,I_p} + \frac{\sin^2\alpha}{E\,I}\right)$$

This agrees with our equation (5) obtained in a different manner.

Angular winding-up movement per unit length

$$= \frac{ef - hj}{R}$$

$$= \theta_T\sin\alpha - \theta_B\cos\alpha$$

$$= W\,R\left(\frac{\sin\alpha\cos\alpha}{G\,I_p} - \frac{\sin\alpha\cos\alpha}{E\,I}\right)$$

$\therefore$ Total winding-up movement

$$= \beta = W\,R\,l\sin\alpha\cos\alpha\left(\frac{1}{G\,I_p} - \frac{1}{E\,I}\right)$$

$$= \frac{32\,W\,R\,l\sin\alpha\cos\alpha}{\pi\,d^4\,G}\left(1 - \frac{2\,G}{E}\right) \quad \text{..........(10)}$$

for a solid circular section

$$= \frac{32\,W\,R\,l\sin\alpha\cos\alpha}{5\,\pi\,d^4\,G} \text{ for } \frac{G}{E} = \frac{2}{5}$$

---

* See p. 249, and note in Fig. 121 that $\theta = \dfrac{C\,C'}{R} = \dfrac{1}{R}$ if C C′ is unity

$$\therefore\ \theta = \frac{1}{R} = \frac{M}{E\,I}.$$

This is a maximum for $a = 45°$.

If p is the pitch of the coil and n is the number of turns

$$l = n\sqrt{p^2 + \pi^2 D^2}$$

Comparison of Close-coiled and Open-Coiled Springs.—Comparing result (7) with equation (3*b*), p. 339, for the close-coiled spring, we have

$$\frac{\delta \text{ for open-coiled spring}}{\delta \text{ for close-coiled spring}} = (1 - \cdot 2 \sin^2 a) = m$$

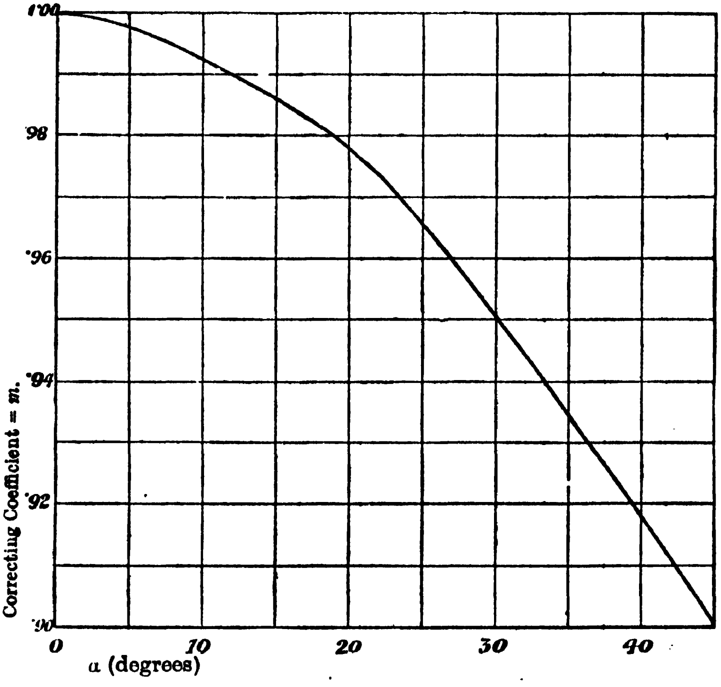

FIG. 151.—Correcting Coefficients for Open-coiled Springs.

m may then be regarded as a correction coefficient, values of which are given in Fig. 151 for various values of a.

STRESSES IN WIRE.—The stresses in the wire can be found by calculating the separate bending and shear stresses and combining them in the manner described to find the equivalent simple direct or shear stress.

The twisting and bending strains both cause a tendency for the free end of the coil to turn about the vertical axis V V, thus altering the effective number of coils.

BENDING SPRINGS

Leaf or Plate Springs.—If we consider a leaf or plate spring of the type shown in Fig. 153 and note that the plates are bent to the same radius so that they contact only at their edges, we see that each plate may be regarded as supported at its point of contact with the one below it, the load transmitted at its overhanging end being $\frac{W}{2}$. (See also Fig. 152.)

The B.M. diagram for each plate comes therefore as shown.

In order that the spring may close practically flat, the curvature of each plate must remain constant after bending, *i. e.* the radius of each plate after bending must be the same.

But from p. 249 $\frac{1}{R} = \frac{M}{EI}$

$\therefore$ Since E is constant $\frac{M}{I}$ is constant.

Between B and B′, the B.M. is constant so that the section is constant, but for A B and A′ B′ the B.M. varies in the triangular manner shown, so that the section must vary so as to keep $\frac{M}{I}$ constant.

This can be done by making the ends triangular in plan, the thickness being constant.

Then at any point at distance x from A

$$I = \frac{b_x d^3}{12}$$

$$M = \frac{W x}{2}$$

$$\therefore \frac{M}{I} = \frac{6 W x}{b_x d^3}$$

$$\text{and } \frac{b_x}{x} = \frac{b}{l'} \text{ by similar } \Delta\text{s}$$

$$\therefore \frac{M}{I} = \frac{6 W l'}{b d^3} = \text{constant}$$

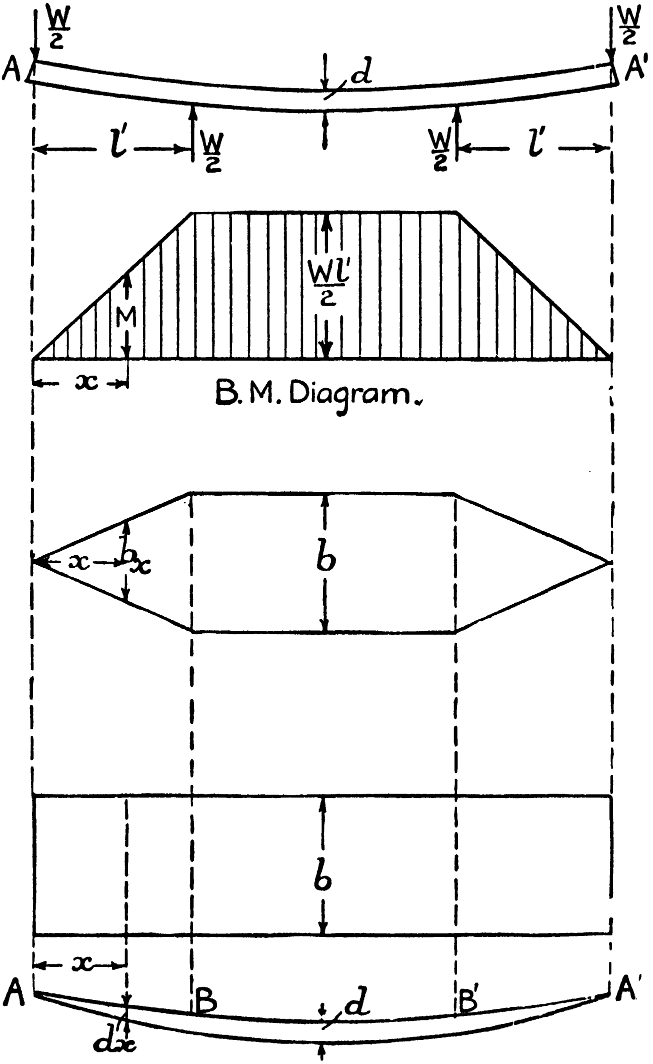

Fig. 152.—Stresses in Plate Springs.

Another way would be to keep the ends square and to vary the thickness as indicated.

$$\text{Then } I = \frac{b\,d_x^3}{12},\ M = \frac{W x}{2}.$$

For $\frac{M}{I}$ to be constant $d_x^3 = \frac{d^3 \times x}{l}$ and if the lap were contoured so that the relation held, the necessary conditions would be satisfied.

Now suppose that there are n plates and that all but the top one are cut longitudinally through the centre and placed as shown in Fig. 153 they would make up the diamond-shaped

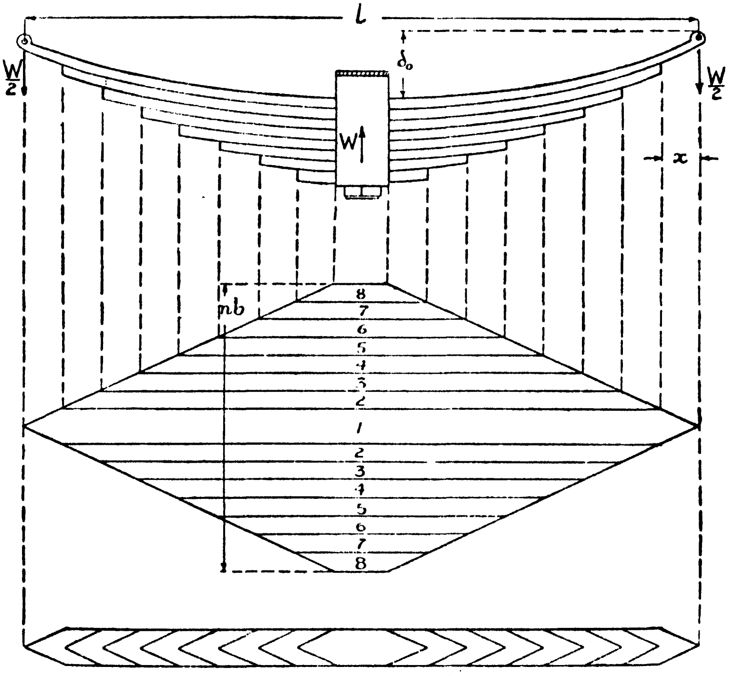

FIG. 153.—Plate Springs.

figure shown. The deflection, *i. e.* the upward vertical movement from its initial curved position, for such a single plate, which bends to a circular one will be very nearly equal to $\frac{M l^2}{8 E I}$

$$\text{Now } I = \frac{n b d^3}{12}$$

$$\therefore \delta = \frac{3 M l^2}{2 E n b d^3}$$

For a centrally loaded beam $M = \frac{W\,l}{4}$

$$\therefore\ \delta = \frac{3\,W\,l^3}{8\,E\,n\,b\,d^3} \dots\dots\dots\dots\dots\dots (1)$$

In a test of such a spring it will be found that the friction between the plates will cause the deflection to be less than this with an increasing load and to be more as the load is reduced.

It is common to test such springs by loading them until the plate is flat; we then have $\delta = \delta_0$, and we get from equation (1) the following value for the test or proof load W_T

$$W_T = \frac{8\,E\,n\,b\,d^3\,\delta_0}{3\,l^3} \dots\dots\dots\dots\dots\dots (2)$$

Stress in Plates.—The stress in the plates will be constant along their length because their depth as well as their moment of inertia is constant.

$$\therefore\ f = \frac{M}{I} \times \frac{d}{2}$$

$$= \frac{W\,l}{4\,.\,\frac{n\,b\,d^3}{12}} \cdot \frac{d}{2}$$

$$= \frac{3\,W\,l}{2\,n\,b\,d^2} \dots\dots\dots\dots\dots\dots (3)$$

Derivation of Deflection from Resilience.—The formula for deflection may be derived from the resilience as follows—

$$\text{Resilience (see p. 273)} = \frac{f^2}{6\,E}$$

$$\therefore\ \text{Total work done in stressing} = \frac{f^2}{6\,E} \times \text{vol.}$$

$$= \frac{f^2}{6\,E} \,.\, l\,d \,.\, \frac{n\,b}{2}$$

$$\therefore\ \frac{W\delta}{2} = \frac{f^2\,l\,n\,d\,b}{12\,E}$$

$$= \frac{9\,W^2\,l^2\,l\,n\,d\,b}{4\,n^2\,b^2\,d^4 \times 12\,E}$$

$$= \frac{3\,W^2\,l^3}{16\,n\,b\,d^3\,.\,E}$$

$$\therefore\ \delta = \frac{3\,W\,l^3}{8\,E\,n\,b\,d^3} \text{ as before.}$$

Numerical Example.—*A laminated plate spring of* 40 *inches span has* 12 *plates, each* ·375 *inch thick and* 3·40 *inches wide. Calculate the deflection when carrying a central load of* 4 *tons, taking* E = 11,600 *tons per sq. in.*

By formula (1) we have

$$\delta = \frac{3 W l^3}{8 E n b d^3}$$

$$= \frac{3 \times 4 \times 40 \times 40 \times 40}{8 \times 11{,}600 \times 12 \times 3{\cdot}40 \times {\cdot}375^3}$$

$$= \underline{3{\cdot}84 \text{ inches.}}$$

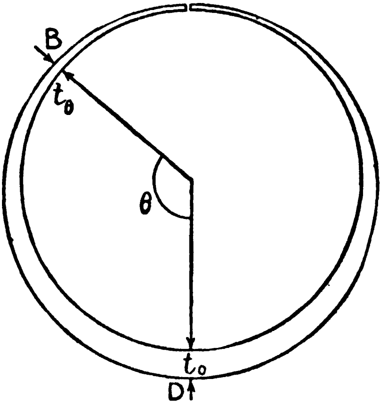

Fig. 154.—Piston Rings.

Piston Rings.—Springs in the form of split-rings are placed around pistons in oil, gas and steam engines to prevent escape of the working fluid past the piston, and such rings should be designed so as to give as constant a pressure as possible all round the cylinder. The necessary variation in thickness has been investigated by Professor Robinson in the following manner.

Let D, Fig. 154, be the point of maximum thickness t_0 at the centre of the ring which was initially circular on the outside and is sprung into position so that it is still circular on the outside.

Consider a length A B of the ring, the thickness of the ring at the point B being t_θ and the breadth throughout being b. θ is the angle which the arc B D subtends at the centre.

Let R be the radius of the spring when bent and let R_0 be the radius at B when in the unstrained condition indicated in dotted lines.

If p is the pressure per sq. in. we have

$$P = p \, . \, AB \, (\text{chord}) \, . \, b = 2 \, p \, b \, R \cos \frac{\theta}{2} \quad \ldots\ldots\ldots(1)$$

and the bending moment at P is equal to

$$M = P \, . \, BO = 2 \, p \, b \, R^2 \cos^2 \frac{\theta}{2} \quad \ldots\ldots\ldots\ldots(2)$$

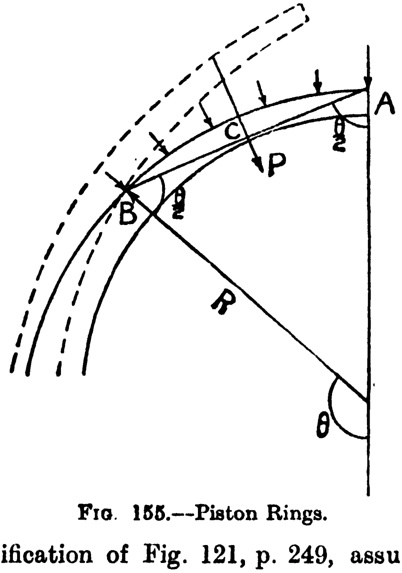

FIG. 155.—Piston Rings.

By a modification of Fig. 121, p. 249, assuming a small initial radius of curvature R_0 we shall get

$$\left(\frac{1}{R} - \frac{1}{R_0}\right) = \frac{M}{EI}$$

as a first approximation.

$$\left(\frac{1}{R} - \frac{1}{R_0}\right) = \frac{M}{EI} = \frac{2 \, p \, b \, R^2 \cos^2 \frac{\theta}{2}}{E \, . \, \frac{b \, t_\theta^3}{12}}$$

$$= \frac{24 \, p \, . \, R^2 \cos^2 \frac{\theta}{2}}{E \, t_\theta^3} \quad \ldots\ldots\ldots\ldots\ldots(3)$$

At the point D, $\theta = 0$ and $t_\theta = t_o$

$$\therefore \frac{1}{R} - \frac{1}{R_o} = \frac{24\, p\, R^2}{E\, t_o^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Now R_o and R have to be the same in each position

$$\therefore \frac{24\, p\, R^2 \cos^2 \frac{\theta}{2}}{E\, t_\theta^3} = \frac{24\, p\, R^2}{E\, t_o^3}$$

If, therefore, the pressure p is to be constant

$$\frac{t_\theta^3}{t_o^3} = \cos^2 \frac{\theta}{2}$$

$$\therefore \frac{t_\theta}{t_o} = \sqrt[3]{\cos^2 \frac{\theta}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Fig. 156 shows values of t_θ in terms of t_o for the various values of θ.

Neglecting the additional stress due to the curvature of the bar (see Chap. XIX.) we have at the point D

$$\frac{f}{\frac{t_o}{2}} = \frac{M}{I} = E\left(\frac{1}{R} - \frac{1}{R_o}\right)$$

$$\therefore f = \frac{E\, t_o}{2}\left(\frac{1}{R} - \frac{1}{R_o}\right)$$

$$i.e.\ \frac{1}{R} - \frac{1}{R_o} = \frac{2f}{E\, t_o} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Putting this result in (4) we get

$$f = 12\, p\, \frac{R^2}{t_o^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

$$\therefore t_o = R\sqrt{\frac{12\, p}{f}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

To find the necessary initial radius we have

$$\frac{1}{R_o} = \frac{1}{R} - \frac{2f}{E\, t_o} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

NUMERICAL EXAMPLE.—*Taking* E = 16×10^6 *lbs. per sq. in. and the working stress* 4000 *lbs. per sq. in., find the necessary thickness and original external diameter for a cast-iron piston ring for a cylinder* 20 *inches in diameter, the necessary pressure being* 3 *lbs. per sq. in.*

From equation (8)

$$t_o = 10 \times \sqrt{\frac{12 \times 3}{4000}} = \cdot 95 \text{ in. nearly.}$$

This is less than is usually used in practice.

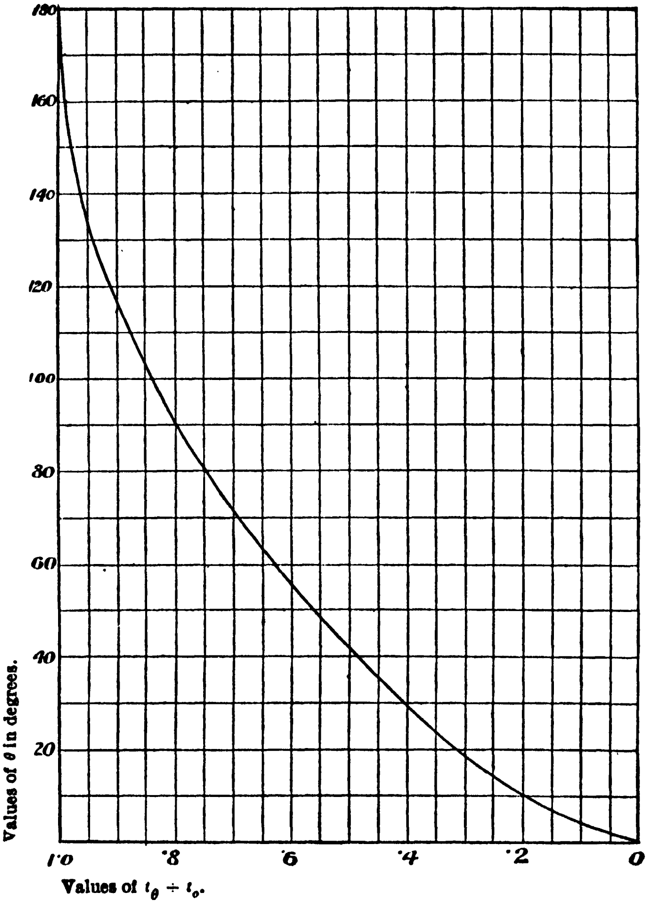

FIG. 156.—Thickness of Piston Rings for Uniform Pressure.

Unwin and Mellanby* give total depth of all packing rings

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* *Elements of Machine Design* (Longmans), Part II (1912).

$$= \frac{D}{15} + \cdot 6 \text{ in. for steam engines}$$

$$= \frac{D}{7} + 2\cdot 4 \text{ ins. for gas and oil engines}$$

$$= \frac{D}{5\cdot 5} \text{ for petrol engines.}$$

Taking, therefore, a steam engine with 3 rings we should have

$$t_o = \tfrac{1}{3}\left(\frac{20}{15} + \cdot 6\right)$$

$$= \underline{\cdot 64 \text{ in.}}$$

From equation (9)

$$\therefore \frac{1}{R_o} = \frac{1}{R} - \frac{2 \times 4000}{16 \times 10^6 \times \cdot 95}$$

$$= \cdot 1 - \frac{1}{1900}$$

$$= \cdot 09947$$

$$R_o = 10\cdot 053$$

$$\therefore \text{ original diameter} = 20\cdot 11 \text{ nearly.}$$

Ring of Uniform Thickness.—If the ring is of the same thickness t throughout, R will be constant, but R_o should vary in accordance with the following treatment—

We have as before in equation (3)

$$\frac{1}{R} - \frac{1}{R_o} = \frac{24\, p\, R^2 \cos^2 \frac{\theta}{2}}{E\, t^3}$$

$$\therefore \frac{1}{R_o} = \frac{1}{R} - \frac{24\, p\, R^2 \cos^2 \frac{\theta}{2}}{E\, t^3}$$

$$\therefore R_o = \frac{E\, t^3\, R}{E\, t^3 - 24\, p\, r^2 \frac{\theta}{2}} \qquad \ldots\ldots\ldots\ldots(10)$$

For given values of t and p, R_o can be found by this formula for different angular positions, and it will be found that the curve for the initial shape of the ring differs considerably from a circle, so that rings made by cutting out parts from circular rings and springing into position will not give a uniform pressure.

* **Plane Spiral Springs.**—Consider a short length A B (Fig. 157) of a plane spiral spring, the free end of which is pulled with a force W.

Then the bending moment acting on this short length $= M = W x$.

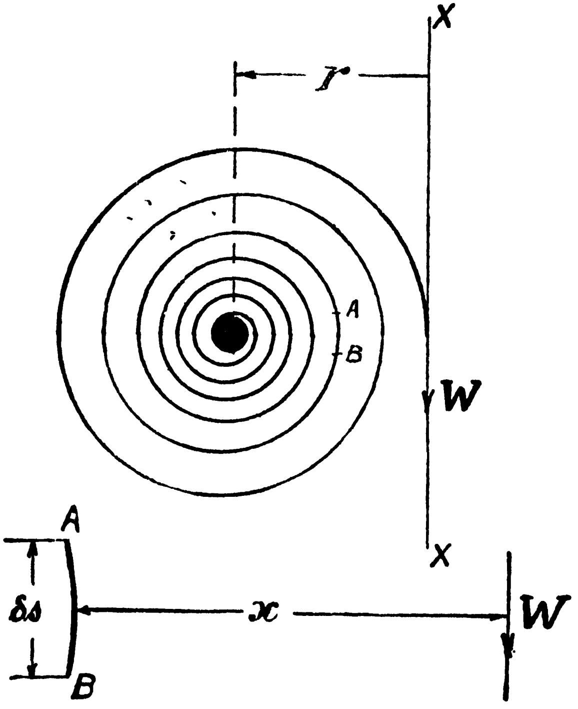

Fig. 157.—Flat Spiral Springs.

∴ If $\delta\theta$ is the change in angle between the tangents at the two ends

$$\delta\theta = \frac{M\,\delta s}{E\,I} \quad \text{(see p. 250)}$$

$$= W x \times \frac{\delta s}{E\,I}$$

∴ Total change in angle

$$= \theta = \sum \frac{M\,\delta s}{E\,I} = \frac{W}{E\,I}\,\Sigma\, x\,\delta s$$

if E and I are constant as is usual.

$$\begin{aligned}\Sigma\, x\,\delta s &= \text{1st moment of spring constant about x x}\\ &= \text{length of spring} \times r \text{ (approx.)}\\ &= l\,r\end{aligned}$$

$$\therefore\ \theta = \frac{W\,l\,r}{E\,I} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

If the spring is wound up to produce a tensile force W at the end, the torque T which must be applied to the shaft will be equal to W r.

Also θ will be the angle turned through by the shaft, so that work stored up in spring

$$\begin{aligned}&= \tfrac{1}{2}\,T\,.\,\theta\\ &= \frac{W^2\,r^2\,l}{2\,E\,I} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)\end{aligned}$$

If the breadth of the spring is b and its thickness t, $Z = \frac{b\,t^2}{6}$

$$\therefore\ \text{Bending stress} = f = \frac{6\,M}{Z} = \frac{6\,M}{b\,t^2}$$

The maximum B.M. occurs at the point and is approximately equal to 2 W r.

$$\therefore\ f = \frac{12\,W\,r}{b\,t^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\therefore\ W = \frac{f\,b\,t^2}{12\,r}$$

Putting this result in (2) we have

$$\begin{aligned}\text{Work stored} &= \frac{f^2\,b^2\,t^4\,l}{144\,r^2\,.\,2\,E\,I}\,.\,r^2\\ &= \frac{f^2\,b^2\,t^4\,.\,l}{288\,E\,.\,\frac{b\,t^3}{12}}\\ &= \frac{f^2\,b\,t\,l}{24\,E}\\ &= \frac{f^2}{24\,E} \times \text{volume}\end{aligned}$$

$$\therefore\ \text{Resilience} = \frac{f^2}{24\,E}$$

This is relatively small because the material is not used very economically, parts of the spring being much more highly stressed than others.

Close-coiled Helical Springs under Bending Stress.—If instead of subjecting a close-coiled helical spring to an axial load we subject it to a twisting action tending to unwind or wind up the spring, the whole spring will be subjected to a bending moment equal to the torque T applied.

Therefore, if we neglect the effect of the curvature of the wire upon the stresses in it, we shall have, if θ is the angle by which the spring winds up or unwinds—

$$\text{Work stored} = \frac{T\theta}{2} = \frac{T^2 l}{2EI} \quad \text{(see p. 331)}$$

$$\therefore \theta = \frac{Tl}{EI}$$

CIRCULAR SECTION.—For a round wire of diameter d,

$$I = \frac{\pi d^4}{64}$$

$$\text{and } f \cdot \frac{\pi d^3}{32} = T$$

$$\text{i.e. } f = \frac{32T}{\pi d^3}$$

$$\therefore \text{Work stored} = \frac{\pi^2 d^6 . f^2}{32 . 32 . 2E} \cdot \frac{l}{\pi d^4} . 64$$

$$= \frac{f^2 . \pi d^2 l}{32E} \cdot = \frac{f^2 \times \text{volume}}{8E}$$

$$\therefore \text{Resilience} = \frac{f^2}{8E}$$

$$\theta = \frac{Tl}{EI} = \frac{64Tl}{\pi d^4 . E} \text{ radians}$$

$$= \frac{1{,}168\,Tl}{d^4 E} \text{ degrees.}$$

RECTANGULAR SECTION.—If the depth of the section is b and the thickness is t, $I = \frac{b t^3}{12}$

$$\text{and } T = \frac{f . b t^2}{6}$$

$$\therefore \theta = \frac{12\,\mathrm{T}\,l}{\mathrm{E}\,b\,t^3} \text{ radians}$$

$$= \frac{688\,\mathrm{T}\,l}{\mathrm{E}\,b\,t^3} \text{ degrees.}$$

$$\text{Work stored} = \frac{\mathrm{T}^2\,l}{2\,\mathrm{E}\,\mathrm{I}}$$

$$= \frac{f^2\,b^2\,t^4\,.\,l}{2 \times 36\,.\,\mathrm{E}\,.\,b\,t^3}\,.\,12$$

$$= \frac{f^2\,b\,t\,l}{6\,\mathrm{E}} = \frac{f^2}{6\,\mathrm{E}} \times \text{volume}$$

$$\therefore \text{Resilience} = \frac{f^2}{6\,\mathrm{E}}$$

SUMMARY OF RESILIENCE OF VARIOUS TYPES OF SPRING.

Type of Spring.	Resilience.
Pure tension	$\frac{f^2}{2\,\mathrm{E}}$
Pure torsion on close-coiled helical spring (circular shaft) .	$\frac{s^2}{4\,\mathrm{G}}$
,, ,, ,, ,, ,, (square shaft) .	$\frac{\cdot 154\,s^2}{\mathrm{G}}$
Plate spring	$\frac{f^2}{6\,\mathrm{E}}$
Plane spiral spring	$\frac{f^2}{24\,\mathrm{E}}$
Close-coiled helical springs with twisting action causing bending stresses—	
Circular section	$\frac{f^2}{8\,\mathrm{E}}$
Rectangular section	$\frac{f^2}{6\,\mathrm{E}}$

CHAPTER XIII

THE TESTING OF MATERIALS

Testing Machines.—In most types of testing machines the loads are applied through a system of levers and are so arranged that the levers are connected to one end of the specimen (or in the case of bending tests to the supports), and that a force is exerted by an hydraulic ram or screw gear to the other end, the lever system " floating " when the force exerted is equal to that applied to the levers. In this way additional weights can be put on to the levers without causing a shock in the specimen, because such additional weight does not come on to the specimen until the hydraulic ram or screw gear is operated further. We will describe some of the most common types of testing machines.

WICKSTEED-BUCKTON SINGLE LEVER VERTICAL TESTING MACHINE.—This type of testing machine, a photograph of which is shown in Fig. 158, was designed by Mr. J. H. Wicksteed, and is manufactured by Messrs. Joshua Buckton & Co., of Leeds. The form shown in the photograph is belt driven, the power being transmitted by toothed gearing to the screw at the base of the machine, but hydraulic rams are commonly employed to exert the necessary test force. This particular machine has a capacity of 30 tons, machines of this type being obtainable for capacities ranging from 5 to 100 tons, and can be employed for tests in tension, compression, bending, shear and torsion.

Fig. 159 shows diagrammatically the action of the machine, an hydraulic ram drive being shown.

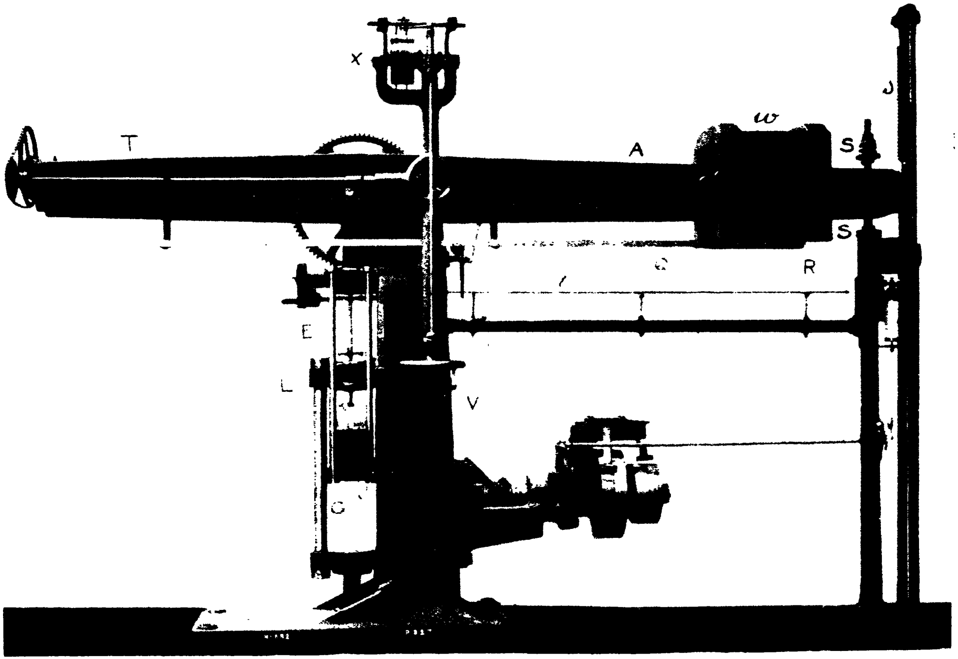

FIG. 158.—Buckton Vertical Testing Machine.

[*To face page* 364

A horizontal lever A, Figs. 158, 159, is provided with a knife-edge B resting upon a strong vertical frame V; a jockey-weight w is movable along this lever and carries a vernier R by means of which the position of the weight can be read off upon a scale Q.

A second knife-edge C, carried by the lever, engages a link O connected to a cross-head. When operating for tension, one end of the specimen E is gripped in this cross-head, the other end being gripped in a cross-head L connected by rods G to an hydraulic ram F.

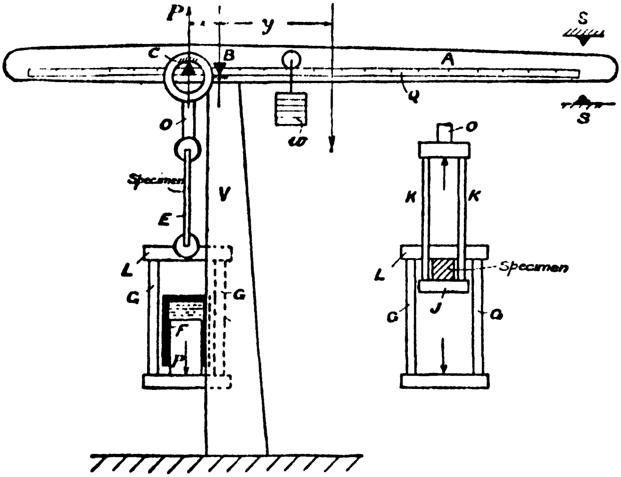

FIG. 159.

If the resultant of the jockey-weight w and the lever A is W and acts at a distance y from the knife-edge B, we have by moments

$$P . x = W y$$

$$P = \frac{W . y}{x}$$

The scale Q is graduated so as to read off values of P direct, because W and x are of fixed value.

Stops S are provided for the lever A, which normally rests on the lower one; as the pressure in the ram is increased the force exerted upon the specimen gradually increases until it reaches the value P, whereupon the lever rises and "floats" between the two stops.

A lower cross-head J is suspended from the cross-head H by rods K and is used for compression and bending tests. The diagram on the right-hand side of Fig. 159 shows how the force is applied in the case of a compression test. In a bending test the arrangement is similar, but the test-beam is placed on supports on the cross-head J and a load point or points is or are connected to the cross-head L.

The jockey-weight w is adjusted along the lever A by a screw which runs through the latter and is driven from a shaft O operated by a hand-wheel U or by power from a countershaft X.

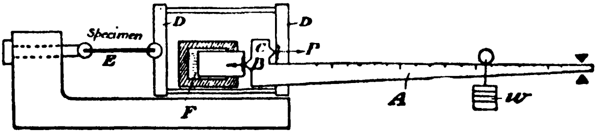

FIG. 160.—Werder Testing Machine.

WERDER HORIZONTAL SINGLE LEVER MACHINE.—This machine is used to a great extent on the Continent, and is shown in diagrammatic form in Fig. 160. The lever A is of bell-crank type and the two knife-edges B C are close together so that the leverage is great and comparatively small weights w can be employed. The knife-edge B is carried by the hydraulic ram F, and the force P is transmitted to the specimen through a cranked lever D. It is quite clear from this diagram that as the specimen stretches the load would go off it if the ram did not follow, *i.e.* if the pressure were not maintained in the cylinder. When, as is common, this pressure is generated by a small hand-pump, the operator goes on pumping until the lever floats between the stops S.

Compound Lever Machines.—RIEHLÉ TYPE.—Fig. 161 shows a vertical type of compound lever testing machine,

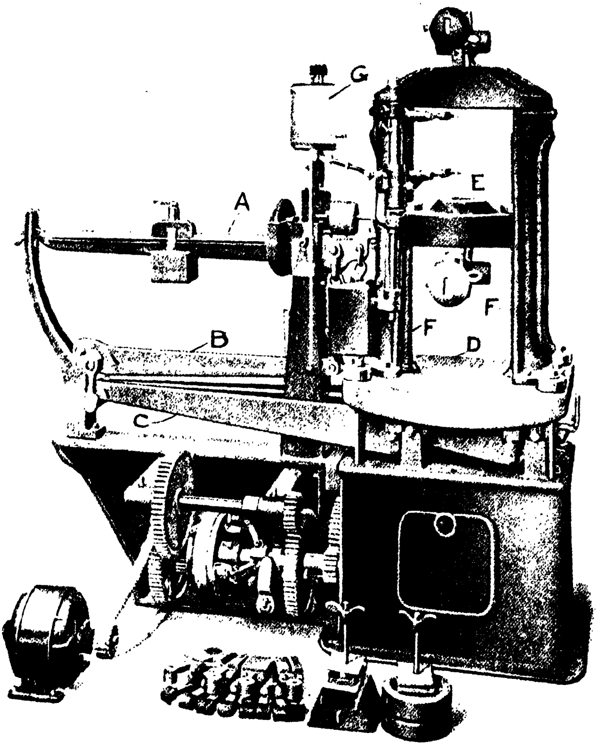

FIG. 161.—Riehlé Testing Machine.

[*To face page* 366.

made by Riehlé Bros., of Philadelphia, U.S.A., and used largely in America.

The steelyard A is connected by a link with lever B, which is in turn connected with a lever C, which presses upwards upon the table or platen D. A cross-head E is operated by screws F, and according as the specimen is placed above or below this cross-head the test will be made in tension or compression. The machine shown is power driven by toothed gearing from an electric motor.

These machines are controlled automatically by an electric contact device. At the outer end of the beam A are two contacts so arranged that when the beam reaches its highest position contact is made; this completes the circuit of an electro-magnet which puts into gear with the driving mechanism the screw for moving the jockey-weight along the beam, but the movement of the jockey-weight can only follow up the extensions because contact is again broken as soon as the extension is more than is necessary to maintain the balance. Means are provided for varying the speed at which the weight is run out. An autographic recorder G is provided (see p. 379).

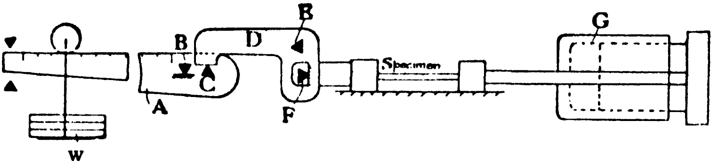

FIG. 162.—Greenwood and Batley Compound Lever Testing Machine.

GREENWOOD AND BATLEY HORIZONTAL TYPE.—This type of machine is made by Messrs. Greenwood and Batley, and was used by Professor Kennedy in the many researches which he carried out while at University College, London, this being one of the first testing machines installed in a college laboratory.

The steelyard lever A, Fig. 162, has a knife-edge B and acts on a knife-edge C of a bell-crank lever D, which is pivoted upon a knife-edge E and is acted upon by a knife-edge F connected to a cross-head connected to the specimen. The other end of the specimen is carried by a cross-head operated by an hydraulic ram G.

The usual leverage of the compound lever is 100 : 1; the jockey-weight *w* is generally moved along the steelyard, which carries a graduated scale, by means of a chain by a hand-wheel.

WICKSTEED-BUCKTON HORIZONTAL TYPE.—This type is shown diagrammatically in Fig. 163 and by a photograph in Fig. 164. The steelyard lever A acts through a link C upon a bell-crank lever D, which connects by shafts shown diagrammatically by G with the specimen. A massive carriage frame J is connected to the hydraulic ram F and carries a number of notches, into any of which can be fitted a cross-head K by which the other end of the specimen is carried. According to the position of the cross-head K, the specimen will be tested in tension or compression. This machine is very convenient for general testing on account of the ease with which it can be adjusted for different lengths of specimen and forms of test.

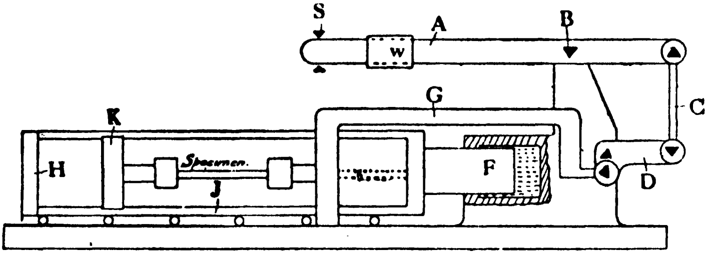

FIG. 163.—Wicksteed-Buckton Compound Lever Testing Machine.

Smaller Testing Machines.—There are a large number of smaller testing machines in use, from which very good results may be obtained in cases in which it is not essential for the specimens to be large ones. The student should remember that a great deal can be learnt with very simple apparatus. Fig. 165 shows a machine, designed by Professors Dixon and Hummel and manufactured by Messrs. W. and T. Avery, Ltd.;

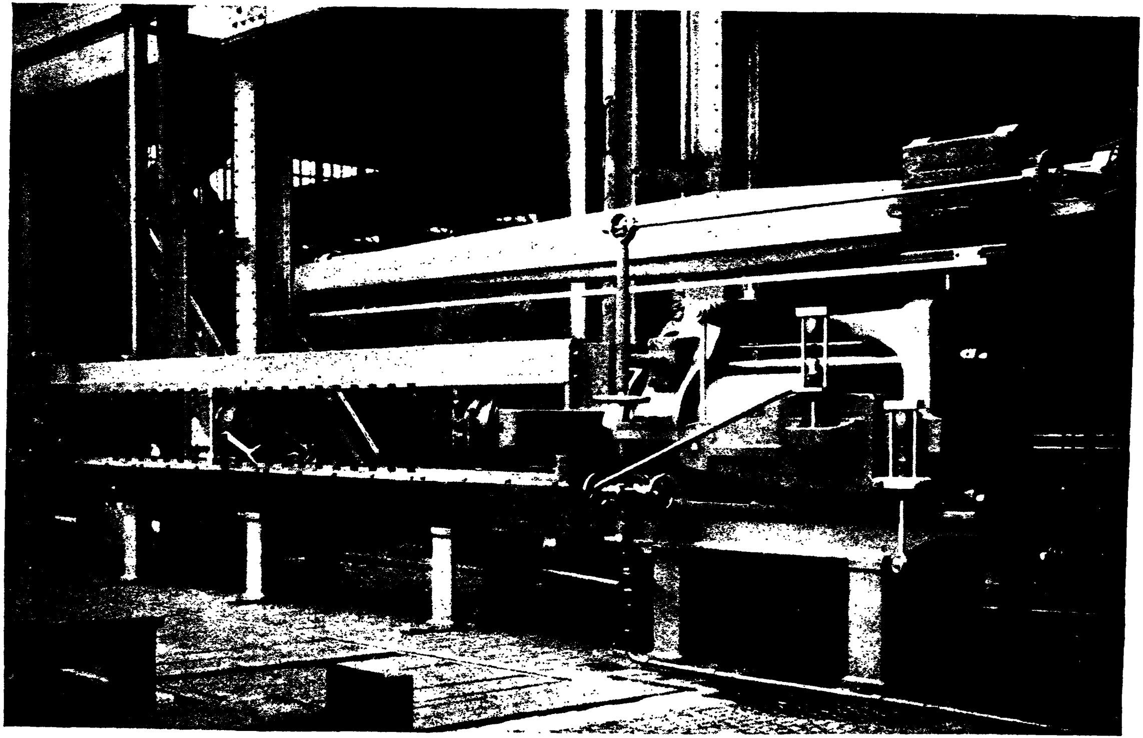

FIG. 164.—Buckton Horizontal Testing Machine.

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it has an automatic load-indicating device in the form of two dished plates connected by a patented flexible metallic diaphragm; the space between the plates is filled with a non-elastic fluid and the pressure is recorded upon a sensitive gauge which is graduated to give the load on the specimen. The gauge can be tested by means of a small plunger which can be loaded with weights supplied with the machine to produce pressures corresponding to the total capacity of the machine.

The machine shown has a capacity of 10,000 lbs. and the force is applied by a capstan acting through worm and wheel gearing to a central screw. This gear can be thrown out for quick return and the screw operated direct by the handle shown.

Calibration of Testing Machines.—To ensure accurate results in the use of testing machines they should be calibrated periodically; the vertical type of machine possesses advantage in this respect because a heavy weight can be hung on direct.

The first test is for *zero error*. This is effected by moving the jockey-weight carefully to the zero mark and seeing if the lever floats; if it does not we can correct for this by an adjustment of the vernier on the jockey-weight by moving the latter until the lever floats and then moving the vernier until it reads zero.

The next point that we may test is the *value of the jockey-weight*. This can be effected without removing it from the machine in the machines shown in Figs. 158, 163, by finding the floating position and then moving the jockey-weight a carefully measured distance l along the lever; then at a distance z from the fulcrum suspend weights w until the lever floats again.

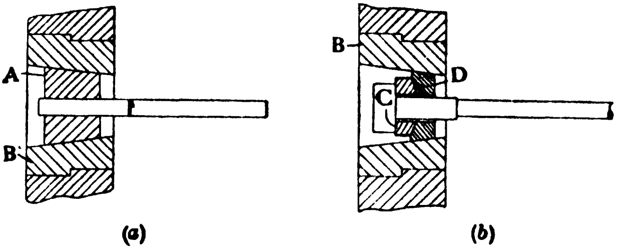
$$\text{Then weight of jockey-weight} = W = \frac{wz}{l}.$$

To test for the accuracy of the *knife-edge distance* x we may proceed as follows: Hang a heavy weight W_1 from the shackles of the machine and note the distance u that the jockey-weight has to move to balance it;

$$\text{then } x = \frac{W_1 u}{w}$$

Another important test is for *sensitiveness*, by which is meant the amount by which the load may vary without causing the lever to come against its stops. This may be tested at zero in vertical machines by placing the jockey-weight at zero and hanging small weights on to the shackles until the lever ceases to " float "; this should be repeated for larger loads and should also be tried by taking weights off as well as by putting them on.

Grips and Forms of Test-Piece in Tension.—When tests are made on flat bars, as is very common for rolled sections, wedge grips are generally employed. Fig. 166 (*a*) shows one form of wedge grip. Wedges A, provided with serrations to grip into the specimen, are driven into a tapered central passage through a block secured to the cross-head of the machine.



(*a*) (*b*)

FIG. 166.

In Fig. 166 (*b*) is shown a grip suitable for a turned specimen provided with a collar. The collar bears against a washer C provided with a spherical end which bears against a tapered bush D, which engages in a similarly tapered central hole in the block B. This construction tends to keep the pull truly axial; a point of great importance. The ends of the specimens are very often screw-threaded, in which case they just screw into the blocks B.

The British Engineering Standard Committee have specified the following rules, see Fig. 167, as to gauge length (cf. p. 55). (*a*) FLAT BARS.—Gauge length = 8″; parallel for 9″.

If the thickness is greater than $\frac{3}{8}$ in., maximum width = $1\frac{1}{2}$ ins.

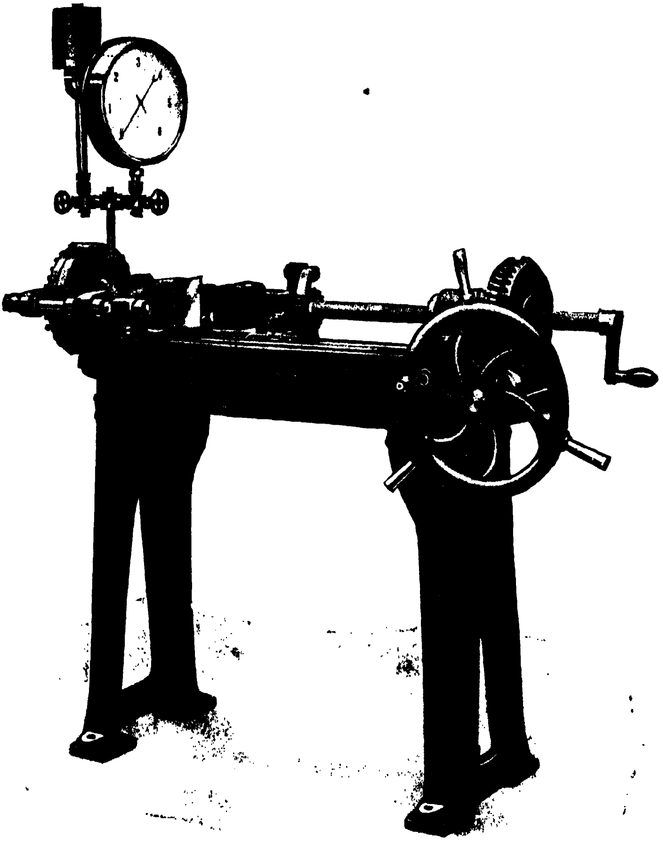


FIG. 165.—Dixon and Hummel's Testing Machine.

[To face page 370.

If the thickness is between ⅜ and ⅞ in., maximum width = 2 ins.

If the thickness is less than ⅜ in., maximum width = 2½ ins.

(*b*) Turned Sections.—Gauge length = 8 *d*; parallel for 9 *d*.

(*c*) Turned Specimens from Forgings.—

Area ¼ in.; gauge length = 2 ins.
Area ½ in.; gauge length = 3 ins.
Area ¾ in.; gauge length = 3½ ins.

Extensometers.—Extensometers are instruments for measuring the elastic strains of materials in tension or compression. In the types in most common use the strains are magnified by an arrangement of levers and are measured by micrometer or by an indicator passing over a scale. We will describe a few of the most common types; for other types a reference may be made to a paper by Mr. J. Morrow, in *Proc. Inst. M.E.* for 1904.

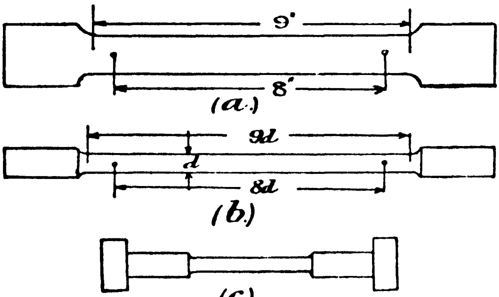


Fig. 167.

An interesting report on the accuracy of various types of extensometers is given in the *Report of the British Association* for 1896. In these tests, different observers had bars of the same material sent for test. The results show very good agreement, some of the nearest results to the mean being obtained by instruments of very simple form.

Goodman's Extensometer.—This extensometer is of very simple form and was designed by Professor Goodman, of

Leeds. It consists of two forked clips, A, B, Fig. 168, which carry pointed screws engaging in centre-punch marks in the specimen and are connected to rods C which join at their ends and carry a scale D on a projecting piece.

Two light rods E, F form a fixed triangle, and the vertical rod E projects and has a small groove at its end which forms a bearing for a knife-edge carried by the pointer P. A second knife-edge on the latter rests upon a second vertical rod H depending from the upper clip A. A small screw is provided for bringing the pointer exactly to zero at the beginning of a test. The strain of the specimen causes the rod H to move slightly relatively to the rod E, this movement being magnified 100 times by the pointer lever.

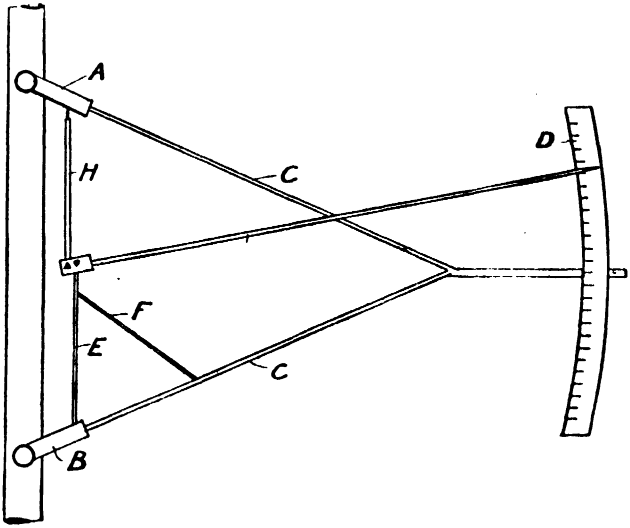


FIG. 168.—Goodman's Extensometer.

This and most other extensometers should be taken off the specimen as soon as the yield point is reached.

KENNEDY'S EXTENSOMETER.—This extensometer was designed by Sir A. B. W Kennedy when professor at University College, London, and was one of the first lever extensometers. The instrument is for use in horizontal testing machines and

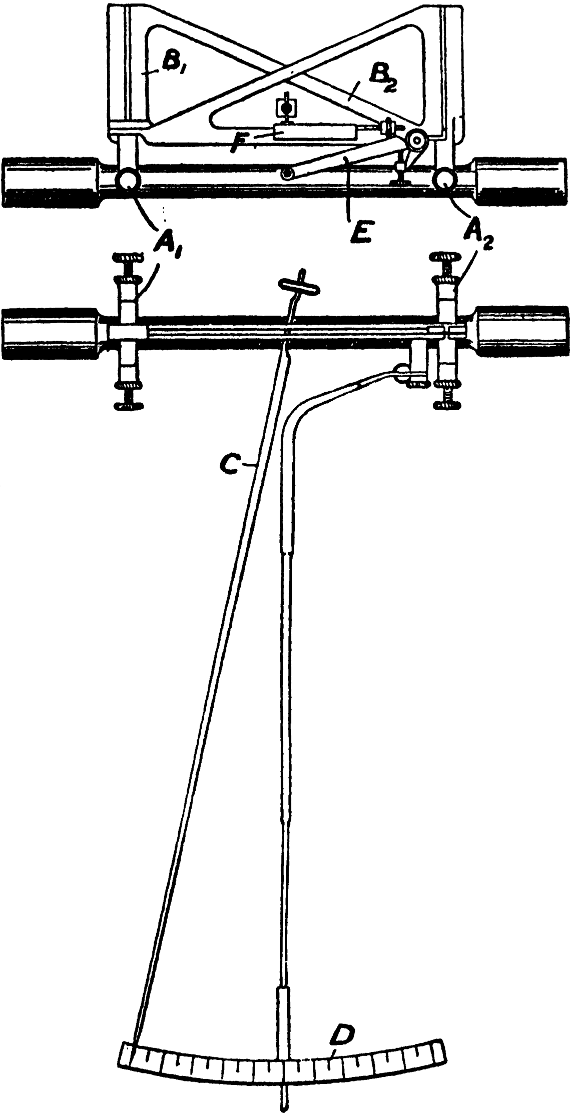


FIG. 169.—Kennedy's Extensometer.

comprises two clips A_1, A_2, Fig. 169, which carry triangular frames B_1, B_2, which slide over and support each other.

The clips are as usual provided with pointed screws for engaging centre-punch marks in the specimen, lock-nuts being provided on the screws. As the specimen stretches, the frames slide relatively to each other, and a pointer-lever C which carries pins resting in depressions in each frame is thus caused to move over a scale D carried on an adjustable arm

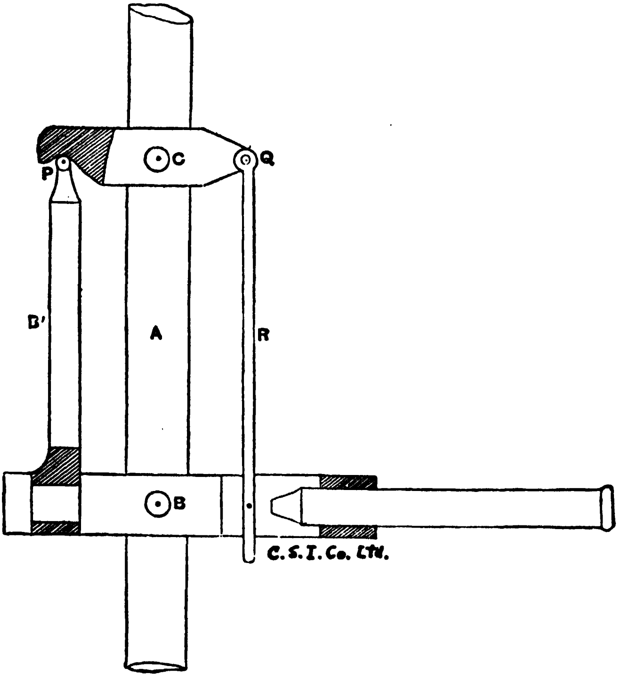


FIG. 170.—Ewing's Extensometer.

E. To give an adjustment for zero, the depression in the front frame is formed in a plate F which can be adjusted by a fine-pitch screw.

EWING'S EXTENSOMETER.—This instrument was designed by Sir J. A. Ewing when professor at Cambridge.

The principle involved is illustrated diagrammatically in Fig. 170. There are two clips B and C each attached to the test-piece A by the points of two set-screws. The slip B has a projection B′ ending in a round point P which engages with a conical hole in C; when the bar extends this rounded point serves as a fulcrum for the clip C, and hence a point Q, equally distant on the other side, moves, relatively to the clip B, through a distance equal to twice the extension. This distance is measured by means of a microscope attached to the clip B. The microscope forms a prolongation of the clip B and the motion of the point Q is brought into the field of view by means of a hanging rod R. The rod R is free to slide on a guide in the clip B, and carries a mark on which the microscope is sighted. The displacement is read by means of a micrometer scale in the eye-piece of the microscope. The pieces B and B′ are jointed to one another in such a way that the bar may twist a little, as it is sometimes liable to do during a test, without affecting the reading C. But the joint between B and B′ forms a rigid connection so far as angular movement in the plane of the paper is concerned. This feature is essential to the action of the instrument: it is only then that P serves as a fixed fulcrum in the tilting of C by extension on the part of the specimen.

Fig. 171 is an illustration of the usual form of the complete instrument. The clips B and C in this standard pattern are set at 8 ins. apart.

The object sighted is one side of a wire stretched horizontally across a hole in the rod R and illuminated by means of a small mirror behind. The distances CP and CQ are in this instance equal, with the effect that the movement of the sighted mark is double the extension of the test-piece. The length of the microscope is adjusted so as to give a constant magnification. This adjustment should be tested with the extensometer mounted on the specimen, and if necessary the length of the microscope tube can be altered by moving out or in the portion carrying the eye-piece. A complete revolution of the screw L, which has a pitch of $\frac{1}{50}$ of an inch, should cause a displacement of the mark through 50 divisions of the eye-piece scale, and when this is the case the eye-piece is at the proper distance from the objective. Readings are taken to tenths of a scale division, so that this displacement, which would also be given by $\frac{1}{100}$ of an inch extension of the test-piece, corresponds to 500 units. Each unit then means $\frac{1}{50000}$ inch in the extension of the test-piece.

A small extensometer based upon the same principle is used for measuring the compressive strain in short cylinders.

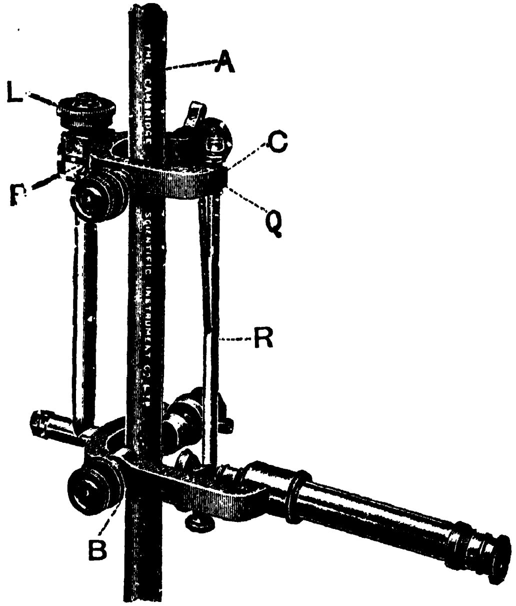


Fig. 171.—Ewing's Extensometer.

Darwin's Extensometer.—This instrument has been designed by Mr. Horace Darwin, F.R.S., and is characterised by simplicity and solidity of construction, which make it suitable for heavy use. Another feature is that if the specimen should break unexpectedly when the extensometer is affixed little damage will result.

The instrument is made in two separate pieces each of which is separately attached to the test-piece M, Fig. 172, by hard steel conical points P, P and P′, P′. The steel rods carrying these points are mounted in slides and after being driven gently into the centre-punch mark in the test-piece are clamped in position by the milled heads R, R. Both parts of the instrument should be capable of rotating quite freely about the points, but there must be no backlash.

The lower piece carries a micrometer screw fitted with a hardened steel point X and a divided head H. It also carries a vertical arm B at the top of which is a hardened steel knife-edge. The upper and lower pieces work together about this knife-edge. A nickel-plated flexible steel tongue A forming a continuation of the upper piece is carried over the micrometer point X. This tongue acts as a lever magnifying the extension of the specimen, so that the movement of the steel tongue to or away from the steel point X is five times the actual extension of the specimen.

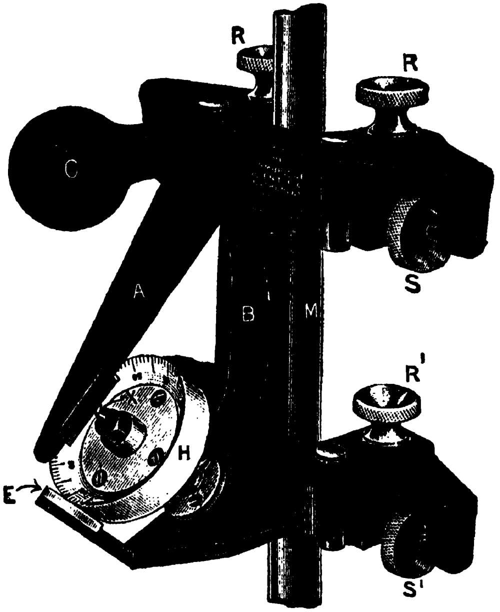


FIG. 172.—Darwin's Extensometer.

To take a reading with the extensometer the thin steel tongue A is caused to vibrate and the divided head then turned till the point X just touches the hard steel knife-edge on the tongue as it vibrates to and fro. This has proved to be a most delicate method of setting the micrometer screw, and the noise produced and the fact that the vibrations are quickly damped out indicate to $\frac{1}{1000}$ mm. the instant when the screw is touching the tongue. After the load is applied a second reading is taken in a similar manner and the difference in the readings gives directly the extension of the test-piece.

If the test-piece is of small diameter the spring does not vibrate in so satisfactory a manner; the cause of this is the flexibility of the test-piece, the instrument itself vibrating as well as the spring. Still, very delicate readings can be taken by simply deflecting the spring with the finger and noting the contact as it passes the point. No damage can be done by advancing the micrometer screw too far forward; all that happens is that the point passes the knife-edge on one side or the other.

In the usual form, the gauge length is 100 mm.; it may be pointed out that over the elastic portion of the test for which extensometers are used, the gauge length is not a matter of importance.

UNWIN'S EXTENSOMETER.—This extensometer, designed by Professor Unwin, is shown in Fig. 173, and makes its measurement by a micrometer acting in conjunction with two spirit-levels.

Two clips *a*, *b*, are secured to the test-bar by pointed set-screws, *c*, *d*, and carry sensitive spirit-levels *g*. The lower clip is first set level by means of an adjusting screw *e*; the upper clip is then levelled by the micrometer screw *f*, on the graduated head of which readings are taken. When placed midway between the two edges of the specimen the extensometer gives the mean strain, but if placed to one side or the other by adjustment of the screws in the clips, the strain at any point in the width can be found in the case of eccentric loading.

MORROW'S MIRROR EXTENSOMETER.—A simple extensometer enabling great magnification of the strain to be obtained is that designed by Mr. J. Morrow. Clamping screws A, B, Fig. 174, pass through rings C, D, to the latter of which a vertical strip is rigidly attached, a pointed rod E acting as a distance piece. Between the ring C and the strip F is placed a small diamond-shaped prism H which carries a mirror M, a light spring clip S maintaining the requisite pressure between the prism and the ring C and strip F. A second mirror N is attached to the strip F and any change in length of the specimen will cause the mirror M to rotate relatively to the mirror N. By observing the images of a scale in both mirrors, we obtain the strain by the difference of the two readings.

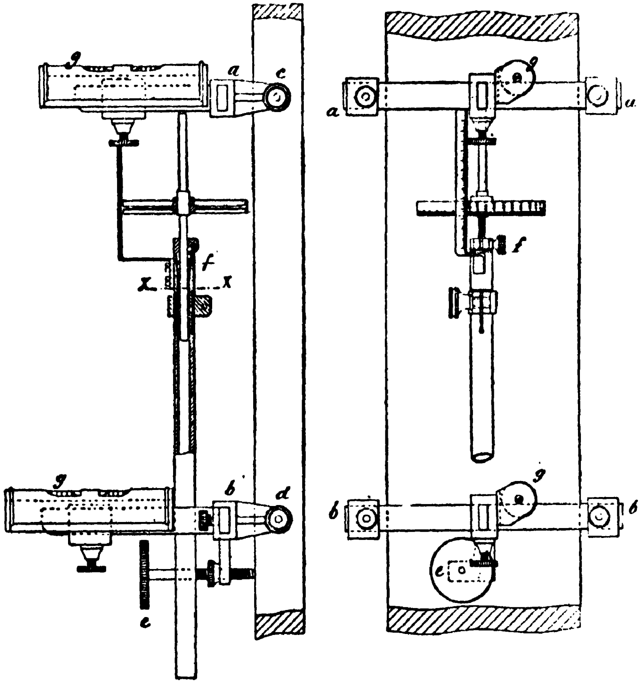


Fig. 173.—Unwin's Extensometer.

Autographic Recorders. — Many testing machines are provided with mechanism for drawing the stress-strain (or more accurately the load extension) diagram automatically as the test proceeds. One of the earliest mechanisms of this kind was one used by Professor Kennedy upon a horizontal compound lever machine. The movements of a pointer upon a piece of smoked glass were obtained in one direction by the actual extensions of the bar, and the movement representing the stress was obtained by multiplying up the strain in a

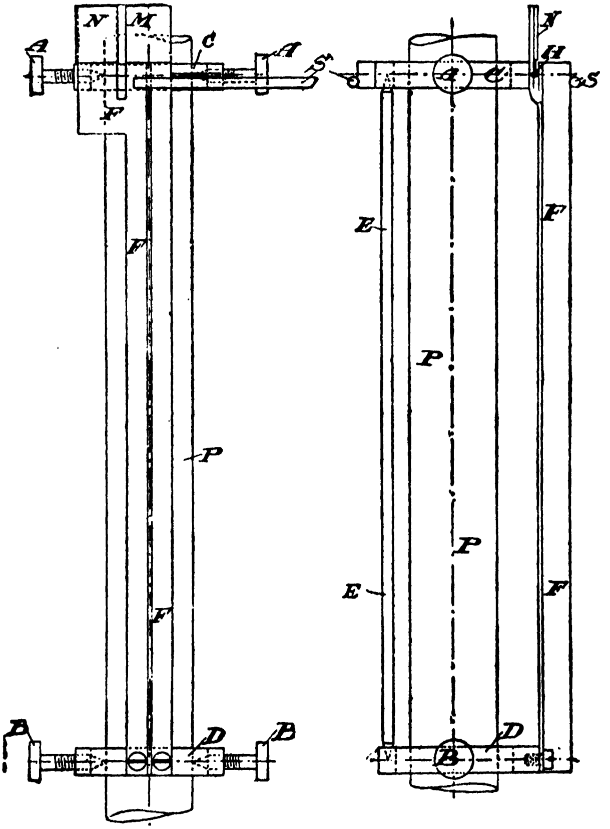


Fig. 174.—Morrow's Extensometer.

longer bar coaxial with the test-bar, this longer bar being always stressed within the elastic limit and the load therefore being proportional to the extension.

Wicksteed-Buckton Recorder.—This autographic recorder is fitted to the Buckton machines and is shown on the

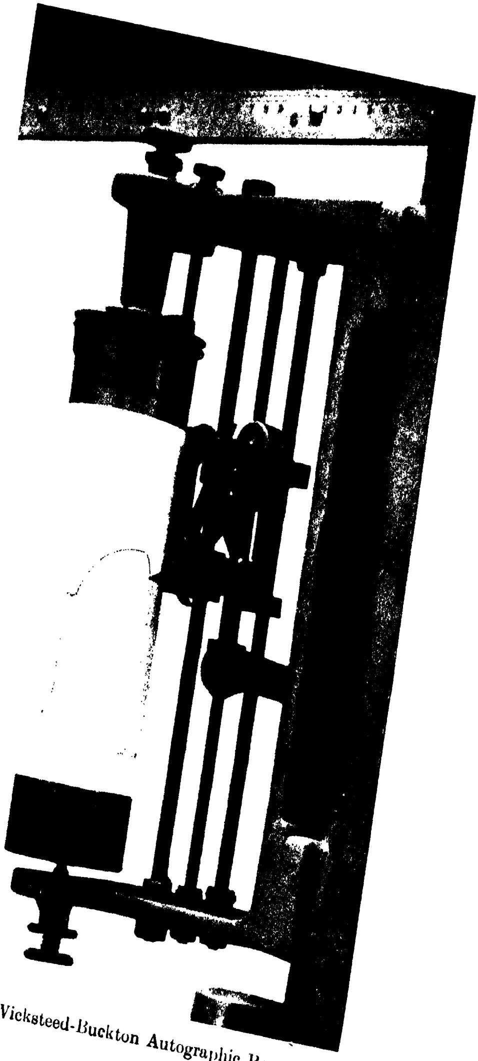


Fig. 175.—Wicksteed-Buckton Autographic Recorder.

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extreme right-hand side of Fig. 158, and also to larger scale in Fig. 175. The record sheet is placed on a drum, around which passes a string which goes through a tube *t* and passes between pulleys upon the cross-heads between which the specimen is gripped. As the specimen becomes strained the distance between these cross-heads varies and this motion is communicated to the drum so that the rotation of the drum is proportional to the strain. The stress is measured by first putting the jockey-weight *w* near its extreme position and preventing the lever A from coming down upon its stop by means of a spring *s* connected up to the pencil carriage. As the specimen becomes stressed, spring *s* is proportionally relieved from load and thus shortens in length by an amount proportional to the load applied to the specimen. The upward movement of the pencil is therefore proportional to the load, and the combined movement of drum and pencil traces out a load-extension curve which is generally—though not quite accurately—called the stress-strain diagram.

Torsion Testing Machines.—Single lever testing machines are often provided with an attachment for enabling testing by torsion to be carried out. Torsion tests to failure can be made upon comparatively small machines.

Fig. 176 shows diagrammatically the form of testing machine used by Professor Kennedy; most other machines are based upon the same principle. A graduated lever A is counter-weighted to balance about a knife-edge B coaxial with the specimen X, which usually has the form shown in Fig. 167 (*b*) with the exception that the ends are not screw-threaded. A jockey-weight *w* runs along the lever and the specimen is clamped in a chuck which is connected to the lever at the point C. The other end of the specimen is secured by a chuck carried by a worm-wheel D which is operated through a worm from a handle E to apply the necessary torque. The jockey-weight is placed so as to exert a given torque, and the handle E is turned until the lever "floats" between the stops S.

Professor Thurston's Torsion Machine. — In this machine, Figs. 177, 178, the specimen is a short one with square ends, one of which is carried by a jaw rotated by worm gear, the other being carried by a jaw connected to a weighted pendulum, the angular movement of which determines the torque applied.

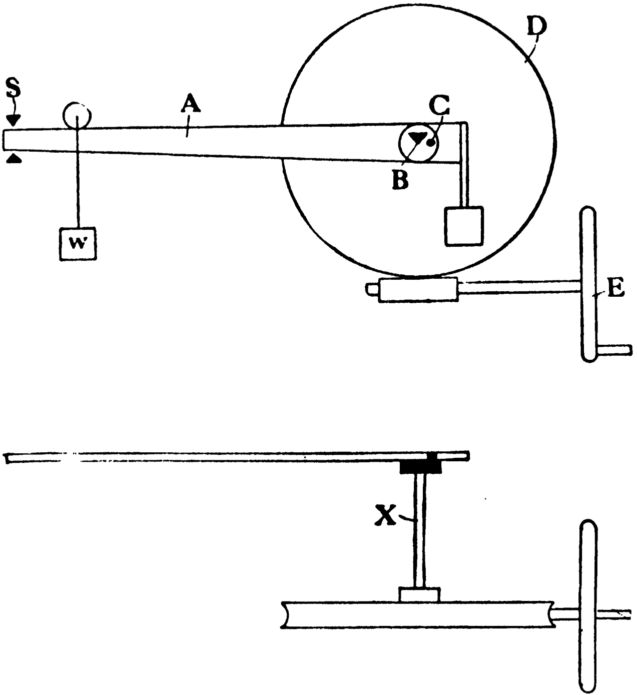

Fig. 176.—Torsion Testing Machine.

An autographic diagram is obtained by securing a pencil to the pendulum in such a manner that the pencil moves parallel to the axis of the specimen as the pendulum swings outwards. A cylinder carrying a paper strip is secured to the jaw carried by the worm-wheel. The paper thus rotates by an amount equal to the angle of torsion, and the pencil moves at right angles by an amount which is a measure of the torque applied.

A templet of the form shown in Fig. 178 is employed to obtain a standard size of specimen.

Avery's Reverse Torsion Machine.—Fig. 179 shows two views of a torsion machine, patented by Messrs. Avery, by means of which a torque can be applied in either direction.

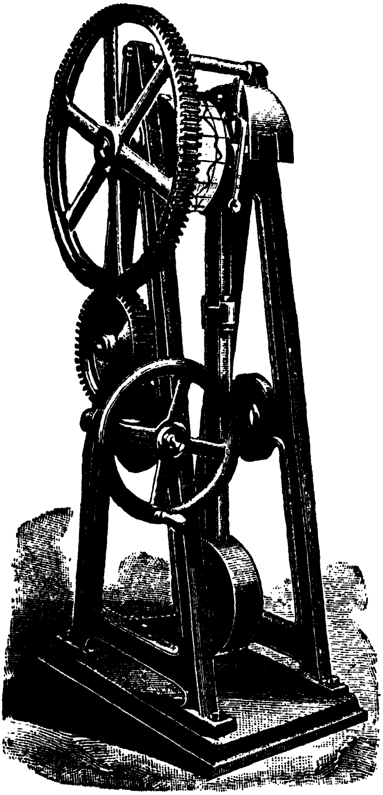

Fig. 177.—Thurston's Torsion Testing Machine.

The specimen x is gripped between special three-jaw chucks g, g', and the torque is applied from a hand-wheel i through a worm-wheel h mounted upon an adjustable standard a^4. The torque is thus communicated to a lever f and thence through a supplementary lever k and rod l to the steelyard b, upon which the usual jockey d is mounted.

The main torsion lever f is fulcrummed on ball-bearings f^1. Passing through the centre of this bearing is a spindle e of cruciform section to which the chuck g is attached. This spindle is connected with the lever f by means of rollers e^1 whereby it has limited longitudinal movement through the lever, but cannot revolve therein. This longitudinal distance is to allow of adjustment due to the shortening of specimens undergoing tests, the collar e^2 on the spindle e preventing the withdrawal or extended movement of the spindle. The main lever f is provided with knife-edges f^2 and f^3 through which connection is made to the supplementary lever k; this lever k is within the main lever f and has a ball-bearing fulcrum k^1 on a bracket a^5. It is suspended by means of the link m from the knife-edge f^2 of the main lever, and at its opposite end it is connected with the knife-edge of the main lever f^3 by a link m^1. The links m and m^1 connect with the lever k through knife-edges k^2 and k^3. Between the knife-edge k^3 and the ball-bearing k^1 is another knife-edge n which forms the connection from the levers f and k to the steelyard b by means of the rod l. The

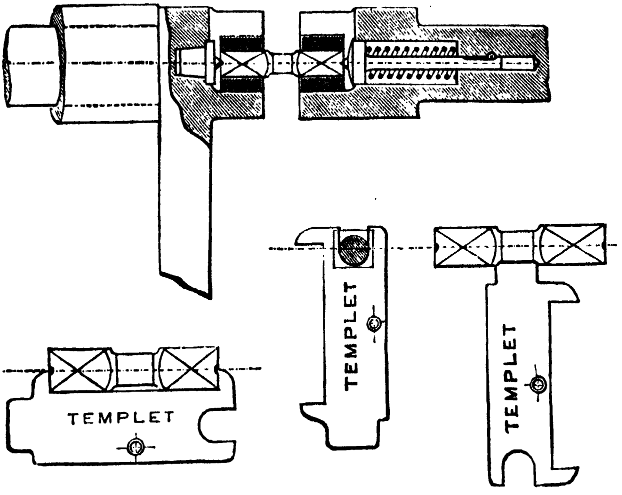

FIG. 178.

lever *f* is counterbalanced by the adjustable weight f^4, and the lever *k* by a similarly arranged weight k^4.

Assuming the torque to be applied as indicated by the arrow *y*, it lowers the end of the lever *f* which is in contact with the lever *k*; the point of greatest depression in contact with the lever *k* will be the knife-edge f^3 which through the link m^1 and knife-edge k^3 lowers this end of the lever *k*, about its fulcrum k^1, so that this movement of the lever *k* lowers the knife-edge *n*, thereby exerting a downward pull on the connecting rod *l* and raising the free end of the steelyard *b*.

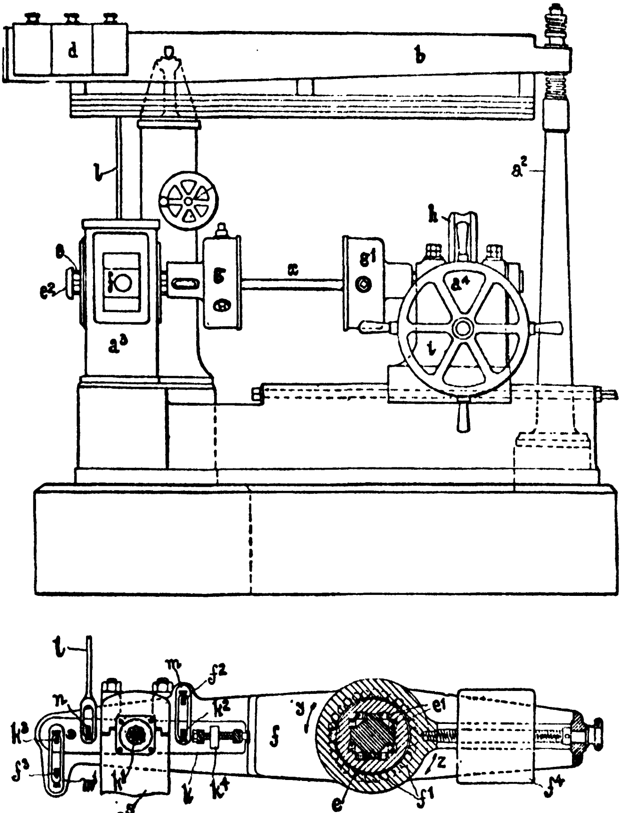

FIG. 179.—Avery's Reverse Torsion Machine.

If the torque is applied in the direction indicated by the arrow *z* the end of the lever *f* which is in contact with the lever *k* is raised and the knife-edge f^2 also raised; this upward movement of the knife-edge f^2 raises the link *m* dependent therefrom and also the knife-edge k^2 of the supplementary lever *k*, causing the lever *k* to move about its fulcrum k^1 as before. The knife-edge *n* is again depressed by this movement of the lever and exerts as before a downward pull on the connecting rod. By this arrangement, in whichever direction the torsional stress is applied to the main lever *f*, the resultant direction of force on the connecting rod *l* is the same.

PROFESSOR LILLY'S REVERSE TORSION MACHINE.—This reverse torsion machine, patented by Professor Lilly of Dublin, is a very simple machine for obtaining autographic diagrams in torsion and is particularly of value when working within the elastic limit.

A circular table A, Fig. 180, is fixed to any convenient bench or stool, and has through its centre a hollow steel cylinder B. In the central part of the cylinder is placed the specimen C to be tested, one end being secured to it by the key at D and the other end passing through the adjustable bearing E; it is secured to the lever G H I by the key at F. The lever consists of a solid shank H, which is rigidly connected to the spring I; the weight G with its connecting arm forms part of the solid shank, and is for the purpose of balancing the lever. Fixed to the spring I at J is a light arm K, at the end of which is an adjustable spring Q carrying the recording pencil L. This pencil is adjusted to slide along the straight edge N which forms part of the frame S. A circular drum M revolves on its outer edge O on the table A, and is connected by adjustable pivot bearings to the frame S which is connected by adjustable pivot bearings R to the arms of the solid shank H.

The manner of carrying out a torsion test with the machine is as follows: The specimen C to be tested is placed in position in the cylinder B, and secured to it by driving up the key at D; the lever G H I is now placed in position on the top end of the specimen O and secured to it by driving up the key at F. A sheet of squared paper is fixed on the drum M, which

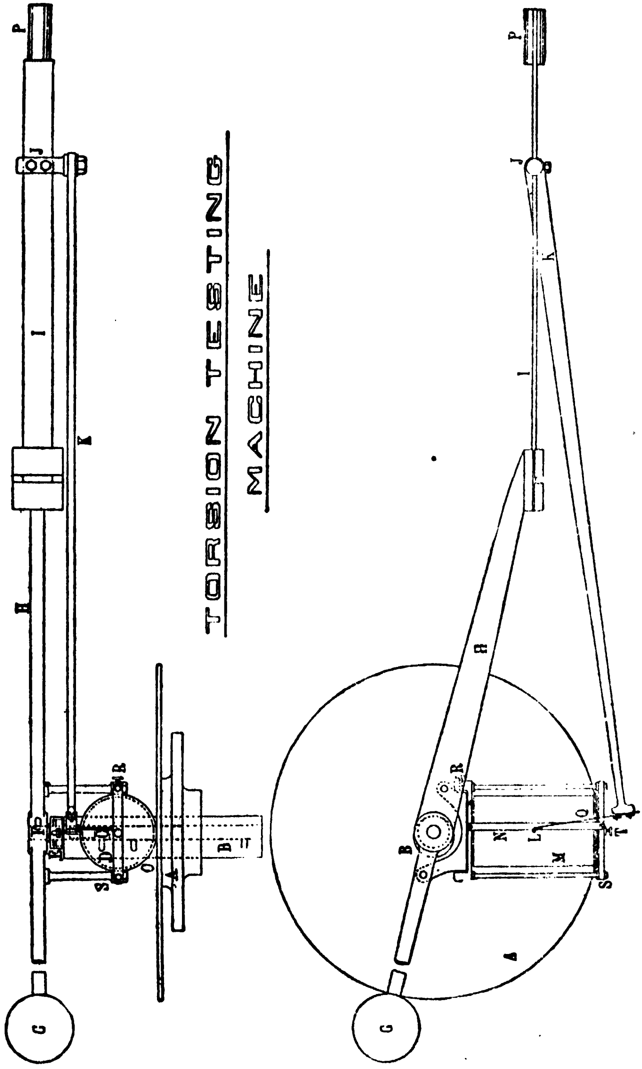

FIG. 180.—Professor Lilly's Reverse Torsion Machine.

is then put in position on the table A, and the pivot bearings adjusted; the pencil L is now placed in the central position on the drum and in contact with the straight edge N by adjusting the arm K and the spring Q. The torsion test on the specimen is carried out by applying a pull or push to the handle P; the pencil L then automatically graphs the stress-strain diagrams on the squared paper. The movement of the pencil L along the straight edge N is proportional to the push or pull on the handle P and gives to scale the magnitude of the torsional or twisting moment; this may be shown as follows—

Regarding the spring I as a cantilever with a load at the free end, the value of y and $\frac{dy}{dx}$ at the point J is proportional to the force applied at P. Calling the length of the arm z, the deflection of the pencil end is $y - z\frac{dy}{dx}$ which is proportional to the applied force. The roll of the drum M under the pencil is proportional to the angle of torsion of the specimen. The pencil graphs the combination of these two movements at right angles to one another, and the resulting stress-strain diagrams are thus obtained to rectangular co-ordinates.

To calibrate the machine a known pull is applied to the handle P by means of a spring or otherwise, and the distance traversed by the pencil along the straight edge gives to scale on the squared paper the magnitude of the applied twisting moment. The scale of the angle of torsion is obtained by observing the number of turns of the drum during one complete revolution on the circular table.

For examples of diagrams taken with this machine the reader is referred to a paper in *Proc. Inst. C. E. Ireland*, Vol. 41.

PROFESSOR COKER'S COMBINED BENDING AND TORSION MACHINE.*—This machine has been devised by Professor Coker for experiments upon combined bending and torsion.

The various parts are supported in a built-up frame consisting of two planished steel shafts, A, Fig. 181, secured in cast-iron cross frames B, mounted on four standards, one of which latter is adjustable in height to secure steadiness on an uneven floor. Upon the steel shafts are two castings C, D, each of which has a cylindrical bearing E encircling one of the shafts and resting with a flat face F in line contact with the other shaft, and secured in position by a cross-bar G threaded on studs. This connection is perfectly rigid, since it removes all degrees of freedom, and it is readily released by simply turning back one of the cross-bar nuts, leaving the casting free to slide into a new position. It also has the advantage

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* *Phil. Mag.*, April 1909.

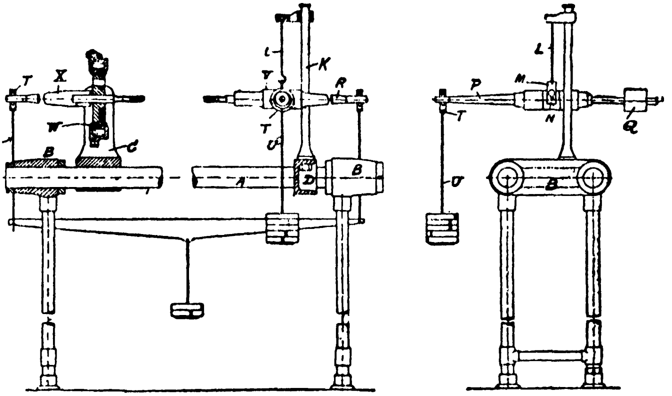

FIG. 181.—Coker's Combined Bending and Torsion Machine.

that no accurate fitting is required for the supporting frame. The casting C carrying the worm-wheel gear W has trunnion bearings H at right angles to and intersecting the axis of the specimen. The bearings are fitted with friction rollers, and when the machine is used simply for torsion the worm-wheel is kept in a vertical position by an arm I keyed to the bearing H and locked in position by a thumb-screw. A weight J attached by an arm to the second bearing balances the pivoted casting in all positions.

The weigh-levers are supported from a vertical standard K of the frame D by a wire L, terminating in a thin plate M with a keyhole slot encircling the spindle N. Formerly a roller bearing was used for this spindle, but this is an unnecessary refinement, as the friction is extremely small and can be easily taken into account. The casting supported in this way has three levers, P, Q, and R, the first two of which are for the application of twisting moments S, and the third R, in the line of the specimen, is for applying a bending moment.

All the loading levers are provided with **knife-edges** of circular form, made by turning an ordinary Whitworth nut down to form a disk with a V-shaped edge. These disks carry rings T with wide-angled V-shaped recesses on the inner sides, and light rods V screwed into these rings carry the weights. This arrangement of knife-edge is very easy to adjust accurately, and when bending and twisting stresses are applied simultaneously the rolling line contact adjusts itself to the bending and twisting of the specimen. The bending of the specimen causes a change in the effective arm of the bending levers, which is generally negligible, but a correction may be necessary with a very long specimen. For if a is the length of the lever-arm, and b is the radius of the circular knife-edge, an angular deviation of amount θ will cause a change of $a - (a \cos \theta + b \sin \theta)$ in the lever-arm, and this is zero when $\theta = 0$ and also when $a = a \cos \theta + b \sin \theta$.

In the machine described a is 10 inches and b is 0·5 inch, and the angles $\theta = 0$ and $\theta = 5{\cdot}75°$ both correspond to an effective length of 10 inches. The maximum correction between these values is easily shown to be at an angle θ given by the equation $b \cos \theta = a \sin \theta$, in the present case 2·9° approximately, for which value the correction is 0·12 per cent. In the majority of tests the angular change at the ends rarely exceeds 5°, and the correction is therefore so very small as to be practically negligible.

The worm-wheel W and the casting V for the weigh-levers are bored out to receive the ends of the specimen, and are provided with fixed keys which slide in corresponding key-ways cut in the specimen. When tubes are subjected to stress they are provided with solid ends secured by transverse pins, thereby avoiding brazed joints, since these latter are troublesome owing to the state of the metal being altered by the brazing. The end of the specimen projecting through the worm-wheel is fitted with a lever X for applying bending moment, and both levers for bending may be loaded independently or by a cross-bar suspended from stirrups as shown.

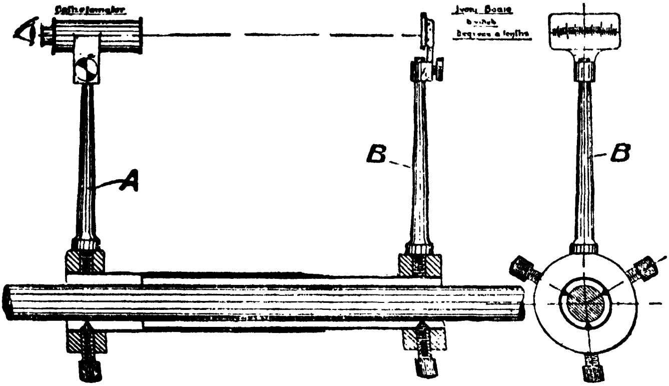

FIG. 182.—Simple Torsion Meter.

Torsion Meters.—The elastic angular strain in torsion requires less magnification than the elastic longitudinal strain in tension, and so comparatively simple apparatus can be used.

Fig. 182 shows a simple apparatus made by Mr. A. Macklow-Smith. Two arms A, B, connected together by an extensible sleeve, are secured by pointed screws to the specimen. The arm A carries a cathetometer or telescope and the arm B carries an ivory scale upon which the angle of torsion is read.

PROFESSOR COKER'S TORSION METER.[*]—This apparatus, which can be used also for measuring the strain in combined bending and torsion, consists of a graduated circle A, Fig. 183, mounted on the specimen B by three screws C in the chuck-plate D. A sleeve E provided with three screws grips the specimen at a fixed distance away from the first set. The spacing of these two main pieces on the specimen is effected by a clamp, not shown in the figure, which grips the double ones F, G, and maintains them at the correct distance apart until the set-screws are adjusted.

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[*] *Phil. Mag.*, April 1909.

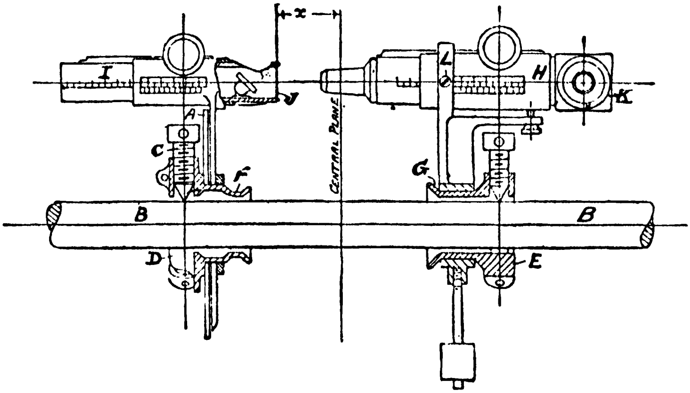

FIG. 183.—Coker's Torsion Meter.

The clamp is afterwards removed, leaving the plane of the graduated circle perpendicular to the axis of the specimen and the sleeve correctly set and ready to receive the reading microscope H.

The vernier plate carries a sliding tube I, on which a wire J is mounted, and the movement of the latter due to bending or twist is measured by a scale in the eye-piece K, the divisions of which are calibrated by reference to the graduated circle. It is found convenient to have the microscope-tube pivoted about an axis perpendicular to its central line at L, so that any slight difference due to imperfect centring can be adjusted by the screw M to make the calibration value agree for a series of specimens.

Torsion Dynamometers (or torsiometers as they are sometimes called) are instruments for indicating the horse-power being transmitted by a shaft rotating at a known speed by measuring the angle of torsion over a given length. They are often provided with autographic record devices. The horse-power is derived by a combination of formula (3), p. 312, and formula (11), p. 324.

Repetition Stress Machines.—We described on p. 85 one of the forms of machine used for rotating beams by Wöhler in his experiments in repetition of stress. Similar machines are in use in many engineering laboratories; in most cases the spring is replaced by a weight which has a spherical socket resting on a spherical bearing fixed to the end of the specimen. By this arrangement the weight remains free from oscillation as the specimen sags under load.

PROFESSOR J. H. SMITH'S MACHINE.—In this machine, which is shown in Fig. 184, the variation of stress is caused by the variation of the longitudinal component of the centrifugal force of the rotating weights E.

The specimen which is to be tested connects the two pieces C and B. The upper piece is of circular cross section, and has a locking arrangement consisting of a cap and set screws. The lower piece B is of circular section in the upper bearing L, and of square section in the lower bearing M

The two pieces C and B are supported by the frame-work F of the machine; and the specimen is inserted without straining it by first locking it to the piece B, then by locking C to the specimen and afterwards locking C to the frame.

The piece B has a bearing N at right angles to its length in which a spindle revolves; there are two plates K, K, and weights E, E, rigidly attached to this spindle.

The rotating spindle is driven by means of a pair of pieces in contact, one, a crank pin diametrically opposite to E, on one of the rotating plates K, and one a radial slot on a plate connected with either another unit or to a shaft rotating in fixed bearings. By this arrangement the driving force is not transmitted to the specimen.

The component of the centrifugal force exerted by the rotating weights E, E, produces an alternating stress in the specimen.

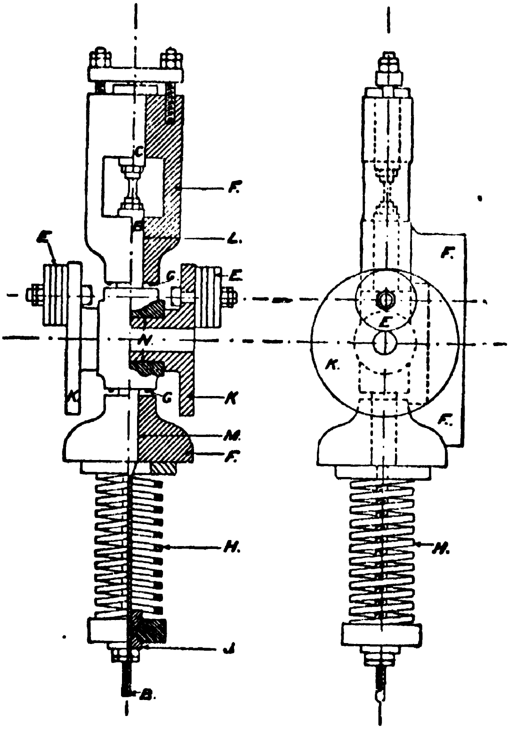

FIG. 184.—Smith's Stress Repetition Machine.

The spring H, and tightening device J, enable the operator to put any desired amount of tension or compression in the specimen. This spring may be replaced by weights and levers or by an hydraulic cylinder. The lead rings or springs at G, G, act as buffers and receive the blow when the specimen breaks.

The complete machine consists of one or more units together with the necessary balancing rotating masses which may be either parts of other units or parts connected to a revolving shaft mounted on the framework.

See also Professor Arnold's machine, p. 399.

CHAPTER XIV

THE TESTING OF MATERIALS (*contd.*)

Impact, Ductility, and Hardness Testing.—As we have indicated on p. 54, the elongation under a tensile test is commonly taken as a measure of the ductility, but experience shows that this is not sufficient in all cases, and in recent years a number of simple machines have been devised for carrying out tests upon small specimens.

Cold and hot bending tests are commonly specified by the various authorities and purchasers of steel and iron, giving the angle through which the specimen must bend without cracking. In the specification for structural steel issued by the British Standards Committee, for instance, there is a clause that test-pieces must without fracture withstand being doubled over until the sides are parallel and the internal radius is not greater than 1½ times the thickness of the test-piece, the latter being not less than 1½ inches wide.

Tests of this kind have the advantage that they can be made in the workshop without special apparatus, but the disadvantage that the results are rather negative.

CAPTAIN SANKEY'S HAND BENDING MACHINE.—In this machine, patented by Captain Sankey, a piece of metal is bent backwards and forwards through a fixed angle until it is broken; the bending moment being measured by the deflection of a spring and recorded upon a paper drum.

The standard angle is 91½°, *i.e.* 1·6 radians, so that the work done to make a complete bend is obtained by multiplying the bending moment by 1·6.

At one corner of the base of the machine there is a grip **A** for securing one end of a flat steel spring **B**. The other end of the spring is fitted with a special grip **C** for holding one end of the test-piece **D**. The other end of the test-piece is fixed into a handle **E**, about 3 feet long, by means of which it is bent backwards and forwards through the standard angle. A graduated arc **F** is provided to show this standard angle. Alongside of the spring, and fixed to the bed-plate, there is a horizontal drum **G**, to carry the recording paper, and the pencil **H** has a horizontal motion actuated by the motion of the grip **C** and conveyed by the steel wires **L** and **M** and the multiplying pulley **N**, the wires being kept taut by a weight. The zero line is in the middle of the paper, and the pencil **H**

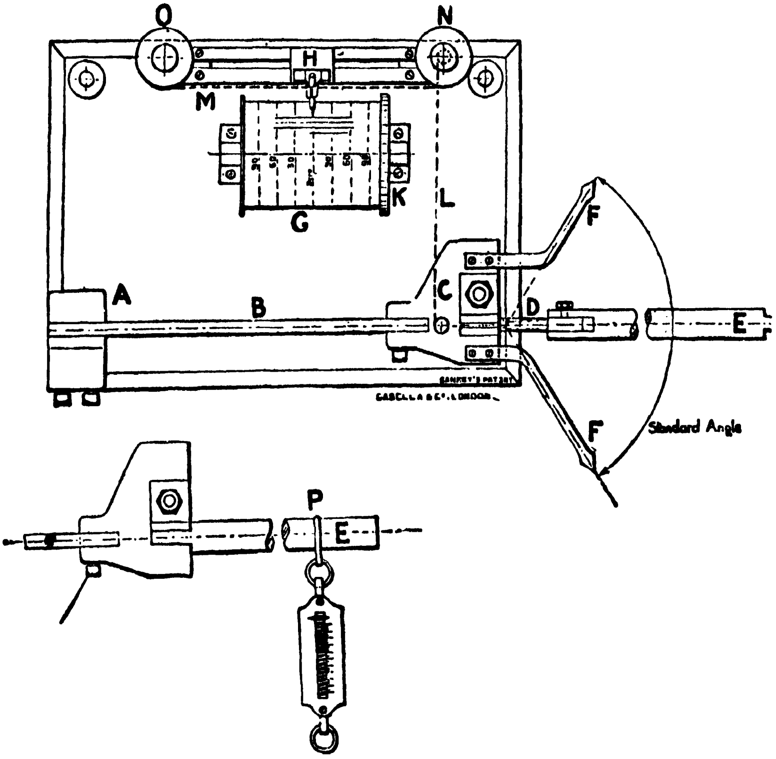

FIG. 185.—Sankey's Hand Bending Testing Machine.

moves in one direction when the bending is from right to left and in the opposite direction when it is from left to right. The drum has a ratchet wheel K, with a detent (not shown) worked by the motion of the pencil carrier. The result of the combined motion of the pencil and of the drum is to produce an autographic diagram such as shown. Obviously, the greater the stiffness of the test-piece the more the flat spring B will have to be bent before its resistance is equal to the resistance to bending of the test-piece. Hence the motion of the pencil is proportional to the bending moment required to bend the test-piece.

In operation, the test-piece is properly secured in the handle E by means of the set screw; it is then inserted into the grip C, and the free length (1¾ inches) is adjusted by means of a gauge provided for the purpose, after which the grip C is tightened. The handle is slowly pulled towards the left until the specimen is felt to be "yielding"—this action can be distinctly felt, and this bend is known as the "yield bend." Without altering the pressure on the handle, the record cylinder is now rotated two teeth by working the detent by hand, and the first bend is completed by making the mark on the handle coincide with the pointer indicating the "standard" angle. The bending is then reversed, and the test-piece is bent until the mark on the handle coincides with the second pointer. The bending is again reversed, and so on until the specimen breaks. The point at which the test-piece breaks should be noted in decimals of one bend, which are marked on the graduated arc.

The machine is calibrated by fixing the handle end of the lever in the jaws and applying a known force by means of a spring-balance and comparing the record made on the strip with the actual moment applied. If there is any discrepancy between the two results, the spring is adjusted until such discrepancy disappears. The number of bends which a given material can endure before fracture is a measure of the ductility, and experiment shows that this is approximately proportional to the percentage elongation multiplied by the percentage reduction in area in a standard tensile test.

The following empirical results have been found from experiment to be approximately true—

Yield-point stress in tons per sq. in.

$$= \frac{\text{First bending moment in lb. ft.}}{1{\cdot}55}$$

Ultimate tensile stress in tons per sq. in.

$$= \frac{\text{Longest line in lb. ft.}}{1{\cdot}55}$$

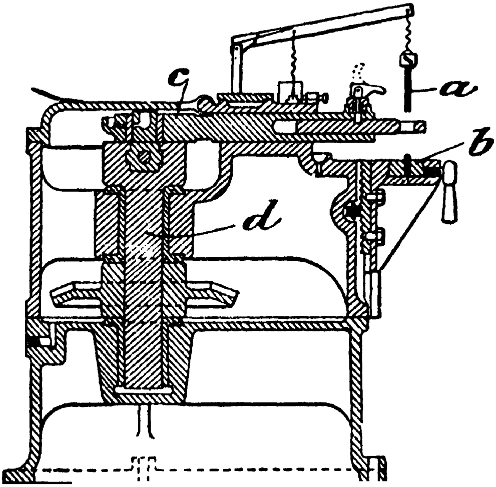

Fig. 186.—Arnold's Testing Machine.

The energy required to cause rupture is equal to 1·6 multiplied by the number of bends, multiplied by the mean range of bending moment in lb. ft.

Professor Arnold's Reverse Bending Machine.—In this machine, which may also be regarded as a repetition of stress machine, a bar *a*, Fig. 186, $\frac{3}{8}$ in. in diameter, is firmly held in a clamp *b* and passes through a slot in a slide *c* which is reciprocated by a shaft *d* running at a standard speed of 650 revolutions per minute. The distance between line of contact of the slot with the specimen and the point where the latter enters the clamp is 3 inches and the slot is adjusted to cause a deflection of $\frac{3}{8}$ in. in the specimen on each side. The number of bends which the specimen endures before fracture is taken as a measure of the capacity of the material to resist failure by shock.

Repeated Impact Testing Machine.—The machine shown in Fig. 187 is made by the Cambridge Scientific Instrument Co., Ltd., and is a modification of a machine described by Messrs. Seaton and Jude * and used by Dr. Stanton † of the National Physical Laboratory.

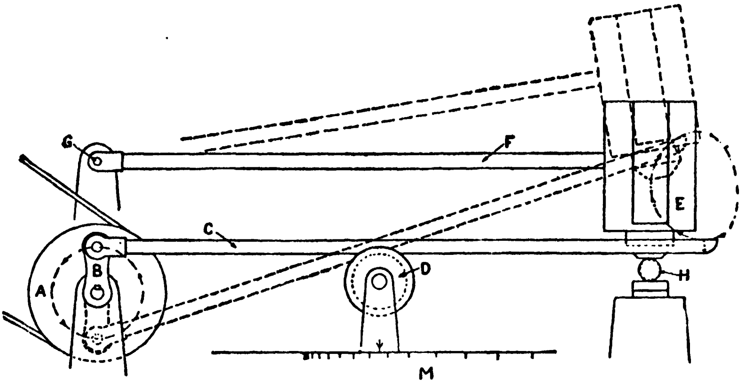

FIG. 187.—Repeated Impact Testing Machine.

The machine is fitted with a cone-pulley A, so that it can be driven by a belt from a line shaft or small electric motor. One end of the spindle driven by this cone-pulley carries a crank B which is connected to the lifting rod C. This lifting rod is supported on a roller D, at some point in its length, so that the circular motion imparted to the rod at the crank end causes it to rock and slide on the roller. Thus an oval path, shown in dotted lines, is traced by the free end of the lifting rod. At this end the rod is bent at right angles so that on the upstroke it engages with and lifts up the hammer head E. This hammer head is fixed to the rod F, which is hinged at the end G. Having reached the top of its path, the lifting rod C moves

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* *Proc. I. Mech. E.*, 1904. † *Proc. I. Mech. E.*, 1908.

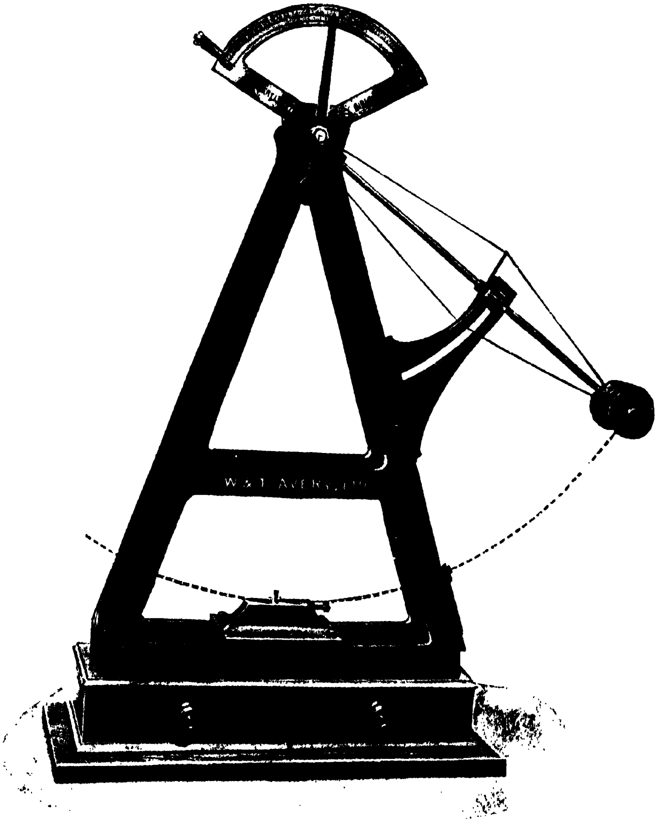

FIG. 188.—Impact Testing Machine.

[To face page 401.

forward and disengages the hammer, which then falls freely on to the specimen H under test.

This cycle is repeated from 70 to 100 times a minute. The height through which the hammer falls can be varied by moving the roller D along a scale M which is calibrated to read directly the vertical height through which the hammer falls. Adjustment can be made by this means up to a maximum of 3½ inches (90 mm.).

The specimen H is usually about ½″ (12 mm.) in diameter, with a groove turned in it at its centre to ensure its fracture at this point in its length. It is supported on knife-edges 4½″ (114 mm.) apart, the hammer striking it midway between these knife-edges. The knife-edges are cut slightly hollow, and a finger spring holds one end of the specimen in place. The other end is held in a chuck which is hinged in such a manner that it does not take any portion of the hammer blow, all of which comes on the knife-edges.

The specimen remains stationary whilst the blow is struck, but between the blows it is turned through an angle of 180°.

A revolution counter to register the number of blows struck is fixed to the bed plate of the instrument. When fracture occurs, the specimen falls away, and the hammer head continues to fall, first tripping an electric switch, and finally coming to rest on a steel stop-pin II.

Izod's Impact Testing Machine.—This machine, which is made by Messrs. Avery, tests the impact-resisting qualities of a material by measuring the energy absorbed from a pendulum which breaks a projecting notched specimen as it swings past it.

The specimen is 2 inches long, $\frac{3}{16}$ inch thick and ⅜ inch broad, and has a notch cut in it by means of a templet; it is held in the vice shown at the base of the machine, Fig. 188, and the pendulum is then released from a fixed height by means of a trigger. The energy required to fracture the specimen takes some of the swing out of the pendulum, and the height to which the latter swings on the other side is indicated by the pointer passing over the scale at the top of the specimen. The scale is graduated to give the energy directly in foot pounds. A brittle material will not absorb much of the energy, whereas a tough material will absorb a good deal.

Inasmuch as the base is not absolutely rigid, the results of tests in this machine are relative rather than absolute, but it gives very useful results in practice and has the advantage that the tests can be made in a very short time.

The same machine can be adapted for testing hardness by impact. A strong cast-iron anvil is provided in which a specimen 1 inch in diameter and 1 inch long is placed. The pendulum strikes a loose plunger which carries a hardened steel ball; this ball is placed in contact with the specimen prior to the impact, which causes an indentation in the specimen. The diameter of this indentation is taken as a measure of the hardness as in the Brinell machine next to be described.

Brinell's Hardness Testing Machine.—In this machine a hardened steel ball is pressed with a predetermined force against the plate whose hardness is required. The diameter of the resulting curved depression is then found and from this the " hardness number " is obtained in the manner described below.

Fig. 189 shows the Brinell machine made by Messrs. J. W. Jackman & Co., Ltd. The specimen is placed upon the top of the stand, which is then adjusted by the hand-wheel to bring the specimen into contact with the hardened steel ball (10 mm. diameter) which projects from the conical end of the plunger of the machine. The upper portion of the machine comprises a small fluid-operated testing-machine, oil being the working fluid. By means of a small projecting pump-handle the pressure of the fluid is increased until the cross-piece " floats," the pressure being indicated on the dial. Weights are provided with the machine, which make the floating occur at a force of 500 to 3000 kg. (increasing 500 kg. at a time).

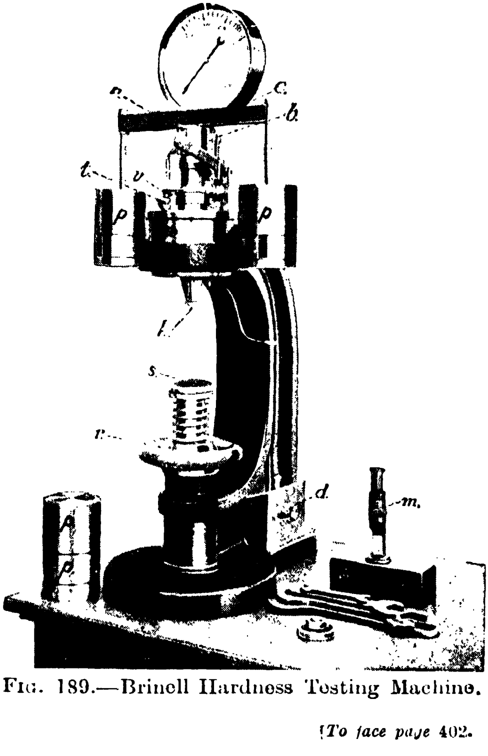

FIG. 189.—Brinell Hardness Testing Machine.

[To face page 402.

The pressure depends only on the weight applied and not upon the accuracy of the gauge.

If P is the load, D the diameter of the ball and d that of the impression, the quantity

$$H = \frac{P}{\frac{\pi D}{2}\left(D - \sqrt{D^2 - d^2}\right)}$$

is called the Brinell Hardness Number. The following table gives values.

BRINELL'S HARDNESS NUMBERS (FOR LOAD 3000 KG.)

Diameter of Steel Ball = 10 mm.

Diameter of Ball Impression mm.	Hardness Number	Diameter of Ball Impression mm.	Hardness Number	Diameter of Ball Impression mm.	Hardness Number	Diameter of Ball Impression mm.	Hardness Number
2·0	946	3·25	351	4·50	179	5·75	105
2·05	898	3·30	340	4·55	174	5·80	103
2·10	857	3·35	332	4·60	170	5·85	101
2·15	817	3·40	321	4·65	166	5·90	99
2·20	782	3·45	311	4·70	163	5·95	97
2·25	744	3·50	302	4·75	159	6·0	95
2·30	713	3·55	293	4·80	156	6·05	94
2·35	683	3·60	286	4·85	153	6·10	92
2·40	652	3·65	277	4·90	149	6·15	90
2·45	627	3·70	269	4·95	146	6·20	89
2·50	600	3·75	262	5·0	143	6·25	87
2·55	578	3·80	255	5·05	140	6·30	86
2·60	555	3·85	248	5·10	137	6·35	84
2·65	532	3·90	241	5·15	134	6·40	82
2·70	512	3·95	235	5·20	131	6·45	81
2·75	495	4·0	228	5·25	128	6·50	80
2·80	477	4·05	223	5·30	126	6·55	79
2·85	460	4·10	217	5·35	124	6·60	77
2·90	444	4·15	212	5·40	121	6·65	76
2·95	430	4·20	207	5·45	118	6·70	74
3·0	418	4·25	202	5·50	116	6·75	73
3·05	402	4·30	196	5·55	114	6·80	71·5
3·10	387	4·35	192	5·60	112	6·85	70
3·15	375	4·40	187	5·65	109	6·90	69
3·20	364	4·45	183	5·70	107	6·95	68

For other test loads, the hardness numbers are proportional to those in the table.

Within certain limits the Brinell Hardness Number of a material gives a very fair indication of its tensile strength. Thus for steels with a hardness number less than 175, the ultimate tensile stress in tons per sq. in. is obtained approximately by multiplying the hardness number by ·23.

TESTING CEMENT AND CONCRETE

Tension Tests.—The form of briquette in accordance with the Specification of the British Engineering Standards Committee is shown in Fig. 190, the cross-section being 1 sq. in. at the weakest point. We have given on p. 78 the requirements as to tensile strength in accordance with this specification. As the strength obtained under test is found to depend upon the rate of loading, being higher for quick loading, the above specification stipulates that the loading shall be at a rate of 100 pounds in 12 seconds.

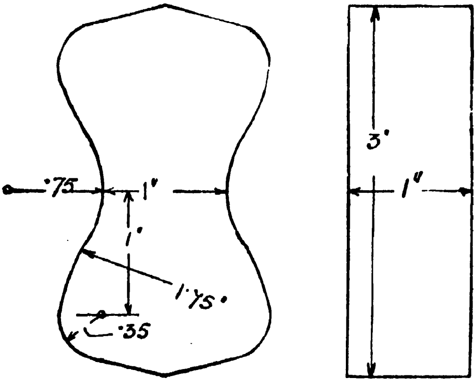

Fig. 190.—Cement Briquette.

A simple form of lever machine, made by W. H. Bailey & Co., is illustrated in Fig. 191. The specimen is gripped in the shackles and the load is applied by allowing shot to fall into the bucket, the leverage being such that the tension applied is fifty times the weight of the shot. The shot-hopper is provided with a valve, the operating arm of which passes over the lever, so that when the specimen breaks the supply of shot is automatically cut off. The shot is then weighed, a spring-balance being often used which gives readings equal to fifty times the weight of the shot, thus giving the breaking stress direct.

A rather more accurate form of machine, made by the same firm, is shown in Fig. 192. In this machine water is allowed to run slowly into a long graduated can placed at the end of the lever. The supply of water is cut off when fracture occurs, and a gauge glass placed outside the can is provided with a scale graduated to enable the breaking stress to be read off direct.

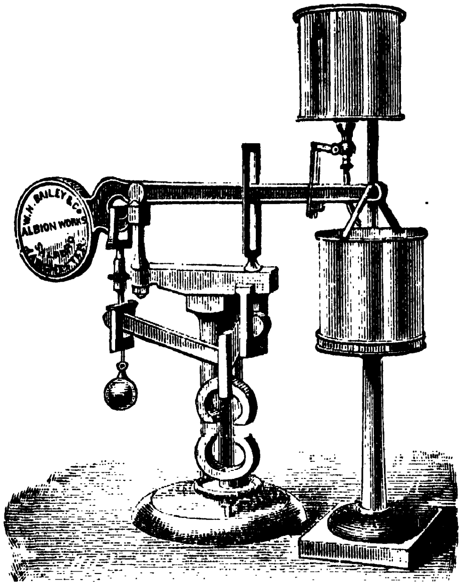

Fig. 191.—Cement Testing Machine (Tension).

In another common form of testing machine a jockey-weight is moved automatically along a lever arm by means of a weight controlled by an adjustable dashpot which enables the rate of loading to be varied. On the fracture of the specimen the weight becomes stationary and the breaking stress is read off on a scale attached to the lever.

Compression Tests.—Compression tests are not usually specified for pure cement, although they are becoming more common. For concrete in reinforced concrete works, however, compression tests are nearly always required.

A common specification is that cubes, the area of each side being 50 sq. cm., of 3 parts sand to 1 part cement by weight shall develop at 28 days at least 10 times the standard tensile strength (*i. e.* 2000 lbs. per sq. in.).

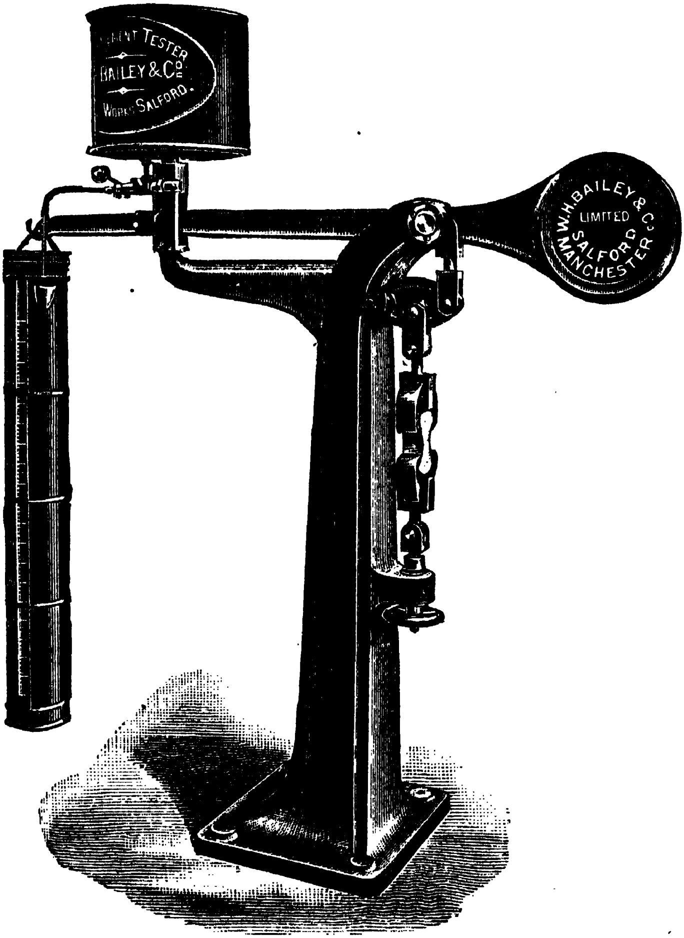

FIG. 192.—Cement Testing Machine (Tension).

The following test results for the concrete cubes (with area of each side 50 sq. cm.) are recommended by the Institution of Structural Engineers.

Proportion by Volume.			Crushing Strength in lbs. per sq. in.	
Cement	Sand	Coarse Material	28 days after mixing	120 days after mixing
1	2	4	1600	2400
1·2	2	4	1800	2600
1·5	2	4	2000	2800
2	2	4	2200	3000

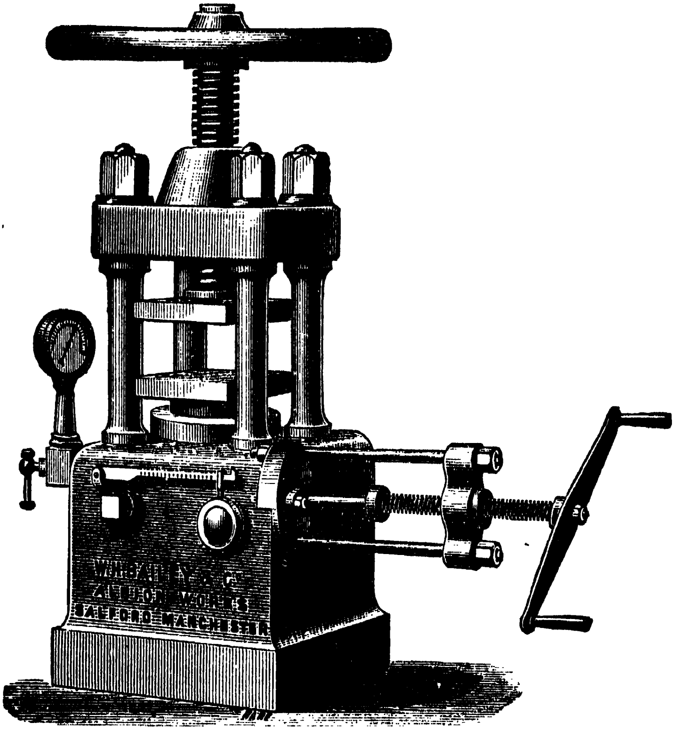

Fig. 193.—Compression Tension Press for Cement, etc.

A common form of special machine for crushing tests of cement, concrete, etc., is shown in Fig. 193. The cube is first fixed by means of the upper hand-wheel and the side-press screw is then operated to compress the operating fluid (oil or glycerine). The crushing pressure is recorded by the pressure gauge which is constructed so as to maintain its reading after fracture has occurred.

Specific Gravity, Fineness, Soundness, and Setting Tests for Portland Cement.

Specific Gravity.—According to the earlier British Standard Specification, the specific gravity of Portland cement shall not be less than 3·15 when fresh or 3·10 after 28 days from grinding.

There is considerable doubt as to the value of this test; some very useful information on this and other matters in cement testing will be found in a paper on "Common Fallacies in Cement Testing," read by Mr. W. L. Gadd, F.I.C., before the Concrete Institute, December 11, 1913. This test has now been omitted from the British Standard Specification test.

Fineness.—The fineness test is applied by means of standard sieves. In the British Standard Specification not more than 14% residue is allowed for a sieve of square mesh with 180 wires, each ·0018 in. in diameter, per inch, and not more than 1% for a 76 mesh with wire ·0044 in. in diameter.

Le Chatelier Soundness Test.—This test is conducted with a piece of split brass tube A, Fig. 194, 30 mm. internal diameter and 30 mm. long, the thickness of metal being ½ mm. Pointers B B are attached to the tube, the length from their points to the centre of the tube being 165 mm.

This tube is used as a mould, and is filled with wet cement, one end being placed previously on a piece of glass; the other end is then covered with a weighted piece of glass and the whole is placed in water at a temperature from 58° to 64° F. and left for 24 hours. The distance between the pointers is then measured and the mould is then placed in water which is heated to boiling point and maintained in that condition for 6 hours. The British Standard Specification stipulates that after boiling the increase in the distance apart of the pointers shall not be greater than 10 mm.

Setting Tests.—Setting tests are often made by finding the time before a standard weighted needle fails to make an impression in the cement. The Standard Specification formerly required that before any sample is submitted to setting test it shall be spread out for a depth of 3 inches for 24 hours in a temperature from 58 to 64° F., and that the setting time shall not be less than 2 hours nor more than 7 hours.

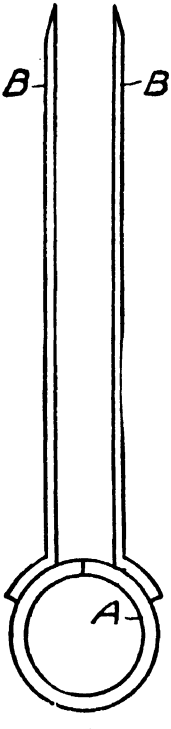

FIG. 194.—Chatelier Cement Test.

Mr. Gadd in the paper referred to above has shown that this spreading for the purpose of aeration leads to very variable results, and it has now been omitted from the specification.

The Thermal Method of Testing Materials.—It was apparently first noticed by Magnus that changes of stress are accompanied by a change of temperature; when a body is stretched its temperature lowers very slightly, and when it is compressed or squeezed its temperature rises. By means of a thermopile and a very delicate galvanometer, therefore, the changes of temperature at various points of a structure, and therefore the stresses, can be determined. Care should be taken to distinguish the phenomenon under consideration from the well-known heating that occurs at the yield point in tension experiments; this is very much greater in magnitude and is reverse in sign, the elastic tension being accompanied by a fall in temperature.

Lord Kelvin has deduced the following formula to deal with the problem—

$$\Delta T = -\frac{T a}{J S d} \cdot \Delta p$$

In this formula

ΔT = change in temperature.
T = mean absolute temperature.
J = the mechanical equivalent of heat.
a = coefficient of expansion of material.
d = density of material.
S = specific heat of material.
Δp = change of stress.

In centigrade units for a temperature of about 20° C. this formula gives for steel

$$\Delta T = -\cdot 000012\, \Delta p$$

Corresponding to a stress change of 20,000 lbs. per sq. in., therefore, we have a temperature change of only ·24° C. The extreme delicacy of the method makes it suitable for use only under circumstances in which great care is taken to exclude draughts.

This subject is dealt with in detail by Professor Coker in his Cantor Lectures before the Society of Arts 1913. In this paper the results of thermal tests upon a channel section were given, and there was a marked departure from the straight-line relation in the stress diagram, this being accounted for by the asymmetry of the section. Other experimental results were quoted which showed that the tensile yield point as determined by the thermal method agree very closely with that found by extensometer. In the curve of temperature plotted against load, there is a sharp cusp at the yield point because the temperature then rapidly rises instead of falling. In these experiments, the observed results have to be corrected for cooling effects. Although of very great interest, the method seems rather too delicate for very extended application.

The Optical Method of Testing Materials.—Sir David Brewster discovered a hundred years ago that when plane polarised light is sent through a piece of glass under stress, an effect is produced upon the light which is detected by the appearance of colour bands when viewed through an analyser. The optical aspects of the subject are beyond our present scope, but the reader will be able to study these from the bibliography given below. The mathematical problems involved were dealt with by Clerk Maxwell and others, and in recent years particular attention has been given to the application of the method to the determination of stresses in various machine and structural details by Professors Alexander, Filon and Coker.

If a beam of plane-polarised light is passed through a specimen of transparent material, such as glass or xylonite, and Nicol's prisms in the polariser and analyser are set with their principal planes at right angles so as to cut off the light, no effect is produced if there is no stress in the material, but if there is any stress, a colour effect is produced, and regions of zero stress or equal and opposite principal stresses, such as the neutral axis of a beam, can be detected by a dark patch or line. If the material is elastic, the colour produced will be a measure of the difference of the principal stresses at any point. The stress corresponding to a given colour is determined numerically by experiment by uniformly loading a small specimen until the colour produced is the same as at any particular point of the model under consideration where the stress is required. In cases where one of the principal stresses is negligibly small, the stress thus obtained can be taken as equal to the greater principal stress required.

In Professor Coker's experiments very successful results have been obtained with models of various structional and machine details cut out of sheet xylonite.

Fig. 195 * shows the results of his experiments upon a tie-bar eccentrically loaded; the resulting curves of stress agree very well with the theoretrical straight-line variation, with the exception of that at the highest load, in which the material begins to yield at the edges and the straight line bends over as shown. In this case the test-piece showed residual stress at this edge after the load had been removed.

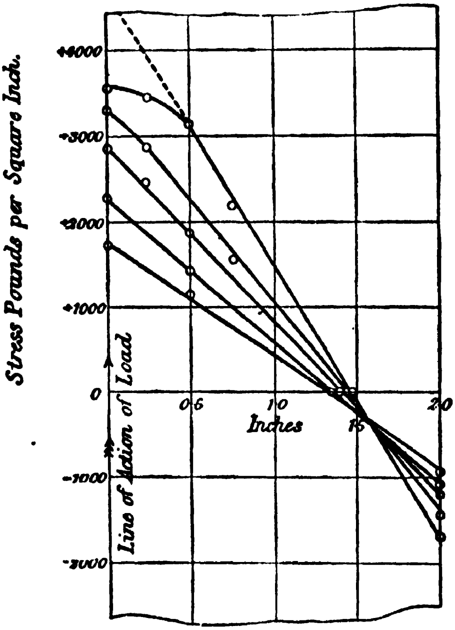

FIG. 195.—Stresses in Eccentrically Loaded Tie-bar.

In experiments upon models of standard cement briquettes Professor Coker found † that the maximum stress was about 1·75 times the mean stress (cf. p. 78).

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* *Engineering*, January 6, 1911.

† *Ibid.*, December 13, 1912.

Fig. 196[*] shows the results of the experiments by the same investigator upon the distribution of stresses in rectangular beams with holes cut through them. In the upper specimen a hole is in the centre and in the lower one holes are formed half way between each edge and the neutral axis. In this case it will be noted that the maximum stress

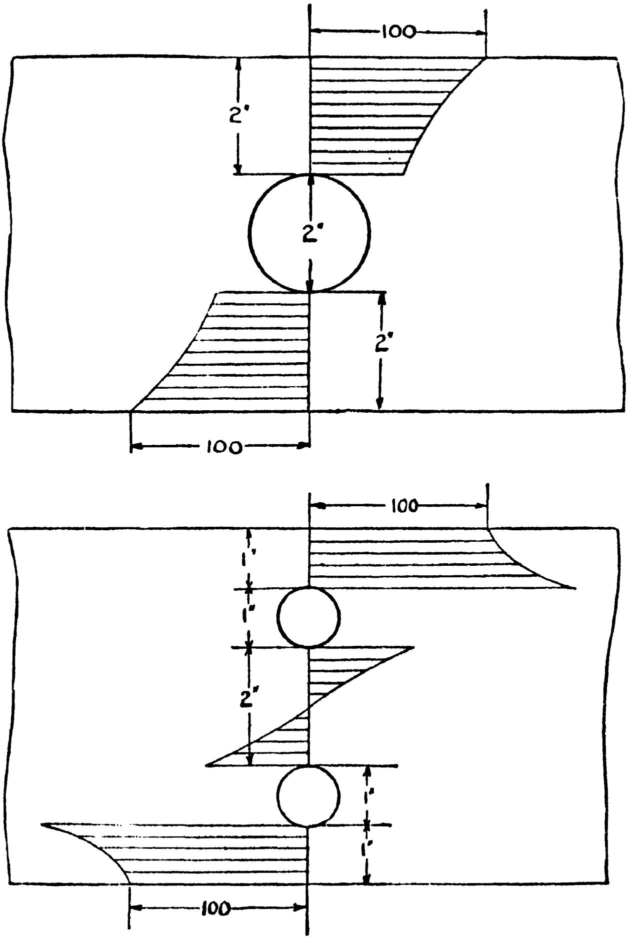

Fig. 196.—Effect of Holes on Stress in Beams.

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[*] *Engineering*, March 3, 1912.

does not occur at the outermost fibre, but at the outer edge of each hole.

Professor Coker and Mr. W. A. Scoble, B.Sc.,* have also experimented upon the stresses in tie-bars with holes in them such as occur in riveted joints. If $c = \frac{\text{width of plate}}{\text{diameter of hole}}$ and p is the mean stress over the whole width of plate, they find that their results for one central hole may be expressed by the formula

$$\frac{\text{maximum stress}}{\text{mean stress}} = \frac{3c}{c+1}$$

Also if a is the radius of the hole, the longitudinal stress at a distance r from the centre of the hole on a normal section of the plate is expressed by the formula

$$f_r = \frac{p}{2}\left(2 + \frac{a^2}{r^2} + \frac{3a^4}{r^4}\right)$$

where p is the stress at a long distance from the hole; while there is also a radial stress given by

$$f_r = \frac{3p}{2}\left(\frac{a^2}{r^2} - \frac{a^4}{r^4}\right)$$

It will be noted that at the edge of the hole $f_r = 3p$, i.e. three times the stress some distance from the hole.

The following results were obtained with a strip 1 in. wide and ·186 in. thick, the load being 100 lbs.

Diameter of Central Hole (inches)	Stresses in lbs. per sq. in.		
	p	mean	maximum
$\frac{1}{16}$	549	584	1470
$\frac{1}{8}$	547	620	1560
$\frac{1}{4}$	568	724	1770
$\frac{3}{8}$	570	868	1850
$\frac{1}{2}$	613	1035	2040

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* *Engineering*, March 28, 1913; *Society of Arts Journal*, January 16, 1914.

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Journal Society of Arts, Cantor Lecture, 1913.

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* Reprinted in Alexander and Thomson's *Elementary Applied Mechanics* (Macmillan).

CHAPTER XV

FIXED AND CONTINUOUS BEAMS

If the ends of a beam are fixed in a given direction so that they are not able to take up the inclination due to free bending, or if a beam rests on more than two supports, the B.M. and shear diagrams will be different from the cases of simply supported beams that we have considered up to the present. In the first case the beam is said to be *fixed*, *built-in*, or *encastré*, and in the second it is said to be *continuous*.

We will consider how the shear and B.M. diagrams can be found for such beams, and will point out their advantages and disadvantages compared with simply supported beams.

FIXED BEAMS

If the ends of a beam are fixed in a horizontal direction, then the beam when bent takes up some form such as A B C (Fig. 197). If the ends were free it would assume the dotted form A′ B C′, and to get it back to the form A B C, negative bending moments, shown diagrammatically as due to forces F_1, F_2, have to be imposed upon it. The ends of the beams will therefore be subjected to bending moments which will be negative because they cause curvature in an opposite direction

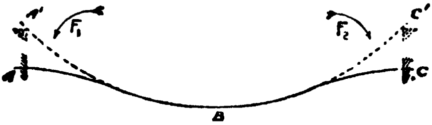


FIG. 197.—Fixed Beams.

to that due to the load. This change in sign of the bending moment means that the tension and compression sides of the beam are reversed. We will consider the cases of fixed beams both from the graphical and the mathematical standpoint, as we did in the case of the deflections of beams.

INVESTIGATION FROM GRAPHICAL STANDPOINT

According to Mohr's Theorem, the deflected form of a beam is the same as that of an imaginary cable of the same span loaded with the bending moment curve of the beam, and subjected to a horizontal pull equal to the flexural rigidity (E I). If the ends of a beam are fixed in a horizontal direction, the first and last links of the link polygon determining the elastic line will be parallel; this means to say that the first and last points on the vector line on which the elemental areas of the bending moment curve are set down must coincide. But this is equivalent to saying that the *total area* of the bending moment curve for the fixed beam must be zero. This enables us to enunciate the following rule—

If the ends of a beam are fixed in a horizontal direction at the same level, and the section of the beam is constant along its length, there will be negative bending moments induced, and the area of the negative bending moment diagram will be equal to that due to the load for the beam if considered freely supported.

We will speak of the negative bending moment diagram as the "end B.M. diagram," and that for the beam freely supported as the "free B.M. diagram."

The problem now divides itself into two cases: (*a*) That in which the loading is symmetrical. (*b*) That in which the loading is irregular or asymmetrical.

Symmetrical Loading.—If the loading is symmetrical, then the beam looks the same from whichever side it is viewed, and so the end bending moments will be equal, and their value can be found by dividing the area of the free B.M. diagram by the span. This will be made more clear by considering the following cases—

(1) **Uniform Load on Fixed Beam.**—Let a uniform load of intensity w cover a span A B (Fig. 198) of length l. The free B.M. curve is in this case a parabola A C B, with maximum ordinate $\frac{w\,l^2}{8}$.

Therefore, since the area of a parabola is two-thirds of the area of the circumscribing rectangle, area of free B.M. curve

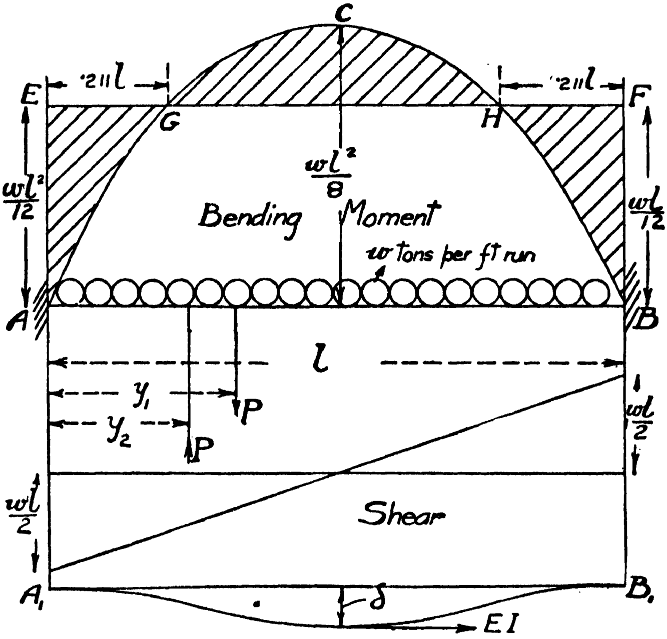


Fig. 198.—Fixed Beam with Uniform Load.

$$= \frac{2}{3} l \times \frac{w\,l^2}{8} = \frac{w\,l^3}{12}$$

$$\therefore \text{ End B.M.} = \frac{w\,l^3}{12} \div l = \frac{w\,l^2}{12}$$

Then setting up A E and B F equal to $\frac{w\,l^2}{12}$ and joining E F we get the end B.M. diagram, and the effective B.M. curve is the difference as shown shaded. At the points G and H the B.M. is zero, and these points are called the *points of contra-flexure*, the curvature of the elastic line changing sign at these points.

Suppose the point G is at distance x from the centre of the beam, then the ordinate of the parabola must be equal to $\frac{w\,l^2}{12}$

$$\text{i.e. } \frac{w}{2}\left(\frac{l^2}{4} - x^2\right) = \frac{w\,l^2}{12}$$

$$\therefore \frac{w\,x^2}{2} = \frac{w\,l^2}{24}$$

$$\therefore x^2 = \frac{l^2}{12}$$

$$x = \frac{l}{2\sqrt{3}}$$

$$\therefore \text{Distance of G from E} = \frac{l}{2} - x = \frac{l}{2} - \frac{l}{2\sqrt{3}}$$

$$= \frac{l}{2}\,\frac{(\sqrt{3} - 1)}{\sqrt{3}} = \frac{l}{6}\left(3 - \sqrt{3}\right)$$

$$= \cdot 211\,l$$

SHEAR DIAGRAM.—With symmetrical loading the shear diagram will be the same as for the simply supported beam. This is because the shear at any point of a beam is equal to the slope of the B.M. curve at that point, and the slope of the B.M. is not altered in the case of symmetrical loading because the base line of the diagram is merely shifted vertically.

DEFLECTION.—The deflection at the centre can be found as before by considering the stability of the imaginary cable A_1 B_1.

Considering the stability of the left-hand half of the cable, then taking moments about A_1, we have

$$E\,I \times \delta = P\,y_1 - P\,y_2$$

$$= P\,(y_1 - y_2)$$

In this case P = area of one-half of the free B.M. curve,

$$= \frac{2}{3}\cdot\frac{l}{2}\cdot\frac{w\,l^2}{8} = \frac{w\,l^3}{24}$$

$$y_1 = \frac{5\,l}{16}$$

$$y_2 = \frac{l}{4}$$

$$\therefore E\,I \times \delta = \frac{w\,l^3}{24}\left(\frac{5\,l}{16} - \frac{l}{4}\right) = \frac{w\,l^4}{384}$$

$$\therefore \delta = \frac{w\,l^4}{384\,E\,I} = \frac{W\,l^3}{384\,E\,I}$$

It will be noted that this is one-fifth of the deflection for a freely supported beam with the same loading.

(2) **Isolated Central Load on a Fixed Beam.**—In this case the area of the free B.M. curve $= \frac{1}{2} l \times \frac{W l}{4} = \frac{W l^2}{8}$

$$\therefore \text{End B.M.} = \frac{W l^2}{8} \div l = \frac{W l}{8}$$

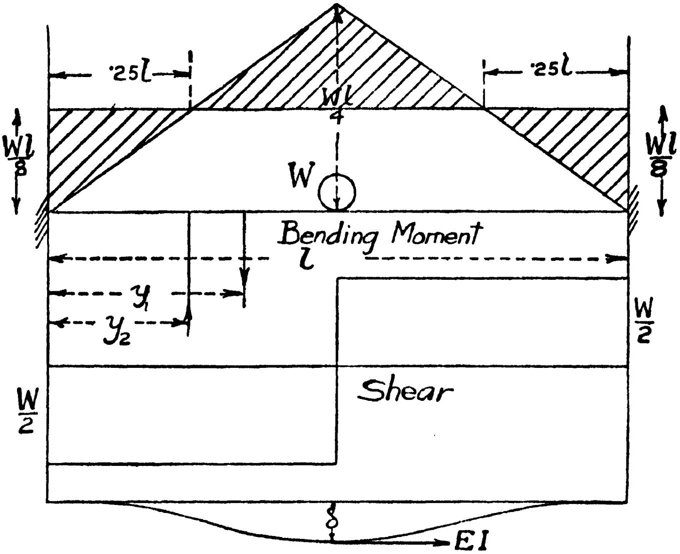


Fig. 199.—Fixed Beam with Central Load.

The B.M. and shear diagrams are as shown in Fig. 199, the points of contraflexure being at ¼ and ¾ span.

Deflection.—As in the previous case we have

$$E I \times \delta = P (y_1 - y_2)$$

$$\text{In this case } P = \frac{W l}{8} \cdot \frac{l}{2} = \frac{W l^2}{16}$$

$$y_1 = \frac{l}{3}$$

$$y_2 = \frac{l}{4}$$

$$\therefore E I . \delta = \frac{W l^2}{16}\left(\frac{l}{3} - \frac{l}{4}\right) = \frac{W l^3}{192}$$

$$\therefore \delta = \frac{W l^3}{192 E I}$$

This is one-fourth of the deflection for a freely supported beam with the same loading.

* **Asymmetrical Loading.**—In this case the end B.M.s will not be equal, and in this case, in addition to the condition that the areas of the end B.M. diagram and free B.M. diagram must be equal, we have the further condition that their centres of gravity must fall on the same vertical line.

This can be proved as follows: Considering the imaginary cable and taking moments about one end, the tension at the other end passes through the point so that its moment is zero Therefore the moment of the B.M. diagrams about this point must be zero. Since the areas of these diagrams are equal, their centres of gravity must be at the same distance from the given point.

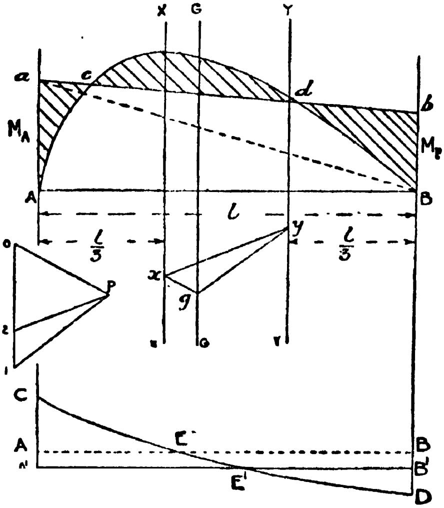


FIG. 200.—General Case of Fixed Beams.

Let a span A B, Fig. 200, of length l, be subjected to any irregular load system which produces a free B.M. curve A c d B, and let the centre of gravity of that diagram lie upon the vertical line G G. Suppose the end B.M.s are M_A and M_B, and A a and B b are set up equal to these end B.M.s, then the trapezium A a b B is the end B.M. diagram, and the conditions that have to be satisfied are that the area of the trapezium shall be equal to the area of the curve A c d B, and that its centre of gravity shall lie upon the line G G. Join a B, thus dividing the trapezium into two triangles, and draw verticals X X and Y Y at distances equal to $\frac{l}{3}$ from A and B. The centres of gravity of the triangles A a B, B a b lie on the lines X X and Y Y respectively, and our problem resolves itself into dividing the total area of the curve A c d B (which area we will denote by a) into two areas acting down the lines X X and Y Y. This is effected by treating the areas as vertical forces, and setting down a vector line 0, 1, to represent the area a. Taking any convenient pole P, we then join 0 P and 1 P and draw x g, g y across the verticals X X, G G, Y Y parallel to 0 P, 1 P respectively, and join x y; then drawing P 2 parallel to x y, 1, 2 gives us the area which must act down the vertical Y Y, and 2, 0 that down X X.

Then $M_A \times \frac{l}{2}$ = area of triangle A a B = 2, 0

$$\therefore\ M_A = \frac{2,0 \times 2}{l}$$

Similarly $M_B = \frac{1,2 \times 2}{l}$

This enables the B.M. diagram to be drawn.

SHEAR DIAGRAM.—In this case as the end B.M.s are not equal the shear diagram will not be the same as for a freely supported beam, but the base line will be shifted. Since the shear at any point is the slope of the B.M. curve, the base line of the shear curve will be shifted downwards by an amount $\frac{M_A - M_B}{l}$ because this is the change in slope of the base line of the B.M. diagram between the freely supported and the fixed beam. If in the figure A C E D B represents the shear diagram for a freely supported beam with the given loading, then the effect of building-in the ends of this diagram is to lower its base line by an amount A A′ = B B′ = $\frac{M_A - M_B}{l}$, thus giving the diagram A′ C E′ D B′.

Special Cases.—1. Fixed Beam with Uniformly Increasing Load.—Let a beam A B of span l be subjected to a load of uniformly increasing intensity, the intensity at unit distance from B being w tons per ft. run, the total load being W. Then, as shown on p. 139 for a freely supported beam, $R_B = \frac{W}{3}$, $R_A = \frac{2W}{3}$ and the free B.M. diagram is a parabola of the 3rd order, the maximum B.M. being equal to ·128 W l and occurring at a distance ·577 l from B. Then the area of this free B.M. diagram is equal to $\frac{W l^2}{12}$ and its centre of gravity occurs at a distance $\frac{8l}{15}$ from B. This can be proved mathematically as follows—

$$\text{Area of B.M. curve} = \int_0^l M\,dx$$

$$= \int_0^l \left(\frac{w l^2 x}{6} - \frac{w x^3}{6}\right) dx$$

$$= \left[\frac{w l^2 x^2}{12} - \frac{w x^4}{24} + c\right]_0^l$$

The area $= 0$ when $x = 0$. ∴ $c = 0$

$$\therefore \text{Area} = \frac{w l^4}{12} - \frac{w l^4}{24} = \frac{w l^4}{24} = \frac{W l^2}{12}$$

$$\text{First mt. of B.M. curve about vertical through B} = \int_0^l M x\,dx$$

$$= \int_0^l \left\{\frac{w l^2 x^2}{6} - \frac{w x^4}{6}\right\} dx$$

$$= \left[\frac{w l^2 x^3}{18} - \frac{w x^5}{30} + c_1\right]_0^l$$

Moment $= 0$ when $x = 0$. ∴ $c_1 = 0$

$$\therefore \text{First moment} = \frac{w l^5}{18} - \frac{w l^5}{30} = \frac{w l^5}{45}$$

$\therefore$ Distance of centroid from vertical through B $= \dfrac{\text{1st moment}}{\text{area}}$

$$= \frac{w l^5}{45} \div \frac{w l^4}{24}$$

$$= \frac{24 l}{45} = \frac{8 l}{15}$$

This fixes the line G G, and the areas that must be considered as acting up X X and Y Y respectively are thus $\dfrac{W l^2}{20}$ and $\dfrac{W l^2}{30}$ since the total area $\dfrac{W l^2}{12}$ acts at distance $\dfrac{3 l}{15}$ from Y Y.

$\therefore$ Taking moments about Y Y we have—

$$\text{Area acting down X X} \times \frac{l}{3} = \frac{W l^2}{12} \times \frac{3 l}{15}$$

$$\therefore \text{Area acting down X X} = \frac{W l^3}{60} \div \frac{l}{3} = \frac{W l^2}{20}$$

$$\therefore M_A = \frac{W l^2}{20} \times \frac{2}{l} = \frac{W l}{10}$$

$$M_B = \frac{W l^2}{30} \times \frac{2}{l} = \frac{W l}{15}$$

The resulting B.M. diagram then comes as shown shaded in Fig. 201.

The amount of shifting of the base line for shear will be $\left[\dfrac{W l}{10} - \dfrac{W l}{15}\right] \div l = \dfrac{W}{30}$ so that the shears at the ends are $\dfrac{7 W}{10}$ and $\dfrac{3 W}{10}$ respectively, the shear curve for the fixed beam then coming as shown.

2. **Non-central Isolated Load.**—The following construction can be used for this case: Let A B, Fig. 202, represent a fixed beam of span l carrying an isolated load W at a point P, the distances of which from A and B are a and b respectively. First draw the "free bending moment" diagram A D B, *i.e.* set up P D $= \dfrac{W a b}{l}$, and join D to A and B. Project D horizontally to meet the vertical through the support B at E and join E A.

Then P F = Reverse or end bending moment at B = M_B; and F D = reverse or end bending moment at A = M_A.

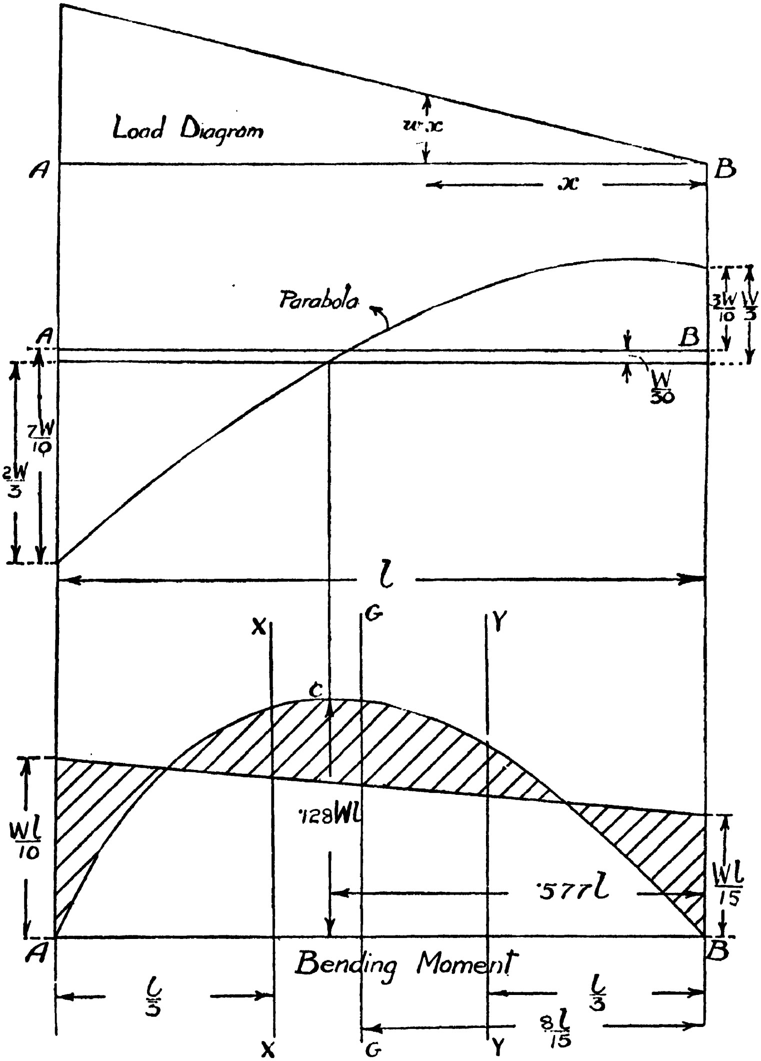


FIG. 201.—Fixed Beam with Uniformly Increasing Load.

Therefore set up B H = F P and A G_1 = F D and join G_1 H, the complete bending moment diagram then coming out as shown shaded. This may be done by projecting F horizontally and drawing D G_1 parallel to F A.

Proof.—To prove this construction we must obtain values for the reverse bending moments for this case. Referring to Fig. 203, it will be remembered that in fixed beams the conditions to be satisfied are that the "free" and "end" bending moment diagrams A B D and A G_1 H B respectively must be equal and opposite in area, and must have their centroids upon the same vertical line.

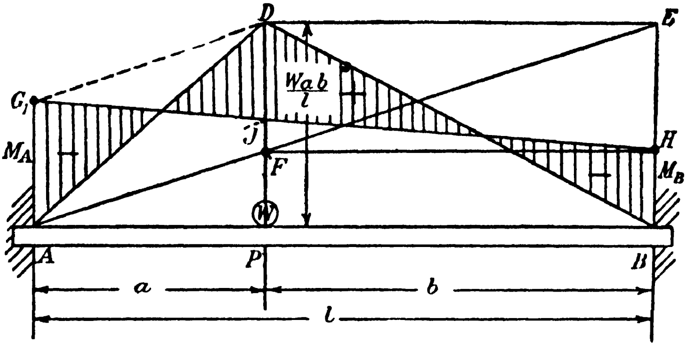


FIG. 202.—Fixed Beam with Isolated Load.

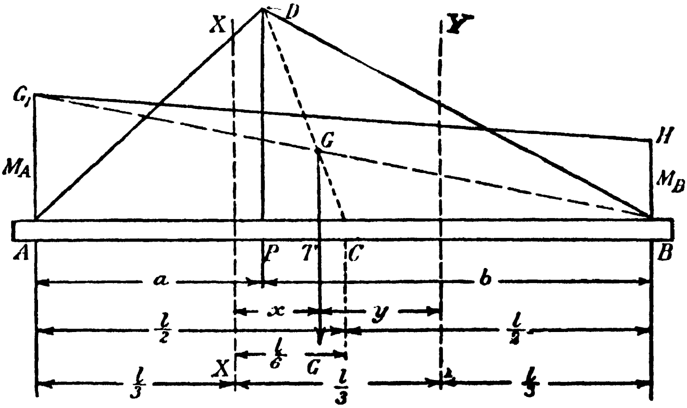


FIG. 203.—Fixed Beam with Isolated Load.

The first condition gives us

$$\frac{(M_A + M_B)\,l}{2} = \text{area of } \Delta \text{ ADB} = \tfrac{1}{2}\,\text{AB}\,.\,\text{PD}$$

$$= \tfrac{1}{2}\,\frac{l\,.\,W\,a\,b}{l} = \frac{W\,a\,b}{2} \quad \dots\dots\dots\dots \quad (1)$$

The end bending moment diagram $\text{A G}_1\text{ H B}$ may be considered as divided up into two triangles $\text{A G}_1\text{ B}$, $\text{B G}_1\text{ H}$, whose centroids act in the "third lines" X X and Y Y respectively.

We have next to calculate the position of the centroid G of the free bending moment diagram. According to the ordinary rule, the centroid G will be one-third of the way up the median line C D.

$$\therefore \text{TC} = \frac{1}{3}\text{PC} = \frac{1}{3}\left(\frac{l}{2} - a\right);$$

$$\therefore x = \frac{l}{6} - \text{TC} = \frac{l}{6} - \frac{1}{3}\left(\frac{l}{2} - a\right) = \frac{a}{3}$$

Regarding the areas of the triangles $\text{A G}_1\text{ B}$, $\text{B G}_1\text{ H}$, A D B as concentrated in the lines X X, Y Y, and G G respectively, we have by taking moments about the line X X

$$\text{Area of } \Delta\ \text{A D B} \times x = \text{area of } \Delta\ \text{B G}_1\text{ H} \times \frac{l}{3}$$

$$i.e. \text{ from (1)}, \frac{W a b}{2} . x = \tfrac{1}{2} M_B l \times \frac{l}{3}$$

$$\therefore \frac{M_B l^2}{6} = \frac{W a b}{2} . x = \frac{W a b}{2} . \frac{a}{3}$$

$$= \frac{W a^2 b}{6}$$

$$\therefore M_B = \frac{W a^2 b}{l^2}$$

$$\text{Similarly, } M_A = \frac{W a b^2}{l^2}$$

$$\text{As a check } (M_A + M_B) = \frac{W a^2 b}{l^2} + \frac{W a b^2}{l^2}$$

$$= \frac{W a b}{l^2}(a + b) = \frac{W a b}{l^2} . l = \frac{W a b}{l}$$

$$\therefore \frac{(M_A + M_B)\, l}{2} = \frac{W a b}{2} \text{ (as in (1))}$$

$$\text{Now, since } \frac{W a b}{l} = \text{P D}$$

$$M_B = \frac{\text{P D} \times a}{l}; \quad M_A = \frac{\text{P D} \times b}{l}$$

but in Fig. 202, by similar Δs

$$\frac{\text{PF}}{\text{BE}} = \frac{\text{AP}}{\text{AB}} = \frac{a}{l},$$

$$\therefore \text{PF} = \frac{\text{BE}\,.\,a}{l} = \frac{\text{PD}\,.\,a}{l} = \text{M}_\text{B}$$

$$\text{Similarly, EH} = \text{DF} = \frac{\text{PD}\,.\,b}{l} = \text{M}_\text{A}$$

POSITION OF LOAD FOR MAXIMUM REVERSE BENDING MOMENT.—The position of the load for a maximum value of the reverse or the end bending moment M_A is obtained by putting $\frac{d\,\text{M}_\text{A}}{d\,b} = 0$ and noting that $a = (l - b)$,

$$i.e.\ \frac{d\left(\frac{\text{W}\,.\,a\,b^2}{l^2}\right)}{d\,b} = 0$$

$$i.e.\ \frac{\text{W}}{l^2}\,\frac{d\,(a\,b^2)}{d\,b} = 0$$

$$i.e.\ \frac{d\,(a\,b^2)}{d\,b} = 0$$

$$i.e.\ \frac{d\{b^2(l-b)\}}{d\,b} = 0$$

$$2\,b\,l - 3\,b^2 = 0$$

$$i.e.\ b = \frac{2\,l}{3},\ \text{or } 0$$

Taking the first value, which is clearly the maximum, then

$$\text{M}_\text{A} = \frac{\text{W}\,.\,\frac{l}{3}\left(\frac{2\,l}{3}\right)^2}{l^2} = \frac{4\,\text{W}\,l}{27} = \frac{\text{W}\,l}{6{\cdot}75}$$

Therefore the maximum reverse bending moment for an isolated load is equal to $\frac{\text{W}\,l}{6{\cdot}75}$, and occurs when the load is at one-third of the span.

MAXIMUM POSITIVE OR INTERMEDIATE BENDING MOMENT.—Referring to Fig. 202, the maximum intermediate bending moment occurs at the load point P and is equal to DJ.

∴ Maximum intermediate bending moment

$$= \text{M}_\text{P} = \text{DJ} = \text{PD} - \text{JP}$$

$$= \frac{\text{W}\,a\,b}{l} - \text{JP}$$

Now, remembering that $(a + b) = l$

$$
\begin{aligned}
\text{JP} &= \text{PF} + \text{FJ} \\
&= \text{M}_B + (\text{M}_A - \text{M}_B)\frac{b}{l} \\
&= \text{M}_B\left(1 - \frac{b}{l}\right) + \frac{\text{M}_A \,.\, b}{l} \\
&= \text{M}_B \,.\, \frac{a}{l} + \frac{\text{M}_A \,.\, b}{l} \\
&= \frac{\text{W}\,a^2 b}{l^2} \,.\, \frac{a}{l} + \frac{\text{W} \,.\, a\,b^2 b}{l^2\,l} \\
&= \frac{\text{W}\,a\,b}{l^3}(a^2 + b^2)
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \text{M}_P &= \frac{\text{W}\,a\,b}{l}\left(1 - \frac{a^2 + b^2}{l^2}\right) \\
&= \frac{\text{W}\,a\,b}{l}\left\{\frac{(a + b)^2 - (a^2 + b^2)}{l^2}\right\} \\
&= \frac{\text{W}\,a\,b}{l^3} \,.\, 2\,a\,b \\
&= \frac{2 \,.\, \text{W}\,a^2 b^2}{l^3}
\end{aligned}
$$

To get the maximum value of this for any position of the load put $\frac{d\,\text{M}_P}{d\,a} = 0$

$$
\begin{aligned}
\textit{i. e.} \quad & \frac{2\,\text{W}}{l^3}\left\{\frac{d\,a^2\,(l - a)^2}{d\,a}\right\} = 0 \\
\textit{i. e.} \quad & \frac{d \,.\, a^2\,(l^2 - 2\,a\,l + a^2)}{d\,a} = 0 \\
\textit{i. e.} \quad & 2\,a\,l^2 - 6\,a^2\,l + 4\,a^3 = 0 \\
\textit{i. e.} \quad & l^2 - 3\,a\,l + 2\,a^2 = 0 \\
& (l - a)\,(l - 2a) = 0 \\
\textit{i. e.} \quad & a = \frac{l}{2}, \text{ or } l.
\end{aligned}
$$

Taking the former value, which is the only one possible, we have

$$
\text{Max. M}_P = \frac{2\,\text{W}}{l^3} \,.\, \frac{l^2}{4} \,.\, \frac{l^2}{4} = \frac{\text{W}\,l}{8}
$$

Therefore the maximum intermediate bending moment occurs when the load is at the centre and is equal to $\frac{\text{W}\,l}{8}$.

Graphical Method of finding G G.—If the nature of the loading is such that the position of the line G G cannot be calculated without difficulty we may proceed as follows: Divide the free B.M. diagram A C B up into a number of vertical strips, not necessarily equal, and draw vertical force lines through the centres of these strips and set down the ordinates on a vector line, and with any pole draw a link polygon. The point where the first and last links meet will be a point on the line G G. This is the same method as adopted in finding the centroid of a figure by Mohr's method (Chap. VI). The area of the B.M. diagram can be found by sum-curve construction, and the problem completed as indicated with reference to Fig. 200.

INVESTIGATION FROM MATHEMATICAL STANDPOINT

As we have previously seen

$$\text{Slope of beam} = \int \frac{M}{EI} dx$$

If the end of the beam is built-in this slope must come zero at the two ends.

Consider the following special cases—

(1) **Uniform Load on Fixed Beam.**—Taking the intensity of load as w and the centre of the beam as origin, then considering a point at distance x from the centre, for the freely supported beam we have

$$M_x = \frac{w}{2}\left(\frac{l^2}{4} - x^2\right) \qquad \text{(See p. 266.)}$$

Let the effect of the building-in be to cause an end B.M. $= M_A$

Then for the fixed beam $M_x = \frac{w}{2}\left(\frac{l^2}{4} - x^2\right) - M_A$

$$\therefore \text{Slope} = \int \frac{M}{EI} dx$$

$$= \frac{1}{EI}\left(\frac{w l^2 x}{8} - \frac{w x^3}{6} - M_A x + c\right)$$

Slope is 0 when $x = 0$. $\therefore$ $c = 0$.

Also slope must be 0 when $x = \frac{l}{2}$

$$\therefore \frac{w\,l^3}{16} - \frac{w\,l^3}{48} - M_A \frac{l}{2} = 0$$

$$i.e.\ M_A \times \frac{l}{2} = \frac{w\,l^3}{16} - \frac{w\,l^3}{48}$$

$$= \frac{w\,l^3}{24}$$

$$\therefore M_A = \frac{w\,l^2}{12}$$

To obtain the deflection we integrate again, and we get

$$\text{deflection} = \iint \frac{M}{E\,I}\,d\,x$$

$$= \frac{1}{E\,I}\left(\frac{w\,l^2\,x^2}{16} - \frac{w\,x^4}{24} - \frac{M_A\,x^2}{2} + c_1\right)$$

$$= \frac{1}{E\,I}\left(\frac{w\,l^2\,x^2}{16} - \frac{w\,x^4}{24} - \frac{w\,l^2\,x^2}{24} + c_1\right)$$

$$= \frac{1}{E\,I}\left(\frac{w\,l^2\,x^2}{48} - \frac{w\,x^4}{24} + c_1\right)$$

This is 0 when $x = \frac{l}{2}$

$$\therefore \frac{w\,l^4}{192} - \frac{w\,l^4}{384} + c_1 = 0$$

$$\therefore c_1 = \frac{-\,w\,l^4}{384}$$

$$\therefore \text{When } x = 0, \text{ deflection} = \frac{c_1}{E\,I}$$

$$= \frac{-\,w\,l^4}{384\,E\,I}$$

$$\therefore \text{Maximum deflection} = \frac{w\,l^4}{384\,E\,I} = \frac{W\,l^3}{384\,E\,I}$$

The B.M. and shear diagrams are then as shown on Fig. 198.

(2) **Isolated Central Load.**—Taking as before a span l and the centre as the origin, if the load is W, for a freely supported beam we have

$$M^x = \frac{W}{2}\left(\frac{l}{2} - x\right)$$

∴ If the end B.M. due to fixing the ends is M_A, we have for the fixed beam

$$M_x = \frac{W}{2}\left(\frac{l}{2} - x\right) - M_A$$

$$\therefore \text{ Slope of beam} = \int \frac{\text{M}}{\text{E I}} d\,x$$

$$= \left(\frac{\text{W}\,l\,x}{4} - \frac{\text{W}\,x^2}{4} - \text{M}_\text{A}\,x + \text{C}_2\right) \times \frac{1}{\text{E I}}$$

When $x = 0$, slope $= 0$. $\therefore$ $\text{C}_2 = 0$.

When $x = \frac{l}{2}$, slope also $= 0$ in this case.

$$\therefore \text{ We have: } \quad 0 = \left(\frac{\text{W}l^2}{8} - \frac{\text{W}\,l^2}{16} - \text{M}_\text{A}\,\frac{l}{2}\right) \times \frac{1}{\text{E I}}$$

$$\therefore \text{M}_\text{A} \cdot \frac{l}{2} = \frac{\text{W}\,l^2}{16}$$

$$\text{M}_\text{A} = \frac{\text{W}\,l}{8}$$

To get the deflection we integrate again, then

$$\text{deflection} = \iint \frac{\text{M}}{\text{E I}} d\,x$$

$$= \frac{1}{\text{E I}}\left(\frac{\text{W}\,l\,x^2}{8} - \frac{\text{W}\,x^3}{12} - \frac{\text{M}_\text{A}\,x^2}{2} + \text{C}_3\right)$$

$$= \frac{1}{\text{E I}}\left(\frac{\text{W}\,l\,x^2}{16} - \frac{\text{W}\,x^3}{12} + \text{C}_3\right)$$

This is 0 when $x = \frac{l}{2}$

$$\therefore \frac{\text{W}\,l^3}{64} - \frac{\text{W}\,l^3}{96} + \text{C}_3 = 0$$

$$\therefore \text{C}_3 = \frac{-\text{W}\,l^3}{192}$$

When $x = 0$, deflection $= \frac{\text{C}_3}{\text{E I}}$

$$\therefore \text{ Maximum deflection} = \frac{\text{W}\,l^3}{192\,\text{E I}}$$

* **(3) Fixed Beam with Uniformly Increasing Load.**— Let a span A B of length l have a uniformly increasing load, of zero intensity at the point B, and let the intensity of load at unit distance from B be w units per ft. run. Then taking the end B as origin, we have in the case of the freely supported beam

$$\text{M}_x = \frac{w\,l^2\,x}{6} - \frac{w\,x^3}{6}$$

Now let M_A and M_B be the end B.M.s, then the negative B.M. at distance x from B is equal to

$$M_B + \frac{(M_A - M_B)}{l} . x$$

∴ for the fixed beam

$$M_x = \frac{w l^2 x}{6} - \frac{w x^3}{6} - M_B - \frac{(M_A - M_B) . x}{l}$$

$$\therefore \text{Slope of beam} = \int \frac{M}{EI} . dx$$

$$= \frac{1}{EI}\left\{\frac{w l^2 x^2}{12} - \frac{w x^4}{24} - M_B x - \left(\frac{M_A - M_B}{l}\right)\frac{x^2}{2} + c_4\right\} \quad ..(1)$$

When $x = 0$, slope $= 0$. ∴ $c_4 = 0$.
Also when $x = l$, slope $= 0$

$$\therefore \frac{w l^4}{12} - \frac{w l^4}{24} - M_B l - \frac{(M_A - M_B) l^2}{2 l} = 0$$

$$\therefore -\frac{M_B l}{2} - \frac{M_A l}{2} = \frac{-w l^4}{24}$$

$$\therefore M_A + M_B = \frac{w l^3}{12} \quad ....(2)$$

To get another relation between M_A and M_B, consider the deflection;

$$\text{then deflection} = \iint \frac{M}{EI} dx$$

$$= \frac{1}{EI}\left\{\frac{w l^2 x^3}{36} - \frac{w x^5}{120} - \frac{M_B x^2}{2} - \left(\frac{M_A - M_B}{l}\right) . \frac{x^3}{6} + c_5\right\} \quad ..(3)$$

Deflection $= 0$ when $x = 0$. ∴ $c_5 = 0$.
Also deflection $= 0$ when $x = l$

$$\therefore \frac{w l^5}{36} - \frac{w l^5}{120} - \frac{M_B l^2}{2} - \left(\frac{M_A - M_B}{l}\right)\frac{l^3}{6} = 0$$

$$\therefore \frac{-M_B l^2}{2} - \frac{M_A l^2}{6} + \frac{M_B l^2}{6} = \frac{w l^5}{120} - \frac{w l^5}{36}$$

$$\therefore \frac{M_A l^2}{6} + \frac{M_B l^2}{3} = \frac{7 w l^5}{360}$$

$$\therefore M_A + 2 M_B = \frac{7 w l^3}{60} \quad ....(4)$$

.·. Combining (2) and (4) we get

$$M_B = \frac{7\,w\,l^3}{60} - \frac{w\,l^3}{12}$$

$$= \frac{w\,l^3}{30} = \frac{W\,l}{15}$$

$$\therefore M_A = \frac{w\,l^3}{12} - \frac{w\,l^3}{30} = \frac{W\,l}{10}$$

The B.M. diagram then comes as shown in Fig. 201. In all the above cases we have assumed that the beam is of constant cross section along its length. If such is not the case, the end B.M.s can be found by taking the corrected B.M. diagram as explained in the chapter on the deflections of beams.

Advantages and Disadvantages of Fixed Beams.—We have seen that, in the examples that have been considered, a fixed beam is stronger than the corresponding freely supported beam, and that the fixed beam has smaller deflections and is thus more rigid. In most cases, moreover, the maximum B.M. occurs at the abutments, where the beam can be strengthened without adding materially to the bending moments and thus increasing the stresses. In the freely supported beam, on the other hand, the maximum B.M. occurs at the centre, where an addition of weight to strengthen the section would add materially to the B.M. The reason why such beams are not more commonly adopted is because, in fixing in the ends securely, the tangents at each end to the beam must be *absolutely* horizontal, and any deviation from this will alter the stresses, and any difference of level at the two ends due to unequal settlement would cause considerable stresses in the beam. There is also considerable stress due to change in temperature if the beam is securely built-in to the masonry, and all these points make the actual stresses in any practical case somewhat uncertain, so that many designers do not use this type of beam. All the above objections can be obviated by cutting the beam through at the points of contraflexure and resting the centre portion on the two end portions. This is the principle of the *cantilever girder* construction and for large spans is very economical. This is shown diagrammatically in Fig. 204, in which a fixed beam A B is shown divided at the points of inflexion C and D and the centre portion is rêpresented as hanging from the end portions. The B.M. in the centre portion will be the same as for a freely supported beam of span l loaded in the given manner. The B.M. for the cantilever portions will be the same as for cantilevers of span l_1 loaded with the given loading and also with loads at the ends equal to the reactions at the ends of the centre portions. In the figure, uniform loading is shown, and in such case these reactions are each equal to $\frac{w\,l}{2}$. It will be

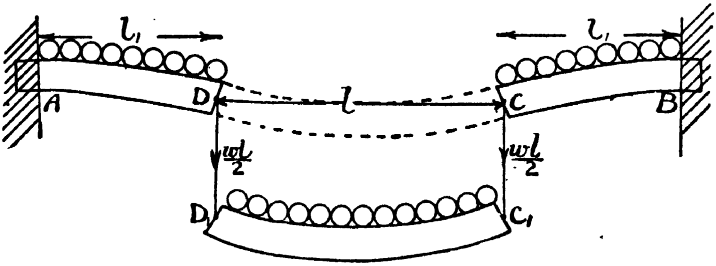


FIG. 204.

found that the resulting B.M. and shear curves obtained in this way will be the same as shown in Fig. 198. The deflections can also be found by adding together the deflections at the centre of the centre portion and at the end of one of the cantilever portions.

Fixed Beam with Ends not at same Level.—Suppose that a fixed beam A B, Fig. 205, has its ends at a different level, then apart from the loading on the beam, the deflected form of the beam will be as shown in the figure, the point of contraflexure being at the centre point C.

The deflection $b\,\delta$ of the portion A C, assuming the beam divided at C, will be equivalent to that due to a weight P hanging downwards at C, but for a cantilever with load at end

$$\delta = \frac{W\,l^3}{3\,E\,I}$$

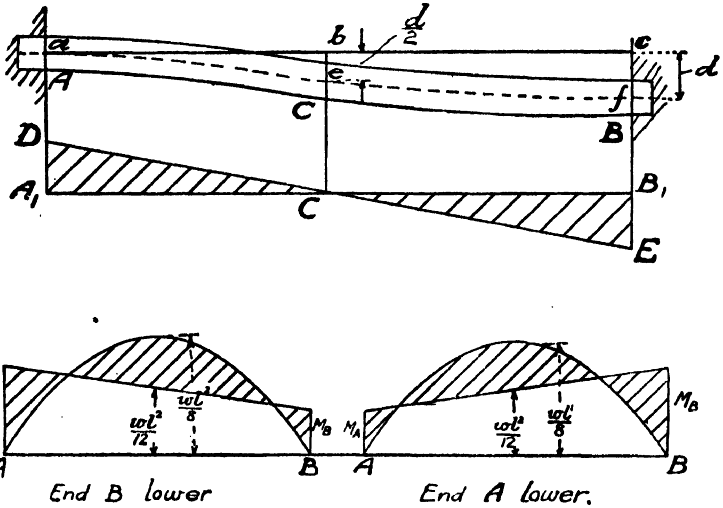
In this case we have

$$e\,b = \frac{P\left(\frac{l}{2}\right)^3}{3\,E\,I}$$

$$\therefore P = \frac{24\,E\,I \times e\,b}{l^3}$$

$$= \frac{12\,E\,I \times c\,f}{l^3}$$

$$= \frac{12\,E\,I \times d}{l^3}$$



End B lower End A lower.

Fig. 205.—Beams with Ends fixed at different Levels.

The B.M. diagram due to this is a triangle C A_1 D, A_1 D being equal to P × $\frac{l}{2}$

$$\therefore A_1 D = \frac{12\,E\,I \times d}{2l^2} = \frac{6\,E\,I\,d}{l^2}$$

Similarly the portion C B is as if it had a load P at its end acting upward, the B.M. diagram for this portion being C B_1 E, B_1 E being equal to A_1 D.

Therefore, this diagram must be combined with the ordinary diagram for a fixed beam if the ends are at different levels, the figure showing the effects for the case in which B is lower than A, and also that in which A is lower than B.

The condition that the end B.M. diagram must be equal in area to the free B.M. diagram still holds in this case, but their centroids are not on the same vertical line because there is a resultant deflection at one end.

It can be shown by considering the stability of the imaginary cable of Mohr's Theorem, that $\mathrm{E\,I} \times d =$ area of B.M. curve × horizontal distance (g) between the centroids of the free and end B.M. curves.

$$i.e.\ \mathrm{E\,I} \times d = \frac{w\,l^2}{12} \times l \times g$$

$$\therefore\ g = \frac{12\,\mathrm{E\,I} \times d}{w\,l^3}$$

Now, if $\mathrm{M_A}$ and $\mathrm{M_B}$ are the end B.M.s, the end B.M. diagram is a trapezium.

$$\therefore\ g = \frac{l}{2} - \frac{l}{3}\left(\frac{2\,\mathrm{M_B} + \mathrm{M_A}}{\mathrm{M_B} + \mathrm{M_A}}\right)$$

$$= \frac{l\,(\mathrm{M_A} - \mathrm{M_B})}{6\,(\mathrm{M_B} + \mathrm{M_A})}$$

$$\therefore\ \mathrm{M_A} - \mathrm{M_B} = \frac{6\,(\mathrm{M_B} + \mathrm{M_A})\,g}{l} = 6 \times \frac{2\,w\,l^2}{12} \times \frac{g}{l}$$

$$= \frac{12\,\mathrm{E\,I}\,d}{l^2}$$

Now, in the figure $\mathrm{M_A} - \mathrm{M_B} = 2\,\mathrm{A_1\,D}$

$$\therefore\ \mathrm{A_1\,D} = \frac{\mathrm{M_A} - \mathrm{M_B}}{2} = \frac{6\,\mathrm{E\,I}\,d}{l^2}$$

This gives the same result as the previous reasoning.

Beams with Cleat Connections, etc.—In building work the girders are usually connected to the stanchions or columns by means of cleat connections, which, owing to their rigidity, make it doubtful whether the girder will act as a freely supported beam, although their strength is almost invariably calculated as such. Neither is an ordinary cleat sufficiently rigid for the girders to be considered as fixed at their ends. The actual B.M. diagram for such beams will be somewhere between that for a freely supported beam and a fixed beam. It has been suggested that these beams should be treated as "half fixed," that is, that the end B.M.s should, in the case of uniform loading, be taken as $\frac{w\,l^2}{24}$. The B.M. diagram then comes as shown in Fig. 206. It will be noted that the maximum B.M. in this case is still $\frac{w\,l^2}{12}$ as in the fixed beam, but such B.M. now occurs in the centre.

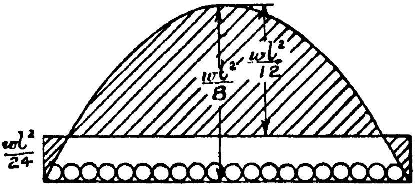


Fig. 206.

In beams where the tensile and compressive strengths of the material are different, as in cast iron and reinforced concrete beams, it must be carefully remembered that at the ends the tension side is at the top, and so the additional strength must be placed at the top at these ends.

It must also be carefully remembered that in all the cases we have assumed that the cross section of the beam is constant along its length, and the results obtained will not be true if such is not the case.

CONTINUOUS BEAMS

If a beam is continuous over a number of supports A, B, C, the deflected form of the beam has to take some shape such as shown in Fig. 207, the curvature changing in direction at the points *a*, *b*, *c*, *d*. As in the case of fixed beams, this change in

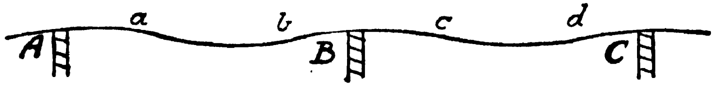


Fig. 207.

the curvature means that a negative bending moment occurs at the supports, such bending moment being called in future the "support B.M."

Consider first the case of a continuous girder, A, B, C, Fig. 208, of two equal spans, each of length l, subjected to a uniform

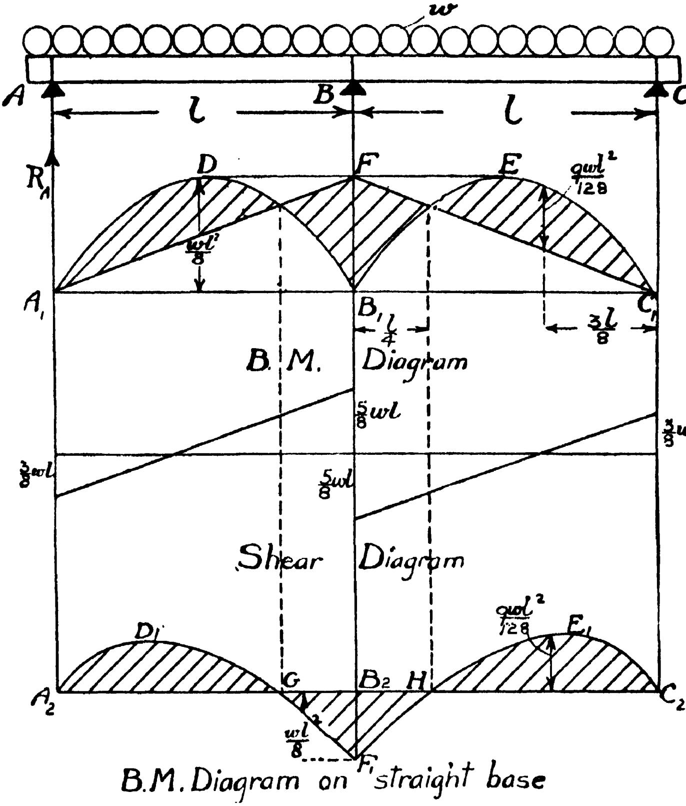


FIG. 208.—Uniformly Loaded Continuous Beam of two equal Spans.

load of w tons per foot run, the supports A, B, and C being on the same level, and the beam being of uniform cross section. Now imagine the centre support removed, then there would be a central deflection δ, given by

$$\delta = \frac{5\,w\,(2\,l)^4}{384\,E\,I}$$

Now, if the centre support be replaced, the pressure R_B on it must be such that as a central load it causes an upward deflection equal to δ

$$\therefore \delta = \frac{R \times (2\,l)^3}{48\,E\,I}$$

$$\therefore \frac{R \times (2\,l)^3}{48\,E\,I} = \frac{5\,w\,(2\,l)^4}{384\,E\,I}$$

$$R = \frac{5\,w \times 2\,l}{8} = \frac{5\,w\,l}{4}$$

or if W is the load *on one span*, $R = \frac{5\,W}{4}$

$\therefore$ Since $R_A = R_C$ from symmetry, and $R_A + R_B + R_C = 2\,W$, we see that $R_A = R_C = \frac{3\,W}{8} = \frac{3\,w\,l}{8}$

In the ordinary case of two separate spans $R_A = R_C = \frac{W}{2}$

$\therefore$ Support B.M. diagram will be as if there were an upward force of $\frac{W}{8}$ acting at A and C. This causes at B a B.M. $= \frac{W}{8} \times l = \frac{W\,l}{8}$ so that the negative B.M. at B $= \frac{W\,l}{8} = \frac{w\,l^2}{8}$ and the B.M. diagram for the continuous beam then comes as shown in Fig. 208.

As the reactions are $\frac{3}{8}\,w\,l$ at A and C, the shear diagram will have an ordinate equal to $\frac{3}{8}\,w\,l$ at these points; the shear then decreases uniformly from C to B until it has a value $-\frac{5}{8}\,w\,l$ at B. It then increases to $+\frac{5}{8}\,w\,l$, since $R_B = \frac{5}{4}\,w\,l$, and then decreases to $-\frac{3}{8}\,w\,l$ again at A, the shear diagram then coming as shown in the figure.

The points of contraflexure G, H, where the B.M. is zero, occur at distances $\frac{l}{4}$ from B.

This can be shown as follows—

Let H be at distance x from C.

Then negative B.M. due to support B.M. $= \frac{W\,x}{8} = \frac{w\,l\,x}{8}$

positive B.M. for freely supported beam $= \frac{w\,l\,x}{2} - \frac{w\,x^2}{2}$

These must be equal, so that

$$\frac{w\,l\,x}{8} = \frac{w\,l\,x}{2} - \frac{w\,x^2}{2}$$

$$\therefore \frac{x}{2} = \frac{l}{2} - \frac{l}{8} = \frac{3\,l}{8}$$

$$\therefore x = \frac{3\,l}{4}$$

$$\therefore \text{distance from B} = l - \frac{3\,l}{4} = \frac{l}{4}$$

If the B.M. diagram be reduced to a horizontal base, the lower diagram shown on the figure will be obtained.

The maximum intermediate B.M.s will occur at distances $\frac{3\,l}{8}$ from C and A.

They will be equal to $\frac{w\,l}{2} \cdot \frac{3\,l}{8} - \frac{w}{2} \cdot \left(\frac{3\,l}{8}\right)^2 - \frac{w\,l}{8} \cdot \frac{3\,l}{8}$

$$= w\,l^2\left(\frac{3}{16} - \frac{9}{128} - \frac{3}{64}\right)$$

$$= \frac{9\,w\,l^2}{128} = \frac{9\,W\,l}{128}$$

* **Two Equal Uniformly Loaded Spans with Supports not on same Level.**—Now consider the case in which the centre support B is at different level from A and C, and let B be at distance *h* below A C (Fig. 209).

As before, if the support B is removed, there will be a central deflection $\delta = \frac{5\,w\,(2\,l)^4}{384\,E\,I}$

The reaction at B is now only sufficient to cause an upward deflection equal to $\delta - h$.

$$\therefore \delta - h = \frac{R_B\,(2\,l)^3}{48\,E\,I}$$

$$\therefore R_B = \frac{48\,E\,I}{(2\,l)^3}\,(\delta - h)$$

$$= \frac{48\,E\,I}{(2\,l)^3}\,\delta\left(1 - \frac{h}{\delta}\right)$$

$$\therefore R_B = \frac{48\,E\,I}{(2\,l)^3} \times \frac{5\,w\,(2\,l)^4}{384\,E\,I}\left(1 - \frac{h}{\delta}\right)$$

$$= \frac{5\,w\,l}{4}\left(1 - \frac{h}{\delta}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$= \frac{5\,w\,l}{4} - \frac{5\,h\,w\,l}{4\,\delta}$$

$$\therefore R_A = R_C = \frac{3\,w\,l}{8} + \frac{5\,h\,w\,l}{8\,\delta}$$

$$= \frac{w\,l}{2} - \frac{w\,l}{8}\left(1 - \frac{5\,h}{\delta}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

∴ Reasoning as before, negative B.M. at B due to the second portion of R_A or R_C

$$= \frac{w\,l^2}{8}\left(1 - \frac{5\,h}{\delta}\right)$$

$$i.\,e.\ M_B = \frac{w\,l^2}{8}\left(1 - \frac{5\,h}{\delta}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

∴ the B.M. curve will be somewhat as shown shaded, the position of D depending on the value of $\frac{h}{\delta}$

Now consider the following special values of *h*.

If $h = 0$, $M_B = \frac{w\,l^2}{8}$ as in the previous case.

If $h = \frac{\delta}{5}$, $M_B = 0$, and the B.M. diagram is the same as for two simply supported beams.

If $h = \delta$, $M_B = \frac{w\,l^2}{8}(1 - 5) = \frac{-\,w\,l^2}{2}$

This is the same as we should have obtained for a simply supported beam of span $2\,l$.

Now let $h = \frac{-\,3\,\delta}{5}$

Then $M_B = \frac{w\,l^2}{8}(1 + 3) = \frac{w\,l^2}{2}$. This is the same as if the supports A and C were removed and the beam were two cantilevers B A and B C. The free deflection at the ends is then $= \frac{w\,l^4}{8\,E\,I}$, and this will be found to be equal to $\frac{3}{5}\,\delta$.

Now h must lie between δ and $\frac{-3\,\delta}{5}$ for the beam to act as a continuous beam, therefore take points E, E′ on the vertical

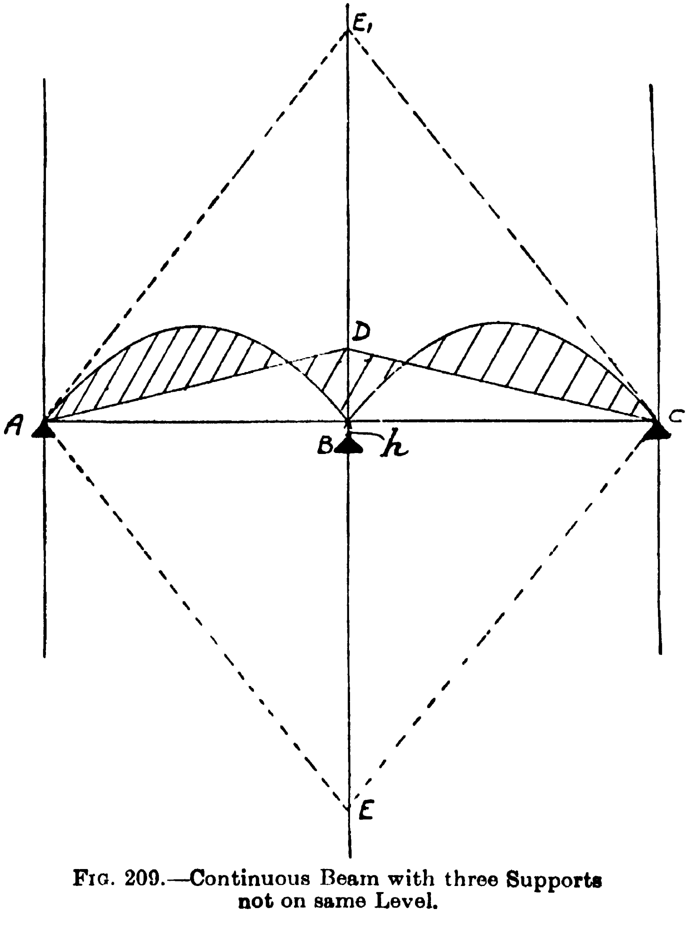


FIG. 209.—Continuous Beam with three Supports not on same Level.

through B, such that $BE' = BE = \frac{w\,l^2}{2}$, then the closing line of the B.M. diagram for the continuous beam with the supports at different levels must lie between A E C and A E′ C.

The following example on this problem is interesting—

A continuous beam of uniform section and two equal spans l has a uniform load of intensity w, and the supports A B C *are* *initially level. The support columns are, however, equally elastic, the force necessary to cause unit compression being e. Find the central reaction and B.M.*

If the centre column is removed, $\delta = \dfrac{5\,w\,(2\,l)^4}{384\,\mathrm{E\,I}}$.

The upward deflection due to $\mathrm{R}_{\mathrm{B}} = \delta_1 = \dfrac{\mathrm{R}_{\mathrm{B}}\,(2\,l)^3}{48\,\mathrm{E\,I}}$

Then $\delta - \delta_1$ = difference in level between final positions of A, B, and C. Now let $\mathrm{R}_{\mathrm{B}} = w\,l + 2\,f$, $2\,f$ being the additional reaction due to the beam being continuous, then

$$\mathrm{R}_{\mathrm{A}} = \mathrm{R}_{\mathrm{C}} = \frac{w\,l}{2} - f$$

$$\therefore \text{Sink of central column} = \frac{w\,l + 2\,f}{e}$$

$$\text{Sink of end columns} = \frac{\frac{w\,l}{2} - f}{e}$$

$$\therefore \text{Difference} = \delta - \delta_1 = \frac{1}{e}\left(\frac{w\,l}{2} + 3\,f\right)$$

$$= \frac{1}{e}\left(\frac{3\,\mathrm{R}_{\mathrm{B}}}{2} - w\,l\right)$$

$$\therefore \frac{1}{e}\left(\frac{3\,\mathrm{R}_{\mathrm{B}}}{2} - w\,l\right) = \delta - \delta_1$$

$$= \frac{5\,w\,l^4}{24\,\mathrm{E\,I}} - \frac{\mathrm{R}_{\mathrm{B}}\,l^3}{6\,\mathrm{E\,I}}$$

$$\therefore \mathrm{R}_{\mathrm{B}}\left(\frac{l^3}{6\,\mathrm{E\,I}} + \frac{3}{2\,e}\right) = \frac{5\,w\,l^4}{24\,\mathrm{E\,I}} + \frac{w\,l}{e}$$

$$\therefore \mathrm{R}_{\mathrm{B}} = \frac{\dfrac{5\,w\,l^4}{24\,\mathrm{E\,I}} + \dfrac{w\,l}{e}}{\dfrac{l^3}{6\,\mathrm{E\,I}} + \dfrac{3}{2\,e}}$$

$$= w\,l\left\{\frac{\dfrac{5\,l^3}{24\,\mathrm{E\,I}} + \dfrac{1}{e}}{\dfrac{l^3}{6\,\mathrm{E\,I}} + \dfrac{3}{2\,e}}\right\}$$

$$= w\,l\left\{\frac{\dfrac{5}{4} + \dfrac{6\,\mathrm{E\,I}}{e\,l^3}}{1 + \dfrac{9\,\mathrm{E\,I}}{e\,l^3}}\right\}$$

Reasoning as before, we then get

$$M_B = \frac{w l^2}{2}\left\{\frac{\frac{5}{4}+\frac{6EI}{e l^3}}{1+\frac{9EI}{e l^3}} - 1\right\}$$

$$= \frac{w l^2}{2}\left\{\frac{\frac{1}{4}-\frac{3EI}{e l^3}}{1+\frac{9EI}{e l^3}}\right\}$$

It will of course be noted that if the piers had been of the same material and of areas proportional to the reactions, the amount of sinking due to their elasticity would have been equal, and the B.M. diagram therefore would remain as shown in Fig. 208.

* **The Theorem of Three Moments.**—We will now find the relation which must exist between the support bending moments and the loading for a continuous beam of any number of spans, the supports all being on the same level.

Let A B and B C be any two consecutive spans of length l_1 and l_2 of a continuous beam of any number of spans, and let A *e* B, B *f* C (Fig. 210) be the free B.M. diagrams for the loading on these spans. Let G_1 and G_2 be the centroids of these free B.M. diagrams, and let them be at distances y_1, y_2 respectively from A and C, the areas of the diagrams being respectively S_1 and S_2. Then, if M_A, M_B, M_C are the support moments at A, B, and C respectively, *Clapeyron's Theorem of Three Moments* states that

$$M_A l_1 + 2 M_B (l_1 + l_2) + M_C l_2 = 6\left\{\frac{S_1 y_1}{l_1} + \frac{S_2 y_2}{l_2}\right\}$$

We can prove this with the aid of Mohr's Theorem* as follows: Let A′ B′ C′ be the deflected form or elastic line of the beam, then if the beam is of the same material throughout, and of constant cross section, the elastic line is of the same shape as that of an imaginary cable loaded with the B.M. diagrams and subjected to a horizontal pull equal to E × I. Now the tangent to the imaginary cable is common at the point B′. Let such tangent be at angle θ to the line A′ B′, and let the perpendicular from A′ on to it be of length p_1, the tension in such cable at B′ being $T_{B'}$; then considering the stability of the imaginary cable we have by taking the span A B and taking moments round A′

---

* See p. 252.

$$T_{B'} \times p_1 = \text{moment of B.M. diagram about A}'$$
$$= S_1 y_1 - \text{moment of support B.M. diagram about A}'$$

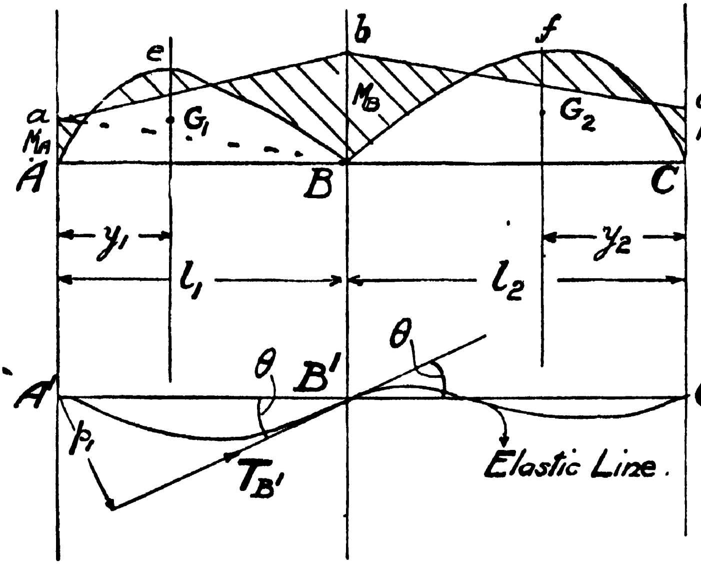


FIG. 210.—Theorem of Three Moments.

$$= S_1 y_1 - M_A \frac{l_1}{2} \times \frac{l_1}{3} - M_B \frac{l_1}{2} \times \frac{2\,l_1}{3}$$

$$= S_1 y_1 - \frac{M_A\, l_1^2}{6} - 2\, M_B \frac{l_1^2}{6} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

because the support B.M. diagram can be divided with two triangles of area $\frac{M_A\, l_1}{2}$ and $\frac{M_B\, l_1}{2}$, the distances of their centroids from A′ being respectively $\frac{l_1}{3}$ and $\frac{2\, l_1}{3}$. Now $p_1 = l_1 \sin \theta$, and $T_{B'} = \frac{E\,I}{\cos \theta}$, E I being the horizontal pull in the cable.

$$\therefore\ T_{B'} \times p_1 = \frac{E\,I\,l_1 \sin\theta}{\cos\theta} = E\,I\,l_1 \tan\theta$$

$$\therefore\ E\,I\,l_1 \tan\theta = S_1 y_1 - \frac{M_A l_1^2}{6} - \frac{2\,M_B l_1^2}{6}$$

$$\therefore\ E\,I \tan\theta = \frac{S_1 y_1}{l_1} - \frac{M_A l_1}{6} - \frac{2\,M_B l_1}{6} \quad \ldots\ldots\ldots\ldots(2)$$

Now by considering the second span, as θ is the same for both spans and E I is constant, we get

$$E\,I \tan\theta = -\left(\frac{S_2 y_2}{l_2} - \frac{M_C l_2}{6} - \frac{2\,M_B l_2}{6}\right) \ldots\ldots\ldots(3)$$

The $-$ sign is used because the moments are taken in opposite directions.

Then, combining equations (2) and (3) we get

$$\frac{S_1 y_1}{l_1} - \frac{M_A l_1}{6} - \frac{2\,M_B l_1}{6} = -\left(\frac{S_2 y_2}{l_2} - \frac{M_C l_2}{6} - \frac{2\,M_B l_2}{6}\right)$$

$$\text{or } M_A l_1 + 2\,M_B (l_1 + l_2) + M_C l_2 = 6\left(\frac{S_1 y_1}{l_1} + \frac{S_2 y_2}{l_2}\right) \ldots\ldots(4)$$

This is the general formula applicable for all loadings.

If the loading is uniform over each span and of different intensities w_1 and w_2, we get

$$S_1 = \frac{2}{3} l_1 \cdot \frac{w_1 l_1^2}{8} = \frac{w_1 l_1^3}{12}$$

$$y_1 = \frac{l_1}{2}$$

Similarly

$$S_2 = \frac{w_2 l_2^3}{12}$$

$$y_2 = \frac{l_2}{2}$$

$$\therefore\ \frac{S_1 y_1}{l_1} + \frac{S_2 y_2}{l_2} = \frac{1}{24}(w_1 l_1^3 + w_2 l_2^3)$$

$\therefore$ In this case we have

$$M_A l_1 + 2\,M_B (l_1 + l_2) + M_C l_2 = \frac{1}{4}(w_1 l_1^3 + w_2 l_2^3) \ldots\ldots(5)$$

If the load is of the same intensity w on the two spans we get

$$M_A l_1 + 2\,M_B (l_1 + l_2) + M_C l_2 = \frac{w}{4}(l_1^3 + l_2^3) \quad \ldots\ldots\ldots(6)$$

***Reactions and Shear Diagrams.**—As in the case of fixed beams, the shear diagrams for continuous beams will have their base lines shifted, due to the change in slope of the B.M. curve.

Consider any support, say B, and let r_1 be the reaction at B due to the span l_1 if the separate spans were simply supported, R_1 being the corresponding quantity for the continuous beam.

Then change in slope of B.M. curve $= \dfrac{M_B - M_A}{l_1}$

$$\therefore \ R_1 = r_1 + \frac{M_B - M_A}{l_1}$$

Similarly if r_2, R_2 are corresponding quantities for the span l_2

$$\therefore \ R_2 = r_2 + \frac{M_B - M_C}{l_2}.$$

∴ Total reaction at

$$B = R_B = R_1 + R_2 = r_1 + r_2 + \frac{M_B - M_A}{l_1} + \frac{M_B - M_C}{l_2}$$

Then R_1 and R_2 give the ordinates of the shear diagrams on either side of B. This will be made clearer in the following numerical example—

A continuous girder, A B C D (*Fig.* 211), *consists of three spans,* 20, 10 *and* 15 *ft. long, and the first span carries* 20 *tons, the second* 15 *tons, and the third* 10 *tons, uniformly distributed. Draw the B.M. and shear diagrams.*

First draw the B.M. diagrams as if the separate spans were freely supported.

Now take the first two spans, then by the theorem of three moments

$$M_A \times 20 + 2\,M_B \times 30 + M_C \times 10 = \frac{1}{4}\left\{\frac{20}{20} \cdot 20^3 + \frac{15}{10} \cdot 10^3\right\}$$

But the end A is freely supported. $\therefore \ M_A = 0$

$$\therefore \text{ We get } 60\,M_B + 10\,M_C = \frac{10^3}{4}\left(8 + \frac{15}{10}\right).$$

$$\text{or } 6\,M_B + M_C = 237{\cdot}5 \quad \dots\dots\dots\dots (1)$$

Now consider the next two spans. Then we have

$$M_B \times 10 + 2\,M_C \times 25 + M_D \times 15 = \frac{1}{4}\left\{\frac{15}{10} \cdot 10^3 + \frac{10}{15} \cdot 15^3\right\}$$

The end D being freely supported, we have

$$10\,M_B + 50\,M_C = \frac{5^3}{4}\left\{12 + 18\right\}$$

$$\text{or } M_B + 5\,M_C = 93{\cdot}75 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Solving the two simultaneous equations (1) and (2) we get

$$M_B = 37{\cdot}75$$
$$M_C = 11{\cdot}20$$

∴ Putting up these values we get the B.M. diagram as shown in the figure.

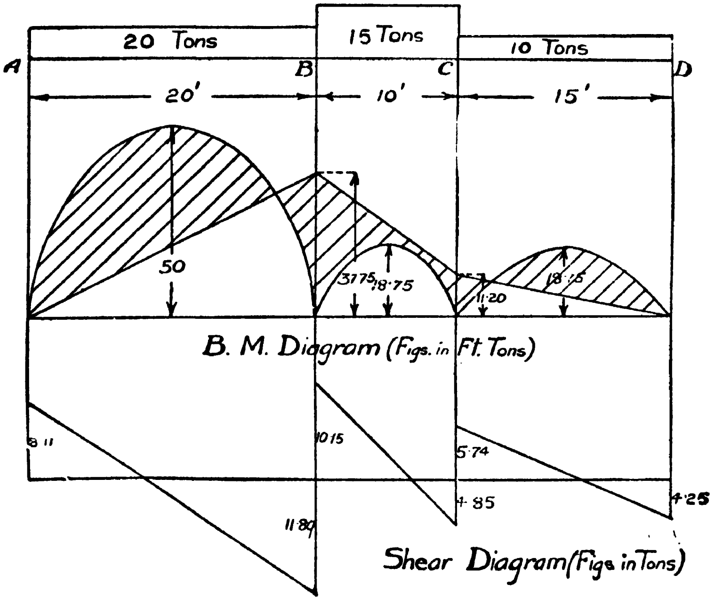


FIG. 211.—Continuous Beam of Three Spans.

To get the shear diagram we first calculate the reactions as follows—

$$R_A = \frac{w_1 l_1}{2} + \frac{M_A - M_B}{l_1} = \frac{20}{2} - \frac{37{\cdot}75}{20} \qquad = 8{\cdot}11 \text{ tons}$$

$$R_B = \frac{20}{2} + \frac{37{\cdot}75}{20} + \frac{15}{2} + \frac{26{\cdot}55}{10} = 11{\cdot}89 + 10{\cdot}15 = 22{\cdot}04 \text{ tons}$$

$$R_C = \frac{15}{2} - \frac{26{\cdot}55}{10} + \frac{10}{2} + \frac{11{\cdot}20}{15} = 4{\cdot}85 + 5{\cdot}74 \qquad = 10{\cdot}59 \text{ tons}$$

$$R_D = \frac{10}{2} - \frac{11{\cdot}20}{15} \qquad = 4{\cdot}26 \text{ tons}$$

Total .. .. **45·00 tons**

The shear diagram then comes as shown in the figure, the continuity of the beam altering only the base lines, and not the form of the curves.

If there are more than three spans, consecutive spans are taken two together, and a series of equations obtained by the theorem of three moments. Further numerical examples will be found at the end of the chapter.

* **Continuous Beams with Fixed Ends.**—If the end of a continuous beam is fixed, the end B.M. is obtained by imagining a beam to exist beyond the fixed end of the same length, and loaded in the same manner as the last beam. This is because the fixing of ends makes the beam horizontal at such ends, and this occurs at the centre of a continuous beam symmetrically loaded. An example of this will be found in the worked examples at the end of the chapter.

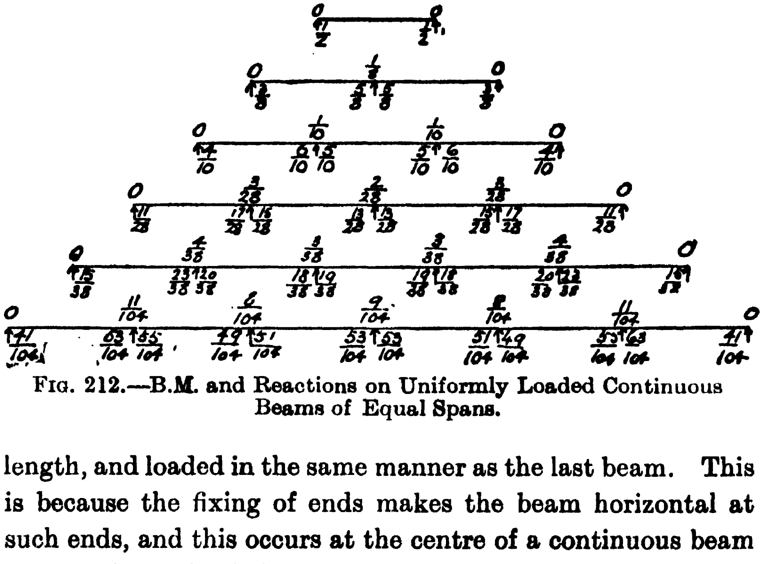


FIG. 212.—B.M. and Reactions on Uniformly Loaded Continuous Beams of Equal Spans.

Equal Spans with constant Uniform Load.—In practice the spans (l) are often equal, and the uniform load (w) per foot run constant, the extreme ends being freely supported. A diagram is shown in Fig. 212, from which the support B.M.s and reactions can readily be obtained for any number of spans up to six.

Above the span lines are the support moment coefficients, which have to be multiplied by $w l^2$

Below the span lines are the reaction coefficients, which have to be multiplied by *w l*.

From these the B.M. and shear diagrams can be readily drawn. The student should check these by working them through by means of the theorem of three moments.

* GRAPHICAL TREATMENT OF CONTINUOUS BEAMS

In dealing with a considerable number of spans with irregular loading, the application of the theorem of three moments becomes a somewhat laborious process. Although the following general graphical method is somewhat involved and takes considerable time to explain, it is interesting and useful, and shows to what extent the graphical method of reasoning can be pursued.

Consider the imaginary cable of Mohr's Theorem which gives the elastic line of a beam. It is a link polygon for the bending moments, drawn with a polar distance equal to E × I.

Now the slope and position of the first and last links of a link polygon are quite independent of the exact distribution of the forces, provided that they have the same resultant in magnitude and direction.

As we shall see later, we shall be able to obtain the support moments if we know the *support tangents* to the elastic line. Let A B (Fig. 213) represent a span of length *l* of a continuous beam, and let A C B represent the free B.M. curve for the loading on it, A *a* and B *b* being the support moments, M_A and M_B. If the centroid of the curve A C B is G, then the vertical G G is called the *centroid vertical*, and if the support B.M. curve be divided into two triangles A *a* B and B *a b*, the areas of such triangles act down the *right and left hand third lines* X X and Y Y. Now replace the actual B.M. curve for purposes of finding the elastic line by single forces acting down and up the lines G G, X X, Y Y.

On a vector line,

set down 1, 2 = area of free B.M. curve A C B = S
" " 0, 1 = area of triangle A a B = S_A
" " 2, 3 = " " a b B = S_B

Then with pole P at polar distance (p) = E I if $A_1 d$ is drawn parallel to 0 P, $d\,h$ to 1 P, $h\,g$ to 2 P and $g\,B_1$ to 3 P, $d\,h$ and $h\,g$ are called the *mid links*, and $A_1 d$ and $g\,B_1$ give the support tangents.

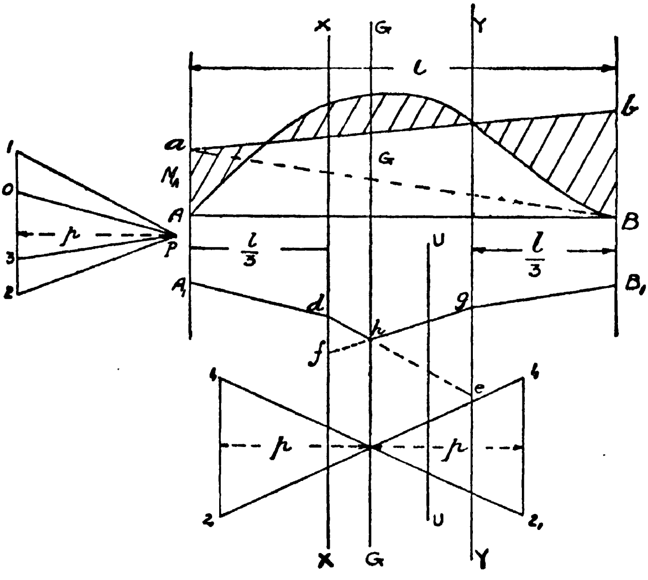


FIG. 213.—Continuous Beams—Graphical Treatment

Now in our problem we do not know the position of the points 0 and 3, and we see that these would be known if the mid links were found, so that our problem now reduces to that of finding these mid links.

On both sides of the centroid vertical G G draw lines at distance p, and set down lengths 1_1, 2_1 equal to 1, 2 and join them across, intersecting on the centroid vertical. These lines are called the ***cross lines***.

Now draw any vertical U U, then clearly the intercepts made by the vertical on the mid links and cross lines are equal.

From this it follows that if a point on one mid link is known, a point vertically below it on the other mid link can be found.

Again, let the right-hand mid link of this span meet the left-hand mid link of the next span in a point j, Fig. 214, on a vertical line Q Q.

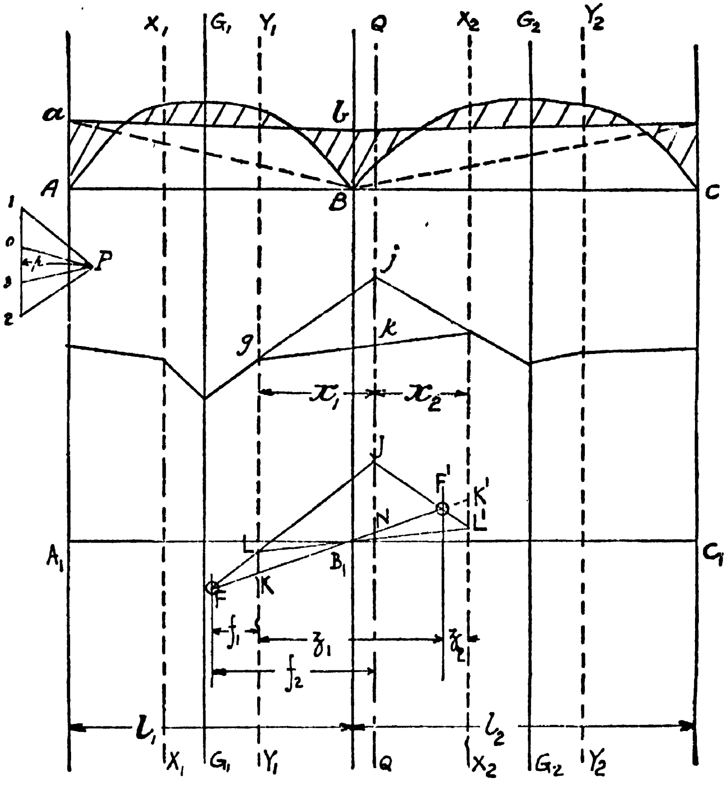


FIG. 214.—Continuous Beams—Fixed Points.

Then consider the triangles $g\ j\ k$, P 2 3.

They are similar $\therefore \dfrac{j\,k}{2,\,3} = \dfrac{x_1}{p}$

$$\therefore p \times j\,k = 2,\,3 \times x_1$$

$$= x_1 \times \text{area } b\,a\,\mathrm{B}$$

$$= \frac{\mathrm{M}_{''}\,l_1}{2} \,.\, x_1$$

Similarly considering the triangle on the other side of Q Q we should have

$$p \times jk = \frac{M_{\text{B}} l_2}{2} . x_2$$

Where l_2 is the length of the next span.

$$\therefore x_1 l_1 = x_2 l_2$$

$$\text{Further, } x_1 + x_2 = \frac{1}{3}(l_1 + l_2)$$

$$\therefore x_1 = \frac{l_2}{3}$$

$$x_2 = \frac{l_1}{3}$$

$\therefore$ Q Q is at a distance $= \frac{l_2}{3}$ from Y Y, and is thus called an *inverted third line.*

Determination of "Fixed Points."—Let A B C (Fig. 214) represent two consecutive spans of a continuous beam, and let the third lines be drawn as shown.

Suppose that we know that the right-hand mid link of the span A B passes through a fixed point F. Let this mid link cut the inverted third line Q Q in J and the third line Y Y in L, then L B′ must be a support tangent. Produce L B′ to meet the first third line of the span B C in L′, then J L′ is the left-hand mid link; and then join F B′ and produce it to meet J L′ in F′, then F′ will be a fixed point on the mid links of the second span. This is shown as follows—

Let the vertical through F′ be at distances z_1, z_2 from the third lines.

Then the triangles F′ J N, F′ K′ L′ are similar.

$$\therefore \frac{\text{J N}}{\text{K}' \text{L}'} = \frac{z_1 - \frac{1}{3} l_2}{z_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and triangles B′ K′ L′, B′ K L are similar.

$$\therefore \frac{\text{K}' \text{L}'}{\text{K L}} = \frac{l_2}{l_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

further, the triangles F L K, F J N are similar.

$$\therefore \frac{\text{K L}}{\text{J N}} = \frac{f_1}{f_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Multiplying together (1), (2) and (3), we get

$$1 = \frac{z_1 - \frac{1}{3} l_2}{z_2} \times \frac{f_1}{f_2} \times \frac{l_2}{l_1}$$

$$\therefore \frac{z_1 - \frac{1}{3} l_2}{z_2} = \frac{l_1 f_2}{l_2 f_1}$$

$$\text{also } z_1 + z_2 = \frac{l_1 + l_2}{3}$$

$$\therefore z_1 - \tfrac{1}{3} l_2 = - z_2 + \tfrac{1}{3} l_1$$

$$\therefore \frac{- z_2 + \frac{1}{3} l_1}{z_2} = \frac{l_1 f_2}{l_2 f_1}$$

$$\therefore \frac{l_1}{3 z_2} = 1 + \frac{l_1 f_2}{l_2 f_1}$$

$$\therefore z_2 = \frac{\frac{1}{3} l_1 l_2 f_1}{l_1 f_2 + l_2 f_1} = \text{constant.}$$

∴ F′ is a fixed point.

In this way a number of fixed points right along the various spans can be found as hereinafter further explained.

A fixed point is found at the terminal spans, as follows—

CASE 1. FREELY SUPPORTED END.—The end B.M. here must be zero, therefore support tangent and mid link must be collinear, so that A′ is the first fixed point.

CASE 2. BUILT-IN OR FIXED END.—Support tangent is horizontal, so that first fixed point is where horizontal through A′ cuts the first third line.

Graphical Construction for any Given Case.—We are now in a position to set out the construction for obtaining the B.M. diagram, which is as follows—

Draw the free B.M. diagrams and the third lines, the inverted third lines and the centroid verticals. Fig. 215 shows a continuous beam of three spans, one end being freely supported and the other fixed, X X representing the left-hand third lines, Y Y the right-hand third lines, Q Q the inverted third lines, G G the centroid verticals.

Now draw the cross lines at the bottom of the paper, such lines being obtained by setting down the areas S_1, S_2, etc., of the free B.M. curves on vertical lines at each side of the centroid verticals at distances representing the value of E I

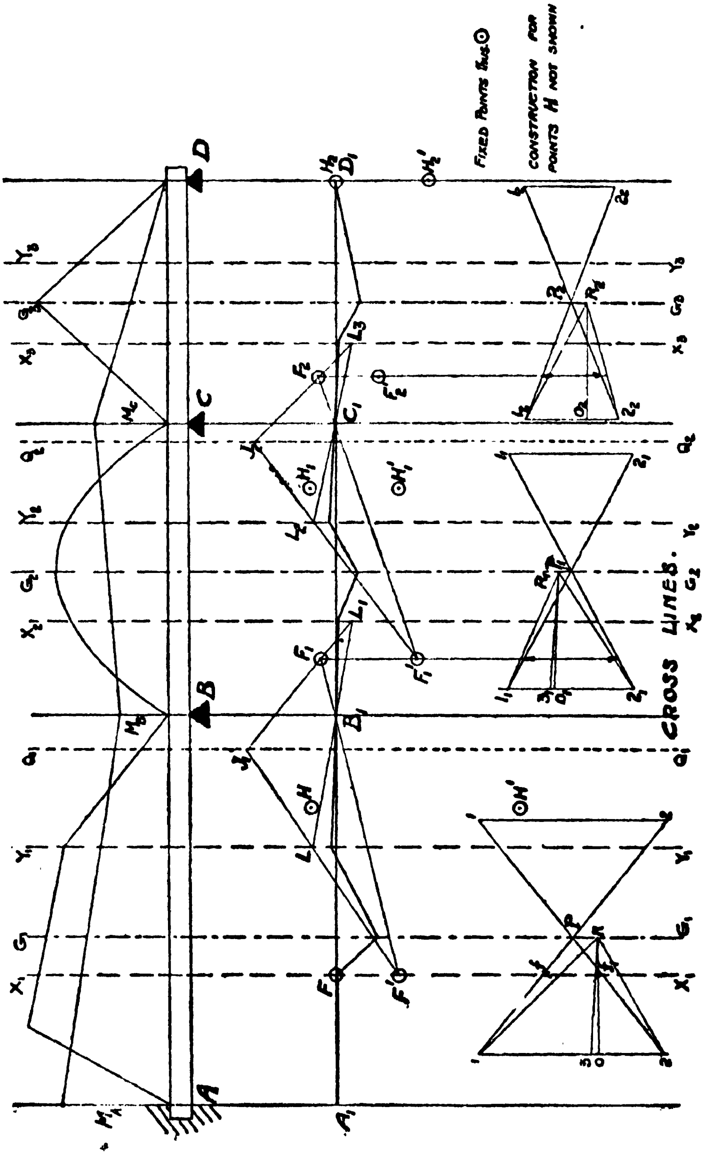


Fig. 215.—Graphical Investigation of Continuous Beam of Three Spans.

reduced in some convenient ratio, the scale of E I being the same as that of the areas. If the support moments only are required and not the deflections, and E I is the same for each span, E I need not be calculated, any convenient polar distance being taken.

P, P_1, and P_2 are the intersections of the cross lines.

Now find the fixed points. The end A is fixed, so that F is the first fixed point; now set down F F′ equal to the intercept $f f_1$ on the cross lines and draw any line F′ J_1 to the inverted third line, cutting Y Y in L; join L B′ and produce to meet the third line $X_1 X_1$ in L_1; then the intersection of $L_1 J_1$ and E′ B′ gives the fixed point F_1 on the second span. This is repeated as shown, and the points F_1', F_2, F_2' found.

Now start the other end D. This is freely supported, therefore, as we have seen before, D′ is the first fixed point H_2. By means of the cross lines, we then get the corresponding fixed point H_2', and by repeating the same construction as for the points F, we get a number of other fixed points, H_1', H_1, H′, H. The mid links and support tangents are now drawn in, and there will be two checks on the accuracy of the construction, viz.—

(*a*) Mid links must meet on centroid verticals.

(*b*) When adjacent mid links are joined, they must pass through points of support.

Now, from the points 1, 2, etc., on the cross lines, draw parallels to the mid links, and obtain the poles R, R_1, R_2 and then draw parallels to the support tangents, thus obtaining the points 0, 3, etc. Then

$$M_A = \frac{2 \times 0,1}{l_1}$$

and so on, the support moments then being set up and the true B.M. curve for the continuous beam thus being found.

Another interesting graphical method of finding the support moments in a continuous beam has been devised by Professor Claxton Fidler, and will be found in his book on *Bridge Construction.*

Advantages and Disadvantages of Continuous Beams.—It will be seen by considering the B.M. diagrams for continuous beams that the maximum B.M. is less than that which would occur if a number of separate simply supported beams were placed across the same supports (except in the case of two uniformly loaded equal spans, when it is the same), and that such maximum B.M. occurs at the abutments. The principal disadvantages are—

(*a*) It is not easy to ensure all the supports remaining at exactly the same level.

(*b*) The method of calculation of the stresses assumes that the beam is of uniform cross section throughout, this condition not being an economical one. Experimental investigations in Germany have shown that if the beam is not of uniform cross section, the method described may still be employed without great error.

(*c*) In the case of rolling loads, which occur frequently in bridge design, the calculations are much more difficult than in the case of separate spans.

Beams Fixed at one End and Freely Supported at the Other.—If a beam is fixed at one end and freely supported at the other, the B.M. and shear diagrams will be the same as for the half of a continuous beam of two equal spans of the same span as the given beam, and loaded in the same manner.

This is because fixing the end of a beam makes such end horizontal, and this is what happens at the central support of a continuous beam with two equal spans loaded in the same manner. The consideration of the following two standard cases should make this clear.

(*a*) BEAM FIXED AT ONE END AND FREELY SUPPORTED AT THE OTHER, SUBJECTED TO A UNIFORM LOAD.—The B.M. and shear diagrams in this case are the same as for one span of the first case of continuous beams that we have considered, and will therefore be as shown in Fig. 216.

(b) Beam Fixed at one End and Freely Supported at the Other, Subjected to a Central Load.—Let the central load be W and the span l.

Then, if B is the fixed end, A the freely supported end, and A′ the imaginary freely supported end existing beyond the fixed end, we have, by the Theorem of Three Moments,

$$M_A l + 2M_B(l+l) + M_{A'} l = 6\left\{\frac{Wl}{4}\times\frac{l}{2}\times\frac{l}{2l}+\frac{Wl}{4}\times\frac{l}{2}\times\frac{l}{2l}\right\}$$

Now $M_A = M_{A'} = 0$

$$\therefore\ 2M_B \cdot 2l = 6\left\{\frac{Wl^3}{16l}+\frac{Wl^3}{16l}\right\}$$

$$\therefore\ M_B = \frac{3Wl}{16}$$

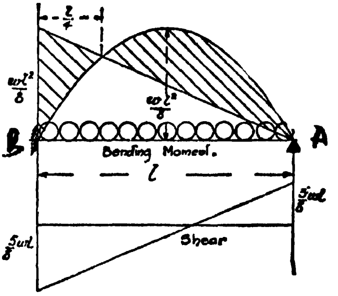


Fig. 216.

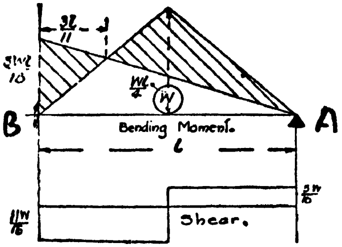


Fig. 217.

Beams Fixed at one End and Supported at the Other.

The B.M. diagram then comes as shown in Fig. 217.

To get the shear diagram we first work out the reactions.

$$R_A = \frac{W}{2} + \frac{M_A - M_B}{l}$$

$$= \frac{W}{2} - \frac{3Wl}{16l}$$

$$= \frac{5W}{16}$$

$$\therefore\ R_B = \frac{11W}{16}$$

The shear diagram then comes as shown in the figure.

(c) To find the Maximum Deflection for a Uniformly Loaded Beam fixed at one End and Supported at the Other.

The bending moment diagram for this case is as shown shaded in Fig. 218. The curve B D C is a parabola of height $\frac{w\,l^2}{2}$ where w is the load per unit length of the beam, this being the B.M. diagram for the downward uniform load on the canti-

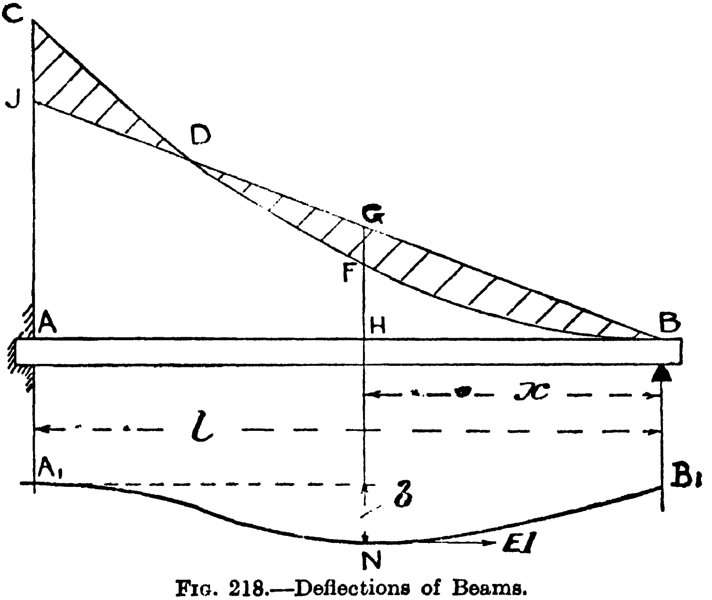


Fig. 218.—Deflections of Beams.

lever, and A J is equal to $\frac{3\,w\,l^2}{8}$, J B being a straight line; this being the B.M. diagram for the reaction at B, which is $\frac{3\,w\,l}{8}$

Our first problem is to find the point N at which the deflection has its maximum value. Consider the position A_1 N of the imaginary cable. The forces acting on it are a horizontal tension equal to E I at N and an equal horizontal tension at the point A_1, since the beam must be horizontal at the fixed end A; also an upward vertical force equal to the negative area O J D, and a downward vertical force equal to the positive area D F G.

If these forces are in equilibrium, since the horizontal forces are equal and opposite, the vertical forces are also equal and opposite, so that we get the following rule—

The maximum deflection will occur at the point where the area D F G *is equal to the area* D J C.

This is the same as saying that the area A H G J is equal to the area A H F C.

Now, if $\text{HB} = x$ and $\text{AB} = l$

$$\frac{\text{HG}}{x} = \frac{\text{AJ}}{l} \qquad \therefore \ \text{HG} = \frac{x\,\text{AJ}}{l}$$

$$\begin{aligned}\text{Area AHGJ} &= \frac{\text{AH}}{2}(\text{AJ} + \text{GH}) \\ &= \frac{l-x}{2}\,\text{AJ}\left(1 + \frac{x}{l}\right. \\ &= \frac{(l-x)}{2}\frac{(l+x)}{l}\cdot\frac{3\,w\,l^2}{8} \\ &= \frac{3\,w\,l}{16}(l^2 - x^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)\end{aligned}$$

$$\begin{aligned}\text{Also Area AHFC} &= \text{Area ABDC} - \text{Area FHB} \\ &= \frac{1}{3}\text{AC}\,.\,\text{AB} - \frac{1}{3}\,\text{FH}\,.\,\text{HB} \\ &= \frac{1}{3}\frac{w\,l^3}{2} - \frac{1}{3}\frac{w\,x^3}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)\end{aligned}$$

If (1) = (2)

$$\frac{3\,w\,l}{16}(l^2 - x^2) = \frac{w}{6}(l^3 - x^3)$$

Factorising, we get

$$\frac{3\,w\,l}{16}(l + x)(l - x) = \frac{w}{6}(l - x)(l^2 + l\,x + x^2)$$

$\therefore$ dividing through by $\frac{w}{2}(l - x)$ and multiplying across we get

$$9\,l^2 + 9\,l\,x = 8\,l^2 + 8\,l\,x + 8\,x^2$$

$$\textit{i.e.}\ \ 8\,x^2 - l\,x - l^2 = 0 \ldots\ldots\ldots\ldots\ldots(3)$$

The general solution of this quadratic equation is

$$x = \frac{l \pm \sqrt{l^2 + 32\,l^2}}{16}$$

$$= \frac{l\,(1 \pm \sqrt{33})}{16}$$

The negative value is inadmissible

$$\therefore\ x = \frac{l\,(1 + \sqrt{33})}{16} = \cdot 422\,l \text{ nearly.}$$

$\therefore$ *The maximum deflection occurs at a distance* $= \cdot 422\,l$ *from the simply supported end.*

We now proceed to find the maximum deflection δ by considering the stability of the portion N B_1 of the imaginary cable. The forces acting on it are a tension at B, the horizontal tension E I at N, and the area of the bending moment diagram B F G.

By taking moments about the point B_1, we eliminate the tension at this point and get $\text{E I} \times \delta =$ moment about B_1 of area B F G.

Now, this area is made up of the difference between the Δ B H G and the parabola B H F.

$$\text{The area of the } \Delta = \tfrac{1}{2}\,\text{G H}\,.\,\text{B H} = \tfrac{1}{2}\,.\,\frac{x}{l}\,.\,\text{A J}\,.\,x$$

$$= \tfrac{1}{2}\,\frac{x^2}{l}\,.\,\frac{3\,w\,l^2}{8} = \frac{3\,w\,x^2\,l}{16}$$

The centroid of the Δ is at distance $\frac{2\,x}{3}$ from B

$$\therefore\ \text{moment of } \Delta \text{ about B}_1 = \frac{3\,w\,x^2\,l}{16}\,.\,\frac{2\,x}{3} = \frac{w\,x^3\,l}{8}$$

$$\text{The area of the parabola} = \tfrac{1}{3}\,\text{F H}\,.\,\text{B H}$$

$$= \frac{1}{3}\,.\,\frac{w\,x^2}{2}\,.\,x = \frac{w\,x^3}{6}$$

The centroid of the parabola is at distance $= \frac{3\,x}{4}$ from B.

$$\therefore\ \text{moment of parabola about B}_1 = \frac{w\,x^3}{6}\,.\,\frac{3\,x}{4}$$

$$= \frac{w\,x^4}{8}$$

$\therefore$ moment about B_1 of area B F G

$$
\begin{aligned}
&= \frac{w x^3 l}{8} - \frac{w x^4}{8} \\
&= \frac{w x^3}{8}(l - x) \\
E I \times \delta &= \frac{w x^3}{8}(l - x) \\
\text{putting } x &= {\cdot}422\, l \\
\therefore E I \times \delta &= \frac{w \times {\cdot}422^3 l^3 ({\cdot}578\, l)}{8} \\
&= {\cdot}00543\, w\, l^4 \\
\therefore \delta &= \frac{{\cdot}00543\, w\, l^4}{E I} \\
\text{putting } w\, l &= \text{total load} = W \\
\delta &= \frac{{\cdot}00543\, W l^3}{E I} \\
&= \frac{W l^3}{184\, E I}
\end{aligned}
$$

For a uniformly loaded beam, simply supported at each end, we should get $\delta = \frac{5 W l^3}{384 E I}$, while for one similarly loaded, but fixed at each end, we should get $\delta = \frac{W l^3}{384 E I}$, so that we see that in the case under consideration the deflection is between these two values. This is, of course, what one would expect.

The same method may be applied to the case of an isolated central load W on a beam similar fixed. In this case the maximum deflection $\doteq \frac{W l^3}{48 \sqrt{5} E I}$ and occurs at $\frac{l}{\sqrt{5}}$ from the simply supported end.

We will conclude this chapter with a further number of worked examples of fixed and continuous beams.

WORKED EXAMPLES.—(1) *A beam of 20 ft. span is built-in at one end and is supported at a point 5 feet from the other end. Draw the B.M. and shear diagrams for a uniform load of $\frac{1}{2}$ ton per foot run.*

Let A B (Fig. 219) be the beam, fixed at the end A and supported at the point C.

The portion B C of the beam acts as a cantilever, and therefore the B.M. at C = $M_c = \frac{1}{2} \times \frac{5 \times 5}{2} = 6{\cdot}25$ ft. tons.

To find the B.M. at A, we imagine a span A C′ exactly similar to A C to exist within the wall.

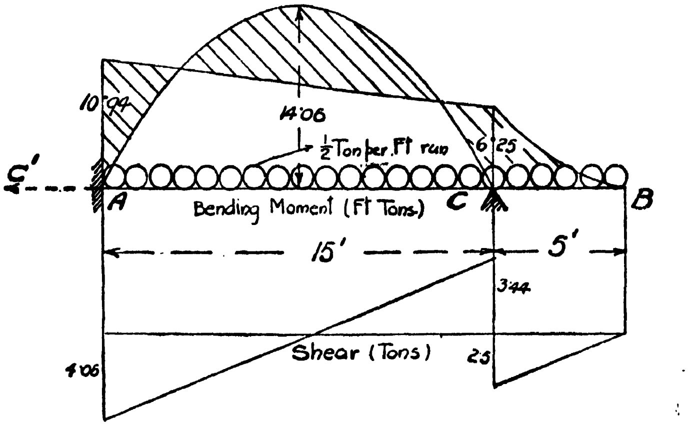


FIG. 219.

Then, by the Theorem of Three Moments, we have

$$M_c \times 15 + 2\,M_A\,(15 + 15) + M_c\,.\,15 = \frac{1}{8}(15^3 + 15^3)$$

But $M_{c'} = M_c = 6{\cdot}25$

$$\therefore\ 60\,M_A + 30 \times 6{\cdot}25 = \frac{1}{8}(2 \times 15^3)$$

$$\therefore\ 4\,M_A + 12{\cdot}5 = \frac{15^2}{4}$$

$$\therefore\ 4\,M_A = \frac{15^2}{4} - 12{\cdot}5$$

$$= 56{\cdot}25 - 12{\cdot}5 = 43{\cdot}75$$

$$\therefore\ M_A = 10{\cdot}94 \text{ ft. tons nearly.}$$

The B.M. diagram is then as shown in the figure. To get the reaction at C we proceed exactly as in the case of continuous beams

$$i.e.\ \mathrm{R}_c = \frac{1}{2} \cdot \frac{15}{2} + \frac{\mathrm{M}_c - \mathrm{M}_A}{15} + \frac{1}{2} \cdot \frac{5}{2} + \frac{\mathrm{M}_c - \mathrm{M}_B}{5}$$
$$= 3{\cdot}75 - {\cdot}31 + 1{\cdot}25 + 1{\cdot}25$$
$$= 3{\cdot}44 + 2{\cdot}5$$
$$= \underline{5{\cdot}94} \text{ tons.}$$

The shear diagram then comes as shown in the figure.

✓(2) *A rolled joist is firmly built-in at one end, and the other end rests freely on the top of a cast-iron column. The span of the joist is* 16 *feet, and it carries a single load of* 10 *tons,* 12 *feet from the column ends. Determine the reaction on the column, and draw the B.M. and shear diagrams.* (*B.Sc. Lond.*)

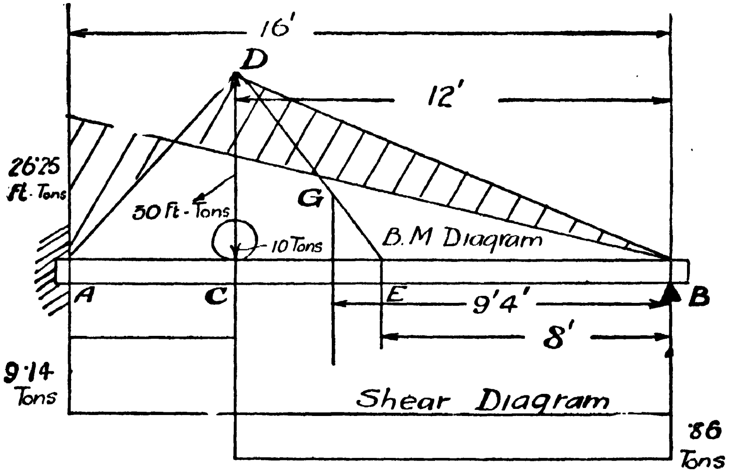


Fig. 220.—Example of Beam Fixed at one End and Supported at Other.

Let A B represent the beam, fixed at the end A, the load being at the point C (Fig. 220).

Then the free B.M. diagram is a triangle A D B, C D being equal to

$$\frac{\mathrm{W}ab}{l} = \frac{10 \times 12 \times 4}{16} = 30 \text{ ft. tons.}$$

Then area of B.M. diagram

$$= \frac{1}{2} \times 30 \times 16 = 240 \text{ sq. ft. tons.}$$

The centroid G of the B.M. diagram occurs at a distance

⅓ E C from E the centre of the beam, *i. e.* at a distance 9⅓ ft. from A.

Then, imagining a span exactly similar to A B to exist beyond the fixed end, we have, by the Theorem of Three Moments

$$16\,M_B + 2\,M_A\,(16 + 16) + 16\,M_{B'} = 6\left\{\frac{240 \times 9\frac{1}{3}}{16} + \frac{240 \times 9\frac{1}{3}}{16}\right\}$$

$$M_{B'} = M_B = 0$$

$$64\,M_A = \frac{6 \times 2 \times 240 \times 28}{16 \times 3} = 7 \times 240$$

$$M_A = \frac{7 \times 240}{64} = \frac{210}{8}$$

$$= 26{\cdot}25 \text{ ft. tons.}$$

The reaction on the end B for a freely supported beam

$$= r_B = \frac{10 \times 4}{16} = 2{\cdot}5 \text{ tons.}$$

$$\therefore \text{ In this case } R_B = r_B + \frac{M_B - M_A}{l}$$

$$= 2{\cdot}5 + \frac{0 - 26{\cdot}25}{16}$$

$$= 2{\cdot}5 - 1{\cdot}64$$

$$= {\cdot}86 \text{ tons.}$$

✓(3) *A continuous girder consists of two unequal spans of* 100 *ft. and* 120 *ft. respectively. The girder is* 300 *ft. long and overhangs the end supports at each end, and is loaded as shown (Fig.* 221*). Draw the B.M. and shear diagrams and show the points of inflexion and magnitude of the supporting forces. (B.Sc. Lond.)*

In this case the end pieces A B, D E act as cantilevers.

$$\therefore\ M_B = \frac{40 \times 1\frac{1}{2} \times 40}{2} = 1{,}200 \text{ ft. tons.}$$ ✓

$$M_D = \frac{40 \times 2 \times 40}{2} = 1{,}600 \text{ ft. tons}$$ ✓

The free B.M. curve for span B C is a parabola with maximum ordinate $= \dfrac{1\frac{1}{2} \times 100 \times 100}{8} = 1{,}875$ ft. tons.

The free B.M. curve for span C D is a parabola with maximum ordinate $= \dfrac{2 \times 120 \times 120}{8} = 3{,}600$ ft. tons.

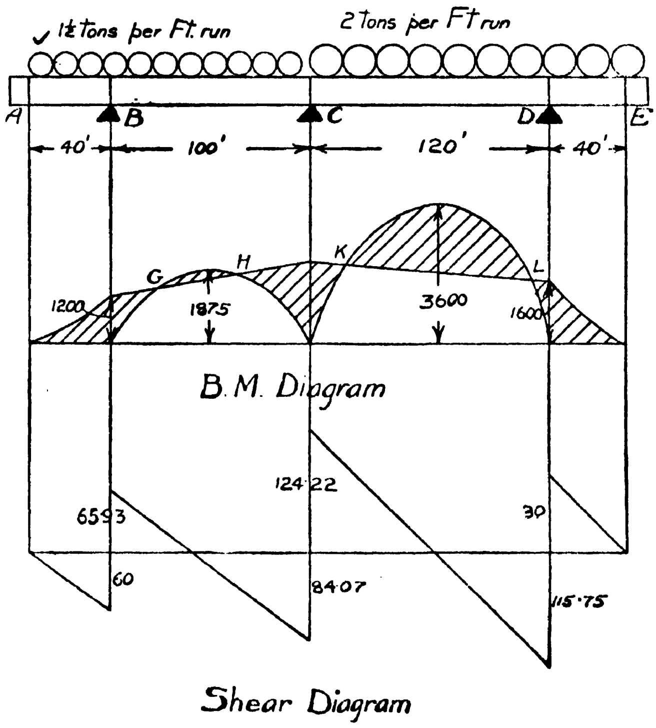


FIG. 221.

Then applying the Theorem of Three Moments we have

$$100\,M_B + 2\,M_C(100 + 120) + 120\,M_D = \frac{1}{4}(1\tfrac{1}{2} \times 100^3 + 2 \times 120^3)$$

$$\therefore\ 120{,}000 + 440\,M_C + 192{,}000 = 375{,}000 + 864{,}000$$

$$440\,M_C = 927{,}000$$

$$M_C = 2{,}107 \text{ ft. tons nearly.}$$

We now proceed to the determination of the reactions.

$$R_B = \frac{1}{2} \times 40 \times 1\frac{1}{2} + \frac{M_B - M_A}{40} + \frac{1}{2} \times 100 \times 1\frac{1}{2} + \frac{M_B - M_C}{100}$$
$$= 30 + 30 + 75 - 9{\cdot}07$$
$$= 60 + 65{\cdot}93 \qquad = 125{\cdot}93 \text{ tons}$$

$$R_C = \frac{1}{2} \times 100 \times 1\frac{1}{2} + \frac{M_C - M_B}{100} + \frac{1}{2} \times 2 \times 120 + \frac{M_C - M_D}{120}$$
$$= 75 + 9{\cdot}07 + 120 + 4{\cdot}22$$
$$= 84{\cdot}07 + 124{\cdot}22 \qquad = 208{\cdot}29 \text{ ,,}$$

$$R_D = \frac{1}{2} \times 2 \times 120 + \frac{M_D - M_C}{120} + \frac{1}{2} \times 2 \times 40 + \frac{M_D - M_E}{40}$$
$$= 120 - 4{\cdot}22 + 40 + 40$$
$$= 115{\cdot}78 + 80 \qquad = 195{\cdot}78 \text{ ,,}$$

Total .. 530 tons

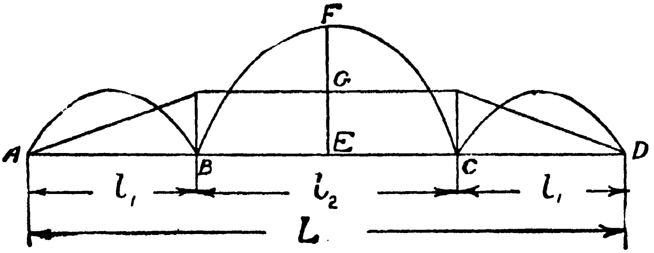


FIG. 222.

The shear diagrams then come as shown on the figure, and the points G H K L are the points of inflection.

(4) *A continuous beam of total length* L *has three spans and is uniformly loaded. Find the most economical arrangement of the spans.*

It follows from symmetry that in the best arrangement the two end spans will be equal. Let the end spans be of length l_1 and the centre span of length l_2 (Fig. 222).

$$\text{Then} \quad L = l_2 + 2\,l_1$$

Now by the Theorem of Three Moments

$$M_A\,l_1 + 2\,M_B\,(l_1 + l_2) + M_C\,l_2 = \frac{w}{4}\,(l_1^3 + l_2^3)$$

From symmetry $M_C = M_B$

also $M_A = 0$

$$\therefore M_B (2 l_1 + 3 l_2) = \frac{w}{4} (l_1^3 + l_2^3)$$

We now require to find the relation between l_1 and l_2 to make M_B a minimum, and then see if M_B is greater than the intermediate B.M.s: if so, this relation will give us the most economical arrangement.

$$M_B = \frac{\frac{w}{4}(l_1^3 + l_2^3)}{(2 l_1 + 3 l_2)}$$

Now $l_2 = L - 2 l_1$

$$\therefore M_B = \frac{\frac{w}{4}\left\{(L - 2 l_1)^3 + l_1^3\right\}}{(3 L - 6 l_1 + 2 l_1)}$$

This will be a maximum when $\frac{d M_B}{d l_1} = 0$

$$\text{i.e. when } \frac{d}{d l_1}\left(\frac{L^3 - 6 L^2 l_1 + 12 L l_1^2 - 7 l_1^3}{3 L - 4 l_1}\right) = 0$$

$$\text{i.e. when } (3 L - 4 l_1)(-21 l_1^2 + 24 l_1 L - 6 L^2) + 4 (L^3 - 6 L^2 l_1 + 12 l_1^2 L - 7 l_1^3) = 0$$

$$\text{i.e. } 56 l_1^3 - 111 L l_1^2 + 72 l_1 L^2 - 14 L^3 = 0$$

The solution of this equation will be found to be $l_1 = \cdot 35\,L$, such solution being found by plotting.

Thus we see that the least value of the support moments occurs when the end spans are each ·35 L and the centre ·3 L. In this case the intermediate B.M.s are less than the support moments, so that this gives the most economical arrangement.

CHAPTER XVI

* DISTRIBUTION OF SHEAR STRESSES IN BEAMS

WHEN a beam is deflected there is a horizontal* shearing stress at every point of the beam, resisting the sliding of one layer over the other. We have already shown (p. 10) that in an elastic material a shear stress must always be accompanied by a shear stress of equal intensity at right angles to it; in the case of the beam we see that the horizontal and vertical shearing stresses at any point of a beam are equal. Now the total shearing force over any vertical cross section of a beam must be equal to the shearing force, obtained, as in previous chapters, by considering the forces on the beam; but the intensity of stress will not be the same across the section, so that by dividing the shearing force S by the area of the cross section A, as is commonly done, we do not get the maximum shear stress.

The existence of the horizontal shearing stress can be seen clearly from the following diagrammatic representation. Fig. 223 (A) shows a short beam deflected under some loading. Now imagine the beam to be replaced by a number of plates placed one above the other. They then take the form shown at (B) on the figure, the plates sliding one over the other as shown. The second case will not be nearly as strong as the first case, and it is clear that in case (A) there must be stresses tending to make one layer slide over the other.

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* We will assume through this investigation that the beam is horizontal. If it is not, the words "parallel to the axis of the beam" and "perpendicular to the axis of the beam" should be substituted for "horizontal" and "vertical."

We will now obtain an expression for finding the shearing stress at any point of a beam, and will consider later certain special cases.

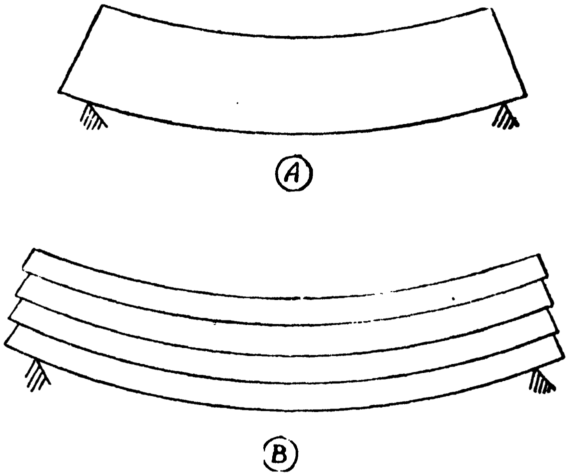


Fig. 223.—Horizontal Shear in Beams.

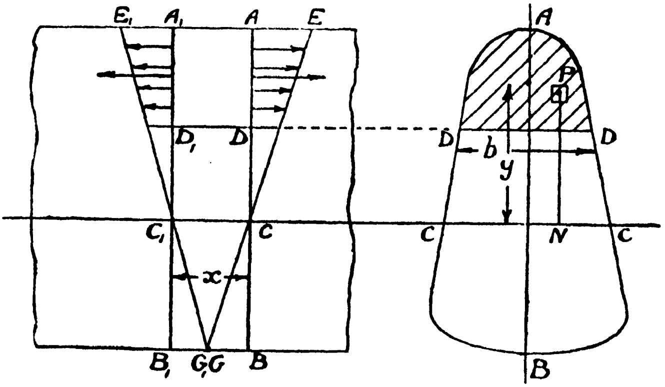


Fig. 224.—Distribution of Shear.

General Case.—Let AB, A_1B_1 (Fig. 224), be two cross sections of a beam at a short distance x apart, and let the cross section of such beam be symmetrical about a vertical axis, and let the loading be wholly transverse. Then ECG and $E_1C_1G_1$, as we have previously seen, give the intensities of transverse stress at any point. Now consider the portion of the section A B above any line D D. Consider an element of area a at a point P at distance P N from the neutral axis.

Then we have by the theory of bending that the intensity of stress at $\text{P} = f_{\text{P}} = \dfrac{\text{M} \times \text{PN}}{\text{I}}$, where M is the B.M. at the point and I the second moment of the section.

$$\therefore \text{ Force on element } a = f_{\text{P}} \times a = \frac{\text{M} \times \text{PN}}{\text{I}} \,.\, a$$

$$\therefore \text{ Total force on area above DD} = \Sigma \frac{\text{M} \times \text{PN}}{\text{I}} \,.\, a$$

$$= \text{F} = \frac{\text{M}}{\text{I}} \Sigma\, a \,.\, \text{PN}$$

$$= \frac{\text{M}}{\text{I}} \times \text{first moment of area above DD about N.A.}$$

$$= \frac{\text{M}}{\text{I}} \times a \,.\, y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Where a is the area above D D and y the distance of its centroid from the N.A.

Similarly taking the section $\text{A}_1\,\text{B}_1$ and taking the force above a line $\text{D}_1\,\text{D}_1$ we have

$$\text{Total force on area above } \text{D}_1\,\text{D}_1 = \text{F}_1 = \frac{\text{M}_1}{\text{I}_1} \times a_1\, y_1$$

Now, if x is small, and the beam has no abrupt change in cross section, we may put $a = a_1$, $y = y_1$, and $\text{I} = \text{I}_1$

$$\therefore \text{F} - \text{F}_1 = \frac{(\text{M} - \text{M}_1)\, a\, y}{\text{I}} \quad \ldots\ldots\ldots\ldots (2)$$

Now this difference in transverse force is the shearing force which has to be carried along the line $\text{D}\,\text{D}_1$. We will write this

$$\text{F} - \text{F}_1 = \frac{(\text{M} - \text{M}_1)}{x} \,.\, \frac{a \,.\, y \,.\, x}{\text{I}}$$

Then, if x is very small $\dfrac{\text{M} - \text{M}_1}{x}$ is the rate of increase or decrease of the B.M., and this we have shown to be equal to the shearing force S at the given point.

$$\therefore \text{ We have F} - \text{F}_1 = \frac{\text{S} \times a \,.\, y \,.\, x}{\text{I}} \quad \ldots\ldots\ldots (3)$$

Now the area over which this shearing force acts is equal to $D_1D \times DD = DD \times x = x \times b$.

$$\therefore \text{ Mean shearing stress along } DD = \frac{F - F_1}{b\,x}$$

$$= \frac{S \times a \, . \, y \, . \, x}{b\,x \, . \, I}$$

$$s_D = \frac{S \, . \, a \, . \, y}{I \, . \, b} \quad \dots\dots\dots\dots (4)$$

We can express this in terms of the mean stress $m = \frac{S}{A}$ over the whole section as follows—

$$s_D = \frac{S \, . \, a \, . \, y}{A \, . \, k^2 b}$$

$$= m \, . \, \frac{a \, . \, y}{k^2 \, b} \quad \dots\dots\dots\dots\dots\dots (5)$$

We may call $\frac{a\,y}{k^2\,b}$ the *shear coefficient.*

It will be noted that $a \times y$ increases up to the neutral axis and then decreases, because the first moment of the area below the N.A. is negative.

We thus see that *the shear stress is a maximum at the neutral axis.*

It must be remembered that s_D gives only the mean shear stress along D D. This stress is not uniform along D D, but for sections which are narrow at the neutral axis, the sections used in practice generally falling under this head, the maximum shear along the neutral axis will be not much greater than the value of s_D at the neutral axis as given by the above result. For sections like the square and the circle the maximum shear along D D will be from 5–10 % greater than the mean shear, while for sections such as an oblate ellipse or a broad rectangle the difference may amount to as much as 25 %. It is beyond our present scope to go further into the question as to the variation of shear stress along D D, but we should remember that such stress is not uniform; the maximum stress for various cases has been worked out by St. Venant.

Consider the following special cases (Figs. 225, 226).

(1) Rectangular Section.—Mean shear along a line at distance x from N.A. of a rectangle of height h and breadth b

$$= s_x = m \cdot \frac{a \cdot y}{k^2 b}$$

In this case $a = \left(\frac{h}{2} - x\right) b$

$$y = x + \frac{1}{2}\left(\frac{h}{2} - x\right) = \frac{1}{2}\left(\frac{h}{2} + x\right)$$

$$k^2 = \frac{h^2}{12}$$

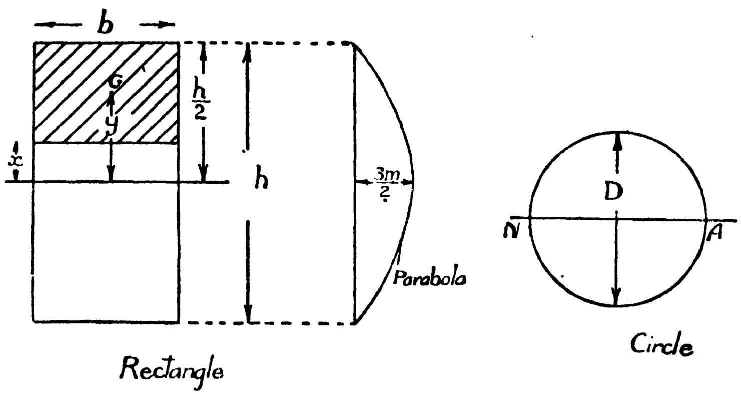


Fig. 225.

$$\therefore s_x = \frac{m \cdot \left(\frac{h}{2} - x\right) b \cdot \frac{1}{2}\left(\frac{h}{2} + x\right)}{\frac{h^2}{12} \cdot b}$$

$$= \frac{6\,m\left(\frac{h^2}{4} - x^2\right)}{h^2}$$

$$= 6\,m\left(\frac{1}{4} - \frac{x^2}{h^2}\right)$$

$$= \frac{3\,m}{2}\left(1 - \frac{4\,x^2}{h^2}\right)$$

This depends on x^2, so that the curve showing the mean shear stress at various depths will be a parabola. The maximum value of s_x occurs when $x = 0$, *i. e.* at the neutral axis. This gives $s_o = \frac{3m}{2} = 1{\cdot}5\,m$. Thus we see that in a rectangular beam the maximum shear stress occurs at the centre, and is equal to 1·5 times the shearing force divided by the area of the section.

(2) CIRCULAR SECTION.—This case is not quite so simple as the previous case, but we can find the shear stress at the N.A. simply as follows—

In this case we have

$$a = \frac{\pi D^2}{8}$$

$$y = \frac{2D}{3\pi}$$

$$k^2 = \frac{D^2}{16}$$

$$b = D$$

$$\therefore s_{N.A.} = m\frac{\frac{\pi D^2}{8} \cdot \frac{2D}{3\pi}}{\frac{D^2}{16} \cdot D}$$

$$= \frac{4m}{3} = 1{\cdot}33\,m$$

So that the mean shear stress along the N.A. is 1⅓ times the mean shear stress over the whole section.

In this case it is interesting to note that the *maximum* shear stress along the N.A. is 1·45 m.

(3) PIPE SECTION.—Let a thin pipe be of mean diameter D and thickness t (Fig. 226).

Then
$$a = \frac{\pi D t}{2}$$

$$y = \frac{D}{\pi}$$

$$k^2 = \frac{D^2}{8}$$

$$b = 2t$$

$$\therefore s_{N.A.} = m \times \frac{\frac{\pi D t}{2} \times \frac{D}{\pi}}{\frac{D^2}{8} \times 2t} = 2m$$

So that the mean stress shear along the N.A. is twice the mean shear stress over the whole section.

(4) **I** SECTION.—To calculate the proportion of the shearing force carried by the flanges and web, respectively.

Take a beam of **I** section of breadth b and height h, and let the thickness of the flanges and the web be t and w, respectively.

First consider a horizontal line P P in the flange at distance x from the top edge (Fig. 226).

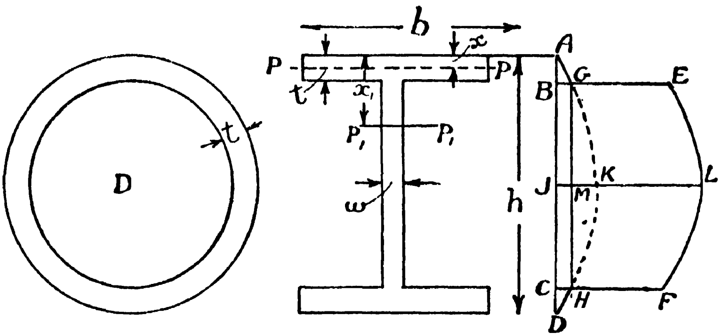


FIG. 226.

$$\text{Then mean shear along P P} = m \cdot \frac{a \cdot y}{k^2 \cdot b}$$

$$= s_x = \frac{m \cdot b\,x\,(h - x)}{b\,k^2\,2}$$

$$= \frac{m}{2\,k^2}(h\,x - x^2) \quad \ldots\ldots\ldots\ldots (1)$$

This depends on x^2, so that the curve showing the variation of stress is a parabola.

When $x = t$, *i. e.* at the junction of web and flange,

$$s_t = \frac{m}{2\,k^2}(h\,t - t^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Now consider a horizontal line $P_1\,P_1$ in the web at distance x_1 from the top.

$$\text{Then mean shear along } P_1\,P_1 = \frac{m \cdot a \cdot y}{k^2\,b}$$

In this case

$$a\,y = \text{first moment of area above } P_1 P_1 \text{ about N.A.}$$

$$= \frac{b\,t\,(h-t)}{2} + w\,(x_1 - t)\left\{\frac{h}{2} - \left(t + \frac{x_1 - t}{2}\right)\right\}$$

$$= \frac{b\,t\,(h-t)}{2} + \frac{w\,(x_1 - t)\,(h - x_1 - t)}{2}$$

also $b = w$ in general expression for shear stress.

$$\therefore\ s_{P_1} = \frac{m}{2\,k^2}\left\{\frac{b\,t\,(h-t)}{w} + w\,\frac{(x_1 - t)\,(h - x_1 - t)}{w}\right\}$$

$$= \frac{m}{2\,k^2}\left\{\frac{b\,t\,(h-t)}{w} + (h\,x_1 - x_1^2 - h\,t + t^2)\right\}$$

$$= \frac{m}{2\,k^2}\left\{(h\,x_1 - x_1^2) + (h-t)\left(\frac{b\,t}{w} - t\right)\right\}$$

$$= \frac{m}{2\,k^2}(h\,x_1 - x_1^2) + \frac{m\,t\,(h-t)\,(b-w)}{2\,k^2\,w} \quad \ldots\ldots\ldots(3)$$

The second term of this expression is constant for all values of x_1 and the first term is the shear stress which would occur if the flanges extended down to $P_1 P_1$.

We thus see that the diagram of distribution stress is obtained as follows—

First draw a parabola AKD, the centre ordinate JK of which is obtained by putting $x = \frac{h}{2}$ in equation (1).

$$i.e.\ \text{JK} = \frac{m}{2\,k^2}\left(\frac{h^2}{2} - \frac{h^2}{4}\right) = \frac{m\,h^2}{8\,k^2}$$

At the points B and C corresponding to the inside edges of the flanges set out GE and HF equal to the expression $\frac{m\,t\,(h-t)\,(b-w)}{2\,k^2 . w}$ and re-draw the portion GKH of the parabola between the points E and F, then the curve AGELFHD gives the shear stress at the various depths of the cross section.

Then total shear carried by web is equal to area of piece BELFC of curve multiplied by width of web.

Now take the case in which $t = \frac{h}{10}$ and $w = \frac{h}{20}$ and $b = \frac{h}{2}$, this being about the proportions for a rolled steel joist, then

$$\mathrm{BG} = s_t = \frac{m}{2\,k^2}(h\,t - t^2)$$

$$= \frac{m}{2\,k^2}\left(\frac{h^2}{10} - \frac{h^2}{100}\right)$$

$$= \frac{m}{2\,k^2} \cdot \frac{9\,h^2}{100}$$

$$\therefore \mathrm{MK} = \frac{m}{2\,k^2}\left(\frac{h^2}{4} - \frac{9\,h^2}{100}\right) = \frac{m}{2\,k^2} \cdot \frac{16\,h^2}{100} = \frac{m}{2\,k^2} \cdot \frac{4\,h^2}{25}$$

also $$\mathrm{GE} = \frac{m\,t\,(h - t)\,(b - w)}{2\,k^2\,w}$$

$$= \frac{m}{2\,k^2} \cdot \frac{h}{10} \cdot \frac{9\,h}{10} \cdot \frac{9\,h}{20} \times \frac{20}{h}$$

$$= \frac{m}{2\,k^2} \cdot \frac{81\,h^2}{100}$$

$$\therefore \mathrm{BE} = \frac{m}{2\,k^2} \cdot \frac{9\,h^2}{100} + \frac{m}{2\,k^2} \cdot \frac{81\,h^2}{100}$$

$$= \frac{m}{2\,k^2} \cdot \frac{9\,h^2}{10}$$

$$\therefore \text{Area of curve } \mathrm{BELFC} = \mathrm{BC}\left(\mathrm{BE} + \frac{2}{3}\mathrm{MK}\right)$$

$$= \frac{4\,h}{5} \cdot \frac{m}{2\,k^2}\left(\frac{9\,h^2}{10} + \frac{8\,h^2}{75}\right) \quad \ldots.(4)$$

Now in this case $$\mathrm{I} = \frac{b\,h^3}{12} - \frac{(b - w)\,(h - 2\,t)^3}{12}$$

$$= \frac{h^4}{24} - \frac{9\,h}{20} \cdot \left(\frac{4\,h}{5}\right)^3 \cdot \frac{1}{12}$$

$$= \cdot 0417\,h^4 - \cdot 0192\,h^4$$

$$= \cdot 0225\,h^4$$

The area of the section $= b\,h - (b - w)\,(h - 2\,t)$

$$= \mathrm{A} = \frac{h^2}{2} - \frac{9\,h}{20} \cdot \frac{4\,h}{5}$$

$$= \cdot 14\,h^2$$

$$\therefore k^2 = \frac{\mathrm{I}}{\mathrm{A}} = \frac{\cdot 0225\,h^4}{\cdot 14\,h^2} = \cdot 1608\,h^2$$

Returning to equation (4) we get area of curve BELFC

$$= \frac{4\,m\,h}{10\,k^2}\left\{\frac{9\,h^2}{10} + \frac{8\,h^2}{75}\right\}$$

$$= \frac{4\,m\,h \times 1{\cdot}007\,h^2}{10 \times {\cdot}1608\,h^2}$$

$$= m\,.\,h \times \frac{4 \times 1{\cdot}007}{1{\cdot}608}$$

$$= 2{\cdot}505\,m\,h \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$\therefore \text{ Shear carried by web} = 2{\cdot}505\,m\,h \times \text{width of web}$$

$$= 2{\cdot}505\,m\,h \times \frac{h}{20}$$

$$= {\cdot}1252\,m\,h^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Now area of whole section $= {\cdot}14\,h^2$

$\therefore$ Total shear S on section $= {\cdot}14\,h^2 \times m$

$$\therefore \frac{\text{Shear carried by web}}{\text{Total shear}} = \frac{{\cdot}1252}{{\cdot}14} = 89{\cdot}4\ \%.$$

It is commonly assumed in practice that in plate and box girders the whole of the shear is carried by the web, and the above calculation shows that in an I beam, in which the flanges are larger in proportion to the depth than in most plate and box girders, this is true within 10 %.

It must, however, be remembered that in girders built up of joists and plates, such as the comparatively shallow and heavy girders used in buildings, this assumption will not be so nearly true.

Shear Stresses in Reinforced Concrete Beams.—The usual treatment is as follows—

Consider two vertical sections of a reinforced concrete beam made at points A B a short distance x apart (Fig. 227), the section not changing appreciably from A to B. At the point A, the total stresses due to the bending moment are C and T, and at B they are C′ and T′; then if the corresponding bending moments are B and B′, we see that

$$\mathrm{T} = \mathrm{C} = \frac{\mathrm{B}}{a}$$

$$\mathrm{T}' = \mathrm{C}' = \frac{\mathrm{B}'}{a}$$

$$\therefore \mathrm{T} - \mathrm{T}' = \frac{\mathrm{B} - \mathrm{B}'}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

ADHESION STRESS DUE TO SHEAR. — But T − T′ is the difference in the pulls in the reinforcement at the two points; that is, it is the force which tends to pull the reinforcement out of the concrete.

$$\therefore \frac{T - T'}{x} = \text{adhesive or shear force per unit length}$$

$$= \frac{B - B'}{x \cdot a}$$

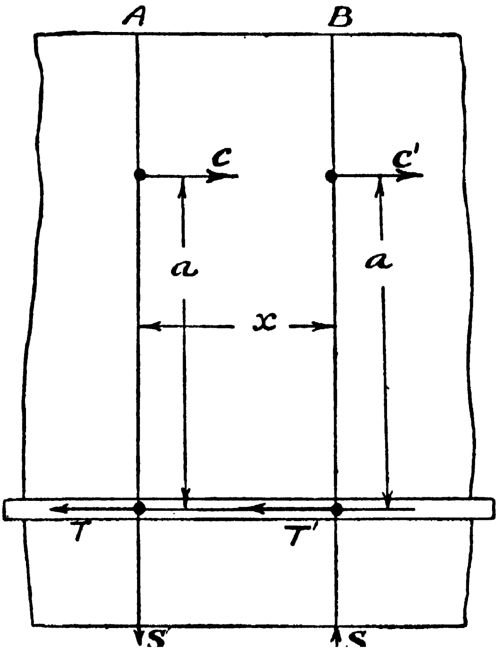


FIG. 227.

$$\text{but } \frac{B - B'}{x} = \text{the rate of change of the bending moment}$$

$$= \text{shearing force} = S$$

and if f_a = the adhesive stress per sq. in. and

O = the total perimeter of the reinforcement,

we have

$$f_a \times O = \text{adhesive stress per unit length} = \frac{T - T'}{x}$$

$$\therefore f_a = \frac{S}{O \cdot a} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

In the case of rectangular beams with tension reinforcement only or with double reinforcement where the top reinforcement is placed at $\frac{1}{3}$ depth of N.A. $a = \left(d - \frac{n}{3}\right)$

$$\therefore \text{ our formula becomes } f_a = \frac{S}{O\left(d - \frac{n}{3}\right)} \quad \ldots\ldots\ldots\ldots\ldots (3)$$

This deals with what is known as the *horizontal shear* as regards the adhesion between steel and tension.

Shear Stress in Concrete.—In addition to this we have to consider the shear stress in the concrete itself, which will be constant from the reinforcement to the N.A. and will then vary towards the top as indicated in Fig. 228.

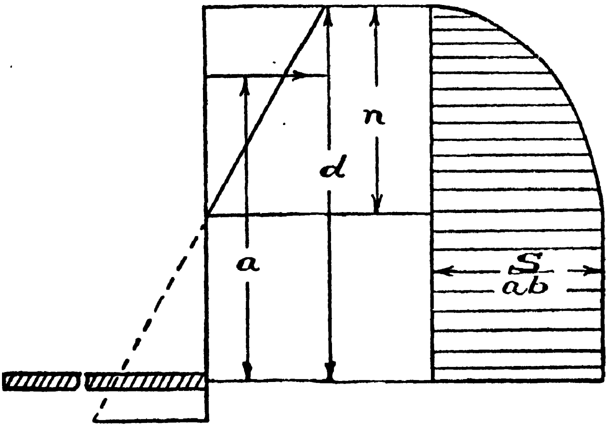


FIG. 228.

The difference of pulls T − T′ is distributed over a horizontal rectangular area one side of which is x and the other side of which is b, the breadth of the beam.

$\therefore$ If s is the shear stress we shall have

$$s \times b \times x = T - T'$$

$$\text{but } T - T' = \frac{B - B'}{a} \text{ (from (1))}$$

$$\therefore s = \frac{(B - B')}{x \, . \, a \, . \, b} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

But as before we may put $\frac{B - B'}{x}$ = shearing force S

$$\therefore s = \frac{S}{a\,b} \dots\dots\dots\dots\dots\dots (5)$$

and for rectangular beams as above

$$s = \frac{S}{b\left(d - \frac{n}{3}\right)} \dots\dots\dots\dots\dots\dots (6)$$

GRAPHICAL TREATMENT FOR FINDING DISTRIBUTION OF SHEAR STRESS ON A CROSS SECTION

Consider the section, composed of joists and plates, shown in Fig. 229. The first step is to "mass the section up" about a vertical centre line: this is done by drawing horizontal lines across the joists, and adding on each side of the centre joist the corresponding horizontal ordinate of the outside joists. This gives the section shown in the figure (i.e. $a\,d = a\,b + b\,c + c\,d$).

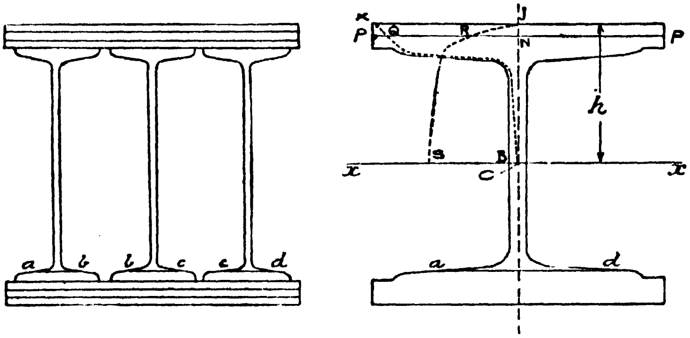


Fig. 229.

Consider any line P P. We have shown that the mean shear stress along P P $= s_{\text{P}} = m \cdot \frac{a \cdot y}{k^2 b}$.

Now $a \cdot y$ = first moment of area above P P about neutral axis $x\,x$. Draw the first moment curve of the section above $x\,x$ about the line $x\,x$, as explained on p. 176. As the section is symmetrical about a vertical axis, we need draw the curve for one half of the area only, X Q C being such curve.

Then $a \times y = 2$ area $\mathrm{JXQN} \times h$.

$$\therefore \text{Mean shear along } \mathrm{PP}_1 = \frac{m}{k^2 b} \cdot \frac{2 \times \mathrm{JXQN} \times h}{1}$$

Now find the sum-curve JRS of the first moment curve, taking the polar distance $p = \frac{k^2}{h}$.

Then $\mathrm{NR} \times p =$ area of first moment curve above PP

$$\therefore \frac{\mathrm{NR} \times k^2}{h} = \text{area } \mathrm{JXQN}$$

$$\therefore \text{Mean shear along PP} = \frac{m}{b\,k^2} \cdot \frac{2\,\mathrm{NR} \,.\, k^2}{h} \times h$$

$$= m \,.\, \frac{2\,\mathrm{NR}}{b}$$

$$\text{But } b = \mathrm{PP} = 2\,\mathrm{NP}$$

$$\therefore \text{Mean shear along PP} = m \,.\, \frac{\mathrm{NR}}{\mathrm{NP}}$$

Then the maximum shear stress, which occurs at the neutral axis, is $m \,.\, \frac{\mathrm{CS}}{\mathrm{CB}}$

NOTE.—Fig. 229 is diagrammatic only and is not drawn to scale. The student should work this case as an example, taking the plates 20″ × ½″ and 16″ × 6″ beams. For accuracy the drawing should be done to a large scale.

Deflection of a Beam due to Shear.—In considering the deflections of beams up to the present we have dealt only with the deflection due to the bending moment. We will now see to what extent the deflection due to shear is comparable with that due to the bending moment.

Let C C (Fig. 230) represent a short length x of the centre line of a beam subjected to a shearing stress s.

Then the shear causes the line C C to take the position C C_1, the slope being σ.

Then if G is the shear modulus, we have $\sigma = \frac{s}{G}$

The deflection C C_1 of the short length of beam is equal to $x \times \sigma$, as σ is small.

$\therefore$ Deflection of short length x of beam $= \dfrac{x \times s}{G}$

$\therefore$ Total deflection due to shear $= \Sigma \dfrac{x \times s}{G}$

Now we have shown that $s = m \cdot \dfrac{a \cdot y}{b k^2}$ where $m = \dfrac{S}{A}$, S being the shearing force at the point, and A the area of the section.

If the section is uniform along its length, $\dfrac{a y}{b k^2}$ will be constant and equal to, say, β.

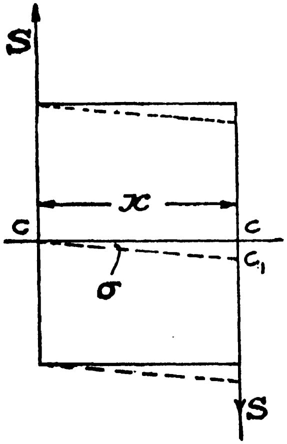


FIG. 230.

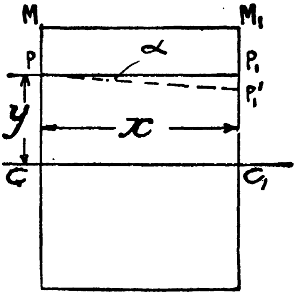


FIG. 231.

$\therefore$ We have: deflection due to shear $= \Sigma x \cdot \dfrac{\beta \cdot S}{A \cdot G}$

$$= \frac{\beta}{AG} \Sigma x \cdot S$$

But $\Sigma \cdot x S$ = area of shear curve up to given point
= B.M. at point
= M

$\therefore$ Deflection due to shear $= \mu = \dfrac{\beta}{AG} \cdot M$ .... (1)

Now consider the following special cases—

(1) Isolated Central Load.

Deflection at centre $= \mu = \dfrac{\beta}{AG} \cdot \dfrac{W l}{4}$

As we have previously shown, the deflection δ in this case due to B.M. is equal to $\frac{W\,l^3}{48\,E\,I}$

$$\therefore \frac{\mu}{\delta} = \frac{\beta}{A\,G} \cdot \frac{W\,l}{4} \div \frac{W\,l^3}{48\,E\,I}$$

$$= \frac{12\,E}{G} \cdot \frac{\beta\,I}{A\,l^2}$$

Taking $\frac{E}{G} = \frac{5}{2}$ and noting that $I = A\,k^2$

$$\frac{\mu}{\delta} = 30\,\frac{\beta\,k^2}{l^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

(2) **Continuous Loading.**

$$\text{In this case } \mu = \frac{\beta}{A\,G} \cdot \frac{W\,l}{8}$$

$$= \frac{5\,W\,l^3}{384\,E\,I}$$

$$\therefore \frac{\mu}{\delta} = \frac{48\,\beta}{5} \cdot \frac{E}{G} \cdot \frac{I}{A\,l^2}$$

Taking $\frac{E}{G} = \frac{5}{2}$ as before,

$$\frac{\mu}{\delta} = 24\,\beta \, . \, \frac{k^2}{l^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

For rectangular section $\beta = 1{\cdot}5$ and $k^2 = \frac{h^2}{12}$, h being the depth of the beam.

$$\therefore \text{ (2) becomes } \frac{\mu}{\delta} = 3{\cdot}75\left(\frac{h}{l}\right)^2$$

$$\text{(3) becomes } \frac{\mu}{\delta} = 3\left(\frac{h}{l}\right)^2$$

It follows from this that if $\frac{h}{l} = \frac{1}{10}$, the deflection due to shear is 3·75 per cent. and 3 per cent. respectively of that due to B.M. in the two cases.

We see, therefore, that for solid rectangular beams in which the span is more than 10 times the depth, the deflection due to shear is negligible.

It must, however, be remembered that for rolled joists, plate girders, and the like, the deflection due to shear will be quite appreciable for sections which are deep compared with their span. Bridge engineers often state that the deflection of a bridge is more than the calculated deflection. Part of this difference may be due to the giving in the riveted connections, but certainly the measured deflection would agree better with the calculated deflection if the latter included the shear deflection. It has been suggested that this could be remedied by taking E about 10,000 tons per sq. in. instead of 12,500 in the ordinary deflection formula.

It should also be noted that we have taken only the strain due to the maximum shear stress, neglecting the fact that it is variable. This gives results a little too high, but is better than taking the mean shear stress.

Distortion of Cross Section of Beam due to Shear, etc.—In finding an expression for the relation between the stresses and the B.M. on a beam, we made use of Bernoulli's assumption that the cross section remains plane after bending.

The two causes tending to distort the cross section are (1) shear stress, (2) differences in lateral compression due to extension in fibres.

Consider two cross sections of a beam at distance x (Fig. 231) apart, and let the B.M. at the sections be M and M_1 respectively, and consider points P and P_1 at distance y from the centre line, the section being the same at the two points.

$$\text{Then stress at P} = \frac{M\,y}{I}, \text{ at } P_1 = \frac{M_1\,y}{I}$$

$$\therefore \text{Lateral compression strain at P} = \eta\,\frac{M\,y}{E\,I}, \text{ at } P_1 = \eta\,\frac{M_1\,y}{E\,I}$$

because longitudinal strain $= \dfrac{\text{stress}}{E}$ and lateral or transverse strain $= \eta \times$ longitudinal strain.

$\therefore$ Difference in lateral compression strain $= \dfrac{\eta}{E\,I} \cdot (M_1 - M)\,y$

$\therefore$ On a short length $d\,y$ of the section, the difference in lateral compression $= P'\,P_1{}' = \dfrac{\eta}{E\,I} \cdot (M_1 - M) \cdot y \cdot d\,y$

$$\therefore a = \text{slope of } \mathrm{P\,P_1}' = \frac{\mathrm{P}'\,\mathrm{P_1}'}{x} = \frac{\eta}{\mathrm{E\,I}} \cdot \frac{\mathrm{M_1} - \mathrm{M}}{x} \cdot y \cdot d\,y$$

but we have shown that when x is very small

$$\frac{\mathrm{M_1} - \mathrm{M}}{x} = \text{the shearing force S}$$

$$\therefore a = \frac{\eta}{\mathrm{E\,I}} \cdot \mathrm{S} \cdot y \cdot d\,y$$

To find the total change in angle between any section and the line originally parallel to the centre line, we must add all the elementary changes in angle.

$$\therefore \text{Total change} = \theta = \frac{\mathrm{S} \cdot \eta}{\mathrm{E\,I}} \int y \cdot d\,y$$

$$= \frac{\mathrm{S} \cdot \eta \cdot y^2}{2\,\mathrm{E\,I}} = \frac{m \cdot \eta\, y^2}{2\,\mathrm{E}\,k^2}$$

$$\text{because } m = \frac{\mathrm{S}}{\mathrm{A}}$$

Now we have previously shown that due to the shear there is a change of angle equal to $\frac{m \cdot a \cdot y}{\mathrm{G} \cdot b\,k^2}$

$\therefore$ Total change due to both causes

$$= \frac{m}{k^2}\left(\frac{a\,y}{b\,\mathrm{G}} + \frac{\eta\,y^2}{2\,\mathrm{E}}\right)$$

$$= \frac{m}{\mathrm{G}\,k^2}\left(\frac{a\,y}{b} + \frac{\eta\,y^2\,\mathrm{G}}{2\,\mathrm{E}}\right)$$

putting $\mathrm{E} = \frac{5\,\mathrm{G}}{2}$ and $\eta = \frac{1}{4}$ this comes to $\frac{m}{\mathrm{G}\,k^2}\left(\frac{a\,y}{b} + \frac{y^2}{20}\right)$

From this relation the slope at any portion of the section can be found, and the distorted form of the cross section can be obtained. Our present scope prevents us from dealing with this interesting problem further, but what we have given should serve as an indication of the method in which the problem may be attacked.

CHAPTER XVII

* FLAT PLATES AND SLABS

In the beams that have been considered up to the present there is a support along two edges only, that is the support is at most along two parallel lines; when a plate or slab is supported upon more than two straight edges we have to consider the strengthening effect of the side supports, and for this purpose slab formulæ are required.

CIRCULAR PLATES AND SLABS

Slab Coefficients.—In many problems it is convenient to use slab coefficients to compare the bending moments in a slab supported on its edges with the corresponding cases in which the slab is supported on two edges only as in the ordinary beam. We then have

$$\beta = \text{slab coefficient} = \frac{\text{B.M. on slab}}{\text{B.M. on corresponding beam resting upon two edges}}$$

Bach's Theory.—Bach obtains formulæ in a very simple manner by assuming the supporting pressure uniformly distributed along the edge of the plate or slab and calculating the bending moment over various sections. In the use of these results it should be remembered that they give the mean bending stress across what corresponds to the breadth of the beam, but that they do not give the absolute maximum. A similar point occurs in considering shear stresses in beams (see p. 473).

We will take the following standard cases.

A. Circular Slabs supported on the Outside Edge.—

(1) UNIFORMLY DISTRIBUTED LOAD W.—The reaction pressure per unit length will be $p = \frac{W}{\pi D} = \frac{W}{2\pi R}$. Then considering the forces in one half of the slab we have a load $\frac{W}{2}$ acting downwards at G_L (Fig. 232), the "load-centre" or centroid of the semi-circular area and a resultant reaction $= \frac{W}{2}$ acting at the "reaction-centre" G_R or centroid of the semi-circular arc.

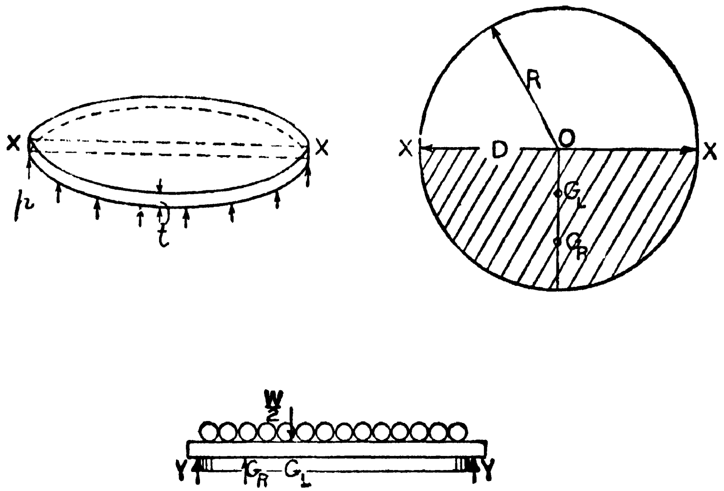


FIG. 232.—Circular Slab supported at the Edge with Uniform Load.

∴ Taking moments about x x we have

$$\text{Bending moment on x x} = M_x = \frac{W}{2} . o\,G_R - \frac{W}{2}\, o\,G_L$$

$$= \frac{W_2}{2}(o\,G_R - o\,G_L)$$

$$o\,G_L = \frac{4\,R}{3\,\pi} \text{ and } o\,G_R = \frac{2\,R}{\pi}$$

$$\therefore o\,G_R - o\,G_L = \frac{R}{\pi}\left(2 - \frac{4}{3}\right) = \frac{2\,R}{3\,\pi}$$

$\therefore M_x = \frac{W\,R}{3\,\pi}$ or, if w is the load intensity of load—

$$= \frac{w\,\pi\,R^2 . R}{3\,\pi} = \frac{w\,R^3}{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The mean stress on x x $= f = \dfrac{M_x}{Z} = \dfrac{M_x}{\dfrac{2\,R\,t^2}{6}} = \dfrac{3\,M_x}{R\,t^2}$

$$\therefore f = \frac{w\,R^2}{t^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

If the slab were freely supported at Y and Y we should have the reaction acting at Y and the load at G_L.

$$\therefore M_x = \frac{W}{2}\left(R - \frac{4\,R}{3\,\pi}\right)$$

$$= \frac{W\,R}{2}(1 - {\cdot}4244)$$

$$= {\cdot}288\ W\,R$$

$$\therefore \text{Slab coefficient} = \beta = \frac{W\,R}{3\,\pi} \div {\cdot}288\ W\,R$$

$$= {\cdot}368$$

(2) Central Load on Radius r.—In this case as before we have

$$\text{B.M. on x x (Fig. 233)} = M_x = \frac{W}{2}\left(o\,G_R - o\,G_L\right)$$

$$= \frac{W}{2}\left(\frac{2\,R}{\pi} - \frac{4\,r}{3\,\pi}\right)$$

$$= \frac{W\,R}{\pi}\left(1 - \frac{2\,r}{3\,R}\right) \quad \ldots\ldots\ldots\ldots (3)$$

$$\therefore f = \frac{3\,M_x}{R\,t^2} = \frac{3\,W}{\pi\,t^2}\left(1 - \frac{2\,r}{3\,R}\right) \ldots\ldots (4)$$

In the limiting case where $r = 0$ we have the point load for which

$$f = \frac{3\,W}{\pi\,t^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

In this case if the supports were at Y Y we should have

$$M_x = \frac{W}{2}\left(R - \frac{4\,r}{3\,\pi}\right)$$

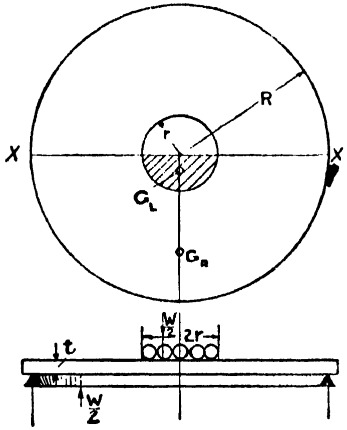
$$= \frac{W\,R}{2}\left(1 - \frac{4\,r}{3\,\pi\,R}\right)$$

$$\beta = \frac{2\left(1 - \dfrac{2\,r}{3\,R}\right)}{\pi\left(1 - \dfrac{4\,r}{3\,\pi\,R}\right)} \quad \ldots\ldots\ldots\ldots (6)$$

$= {\cdot}636$ for the point load when $r = 0$

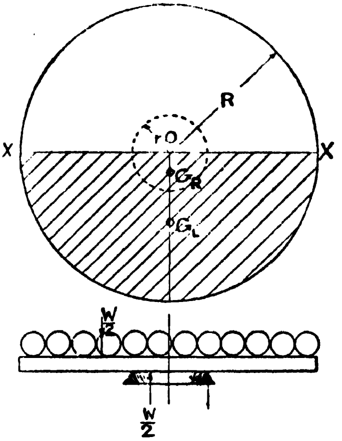
B. Circular Slab supported on a Circular Pillar.

(1) Uniformly Distributed Load W.—In this case the tension and compression edges will be on opposite sides of those in the previous case, the present one being the equivalent of the uniformly loaded cantilever.



Supported at edge; central load.

Fig. 233.



Supported at centre; uniform load.

Fig. 234.

Circular Slabs.

By moments as before about x x (Fig. 234) we have

$$M_x = \frac{W}{2}\, O\, G_L - \frac{W}{2}\, O\, G_R$$

$$= \frac{W}{2}\left(\frac{4\,R}{3\,\pi} - \frac{2\,r}{\pi}\right)$$

$$= \frac{W\,R}{\pi}\left(\frac{2}{3} - \frac{r}{R}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

If the load is w per unit area, $W = w\,\pi\,R^2$

$$\therefore\ M_x = w\,R^3\left(\frac{2}{3} - \frac{r}{R}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

In the limiting case of $r = 0$, which corresponds to a point support, this gives

$$M = \frac{2\,w\,R^3}{3}$$

Taking the corresponding beam we should have G_R acting on the edge of the supporting circle

$$\therefore M_x = \frac{W}{2}\left(\frac{4R}{3\pi} - r\right)$$

$$= \frac{WR}{\pi}\left(\frac{2}{3} - \frac{\pi r}{2R}\right)$$

$$\therefore \beta = \frac{\frac{2}{3} - \frac{r}{R}}{\frac{2}{3} - \frac{\pi r}{2R}}$$

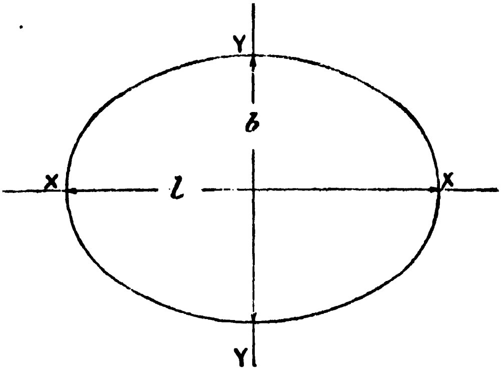


FIG. 235.

There is no slab coefficient corresponding to $r = 0$, or rather it would be more correct to say that it will be 1 in this case

$$\therefore f = \frac{6M}{2R \,.\, t^2} = \frac{2wR^2}{t^2} \quad \ldots\ldots\ldots\ldots\ldots(9)$$

(2) LOAD DISTRIBUTED UNIFORMLY ALONG THE EDGE.—This case is the same as case A (2) with the loads and reactions reversed.

C. Oval Plate supported on Edge (Fig. 235).—LOAD UNIFORM.—In this case we can obtain an approximate solution by assuming that the points G_R and G_L will be the same for M_x as for a circle of radius $\frac{b}{2}$ and for M_y, as for a circle of radius $\frac{l}{2}$

This gives $M_x = \dfrac{W\,b}{6\,\pi}$.

$$\therefore f_x = \frac{6\,M_x}{l\,t^2} = \frac{W\,b}{\pi\,l\,t^2}$$

$$M_y = \frac{W\,l}{6\,\pi}$$

Similarly $f_y = \dfrac{6\,M_y}{b\,t^2} = \dfrac{W\,l}{\pi\,b\,t^2}$

If we put $W = \dfrac{w\,\pi\,l\,b}{4}$ we have

$$M_x = \frac{w\,l\,b^2}{24}$$

$$M_y = \frac{w\,b\,l^2}{24}$$

Corresponding to these we have

$$f_x = \frac{w\,b^2}{4\,t^2}$$

$$f_y = \frac{w\,l^2}{4\,t^2}$$

It will be noted that the stress is greatest across the short axis. This should not be used for ovals with $l > 2\,b$ which should be treated as ordinary beams of span b, giving $M_y = \dfrac{W\,b}{8}$

Another way of dealing with this problem is as follows: If l is so great that the effect of the edges is negligible, we have for the short span of a unit width B.M. $= \dfrac{w\,b^2}{8}$ and $Z = \dfrac{t^2}{6}$

$$\therefore f_x = \frac{w\,b^2}{8} \div \frac{t^2}{6} = \frac{\cdot 75\,w\,b^2}{t^2}$$

For the circle, we have by Grashof's theory (table on p. 495)

$$f_x = \frac{39\,w\,R^2}{32\,t^2} = \frac{39\,w\,b^2}{128\,t^2} = \frac{\cdot 3\,w\,b^2}{t^2} \text{ approx.}$$

$\therefore$ We may make up an empirical formula

$$f_x = \cdot 375\,\frac{w\,b^2}{t^2}\left(2 - \frac{6\,b}{5\,l}\right)$$

which is correct for the extreme values $b = l$ and $\dfrac{b}{l} = 0$

Grashof's Theory.—Grashof investigated the strains in flat plates by an investigation of the deflected form and adopted the stress equivalent to the maximum strain as the criterion of the resultant stress (see p. 44), the result being different from that obtained by means of the calculation of the principal stresses.

The derivation of the formulæ is too complicated for our present scope, so that we will give the results only and refer the reader to Professor Morley's *Strength of Materials* (Longmans) for the mathematical deduction of the formulæ.

SQUARE AND RECTANGULAR SLABS

Square and rectangular slabs are of importance in several cases in practical design, particularly in reinforced concrete construction and in tanks and valve-chest covers.

Rigorous methematical methods cannot be applied to these cases, so that we have to fall back on approximate methods of which the two following are the most common. These hold for uniformly distributed loads only.

Grashof-Rankine Theory.—These formulæ for slabs supported on their edges are obtained by the following reasoning and should not be confused with the rigorous Grashof treatment for circular slabs.

Consider two narrow central strips (Fig. 236) of width x parallel to the axes X X and Y Y, thus forming one strip passing over the other so that the two strips must have the same deflection. The whole slab is considered as divided up into strips at right angles to each other, the strips passing one over the other. The load w per unit area may then be considered as divided up into two portions w_l, w_b carried respectively on the long and short spans.

For the long span we have $\delta = \frac{5\,w_l\,l^4}{384\,\mathrm{E\,I}}$ ....(1)

„ short span we have $\delta = \frac{5\,w_b\,b^4}{384\,\mathrm{E\,I}}$ ....(2)

GRASHOF'S RESULTS FOR CIRCULAR PLATES OR SLABS

Support and Loading.	Principal Stress.	Grashof's Value. Stress equivalent to Maximum Strain.	Point of Maximum Stress.	Maximum Deflection.
Edge freely supported. Load uniform	$\frac{39}{32}\frac{w\,R^2}{t^2}$	$\frac{117}{128}\frac{w\,R^2}{t^2}$	Centre	$\frac{189}{256}\frac{w\,R^4}{E\,t^3}$
Edge securely clamped. Load uniform	$\frac{3}{4}\frac{w\,R^2}{t^2}$	$\frac{45}{64}\frac{w\,R^2}{t^2}$	Edge	$\frac{45}{256}\frac{w\,R^4}{E\,t^3}$
Edge freely supported. Central load	$\frac{15\,W}{8\,\pi\,t^2}\left(\frac{4}{5}+\log_e\frac{R}{r}-\frac{3\,r^2}{20\,R^2}\right)$	$\frac{45\,W}{32\,\pi\,t^2}\left(\frac{4}{5}+\log_e\frac{R}{r}-\frac{3\,r^2}{20\,R^2}\right)$	Centre	$\frac{117\,W\,r^2}{64\,\pi\,E\,t^3}$
Edge securely clamped. Central load	$\frac{15\,W}{8\,\pi\,t^2}\left(\log_e\frac{R}{r}+\frac{r^2}{4\,R^2}\right)$ R must be > 1·7 r	$\frac{45\,W}{32\,\pi\,t^2}\left(\log_e\frac{R}{r}+\frac{r^2}{4\,R^2}\right)$ R must be > 2·4 r	Centre	$\frac{45\,W\,r^2}{64\,\pi\,E\,t^3}$
Supported on central column. Load uniform	$\frac{3\,w\,R}{8\,t^2}\left\{5\log_e\frac{R}{r}+\frac{3}{4}\left(1-\frac{r^2}{R^2}\right)\right\}$	$\frac{45\,w\,R^2}{32\,t^2}\left\{\log_e\frac{R}{r}+\frac{3}{20}\left(1-\frac{r^2}{R^2}\right)\right\}$	Centre	$\frac{279}{256}\frac{w\,r^4}{E\,t^3}$

Notation.—R = radius of slab; r = radius of column or central load; t = thickness of slab; W = central load; w = uniform load per unit area. Poisson's ratio has been taken ¼.

These must be equal $\therefore \frac{w_b}{w_l} = \frac{l^4}{b^4}$

$$\text{also } w_b + w_l = w$$

$$\text{i.e. } w_l\left(1 + \frac{l^4}{b^4}\right) = w$$

$$w_l = \frac{w}{\left(1 + \frac{l^4}{b^4}\right)}$$

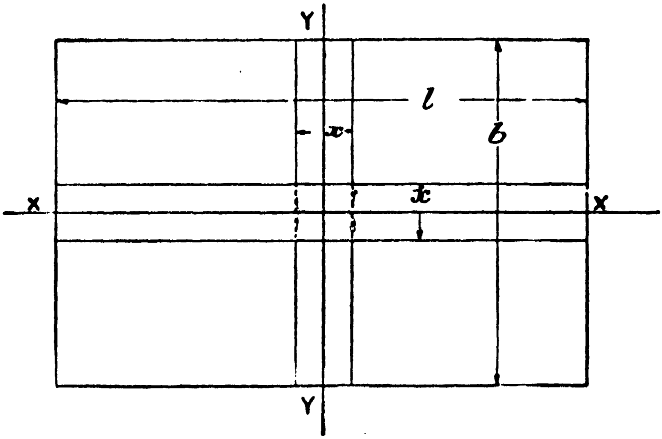


FIG. 236.—Grashof-Rankine Theory of Rectangular Slabs.

$\therefore$ M_Y = B.M. on short section, *i.e.* B.M. on long span.

$$= \frac{w_l\, l^2}{8} = \frac{w\, l^2}{8\left(1 + \frac{l^4}{b^4}\right)} \quad \ldots\ldots\ldots\ldots\ldots(3)$$

$$\text{Similarly } w_b = \frac{w}{\left(1 + \frac{b^4}{l^4}\right)}$$

$\therefore$ M_X = B.M. on long section, *i.e.* B.M. on short span.

$$= \frac{w_b\, b^2}{8} = \frac{w\, b^2}{8\left(1 + \frac{b^4}{l^4}\right)} \quad \ldots\ldots\ldots\ldots\ldots(4)$$

$\therefore$ For long span, *i.e.* short axis—

$$\beta_l = \frac{\text{B.M. on slab}}{\text{B.M. on corresponding beam}} = \frac{b^4}{l^4 + b^4} \quad \ldots.(5)$$

For short span, *i.e.* long section

$$\beta_b = \frac{l^4}{l^4 + b^4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Some confusion is likely to arise if we do not clearly keep in mind the fact that if we are considering the stresses across the long section we take the B.M. on the short span.

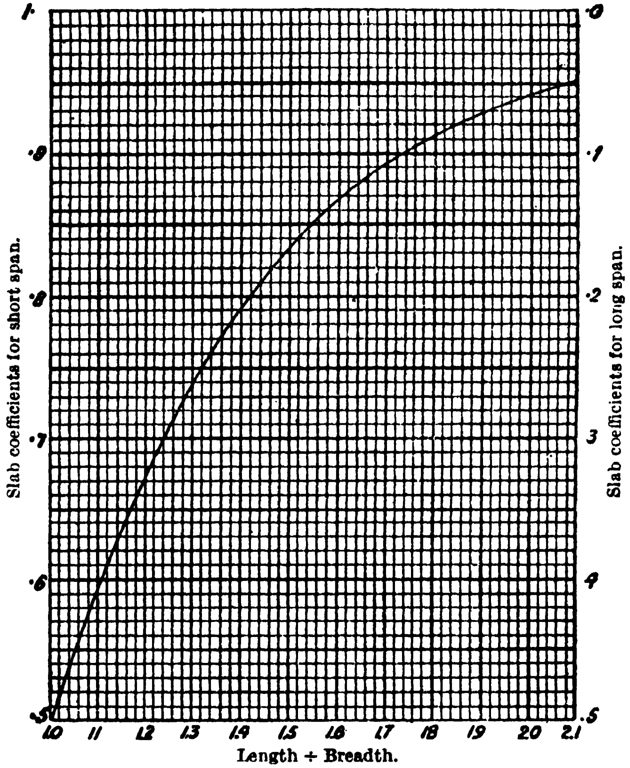


FIG. 237.

As a check we may remember *that the greater B.M. always comes on the short span.*

$\frac{l}{b}$	Slab Coefficients	
	Short Section and Long Span β_l	Long Section and Short Span β_s
1	·500	·500
1·25	·291	·709
1·5	·164	·836
1·75	·096	·904
2	·059	·941

Intermediate values may be obtained from Fig. 237.

In comparing this theory with Bach's it should be remembered that the present gives the maximum stress, whereas Bach's gives the mean stress across any section.

NUMERICAL EXAMPLE.—*A rectangular slab* 18 *ft. long and* 12 *ft. wide carries a uniformly distributed load of* 200 *lbs. per sq. ft. What B.M. should it be designed for across the long and short section respectively?*

In this case $W = 200 \times 18 \times 12$

$\therefore$ B.M. on long span, neglecting slab action

$$= \frac{Wl}{8} = \frac{(200 \times 18 \times 12) \times (18 \times 12)}{8}$$

$$= 1{,}166{,}400 \text{ in. lbs.}$$

B.M. on short span, neglecting slab action

$$= \frac{Wb}{8} = \frac{(200 \times 18 \times 12) \times (12 \times 12)}{8}$$

$$= 777{,}600 \text{ in. lbs.}$$

From our table for $\frac{l}{b} = 1{\cdot}5$, $\beta_l = {\cdot}164$, and $\beta_b = {\cdot}836$

$$\therefore \text{ B.M. on long span} = {\cdot}164 \times 1{,}166{,}400$$

$$= \underline{191{,}000 \text{ in. lbs. nearly}}$$

$$\text{B.M. on short span} = {\cdot}836 \times 777{,}600$$

$$= \underline{650{,}000 \text{ in. lbs. nearly}}$$

$\therefore$ On short span if $f = 16{,}000$ for a metal slab we have for short span

$$16{,}000 \,.\, \frac{18\,t^2}{6} \times 12 = 650{,}000$$

$$t^2 = \frac{650{,}000}{576{,}000}$$

$$= 1{\cdot}13$$

$$t = \sqrt{1{\cdot}13} = \underline{1\tfrac{1}{8}'' \text{ nearly.}}$$

The other span would require a smaller thickness.

In metal slabs the modulus of section is probably the same in both directions, but in reinforced concrete slabs the modulus generally varies.

Bach's Theory.—The pressure is taken as uniform along the edges.

(1) DIAGONAL SECTION.—Assuming that the diagonal sections are the weakest because tests indicate that failure often occurs diagonally, consider the bending moment about the line A C (Fig. 238).

Let p be the pressure per unit length along the supports and let W be the total uniformly distributed load and w its intensity per unit area.

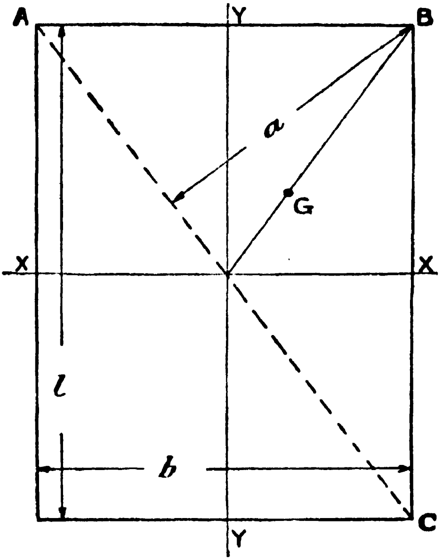


FIG. 238.—Bach's Theory of Rectangular Slabs.

Then
$$p = \frac{W}{2(l + b)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The supporting forces or reactions may be taken as a force equal to $p\,b$ acting at Y and one equal to $p\,l$ acting at X; the load on the Δ A B C $= \frac{W}{2}$ and acts at the centroid G of the Δ.

The perpendicular distance of X and Y from A C are each $\frac{a}{2}$ and the perpendicular distance of G from A C is $\frac{a}{3}$

∴ Taking moments about A C we have

$$\text{Bending moment} = M_{AC} = \frac{pb \times a}{2} + \frac{pl \times a}{2} - \frac{W}{2} \cdot \frac{a}{3}$$

$$= \frac{a}{2}\left\{p(l + b) - \frac{W}{3}\right\}$$

$$= \frac{a}{2}\left\{\frac{W}{2} - \frac{W}{3}\right\}$$

$$= \frac{Wa}{12} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

But $a \times AC = 2$ area of $\Delta ABC = lb$.

$$\therefore a = \frac{lb}{AC},$$

$$\therefore M_{AC} = \frac{W . lb}{12 AC}$$

$$= \frac{W l b}{12\sqrt{l^2 + b^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Neglecting the support on the short side we should have

$$M_Y = \frac{Wb}{8} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

To get a reasonable comparison between M_{AC} and M_Y we ought to compare the B.M.s on the same length because, of course, A C is greater than Y Y.

∴ B.M. per unit length along A C for slab $= \frac{M_{AC}}{AC} = \frac{M_{AC}}{\sqrt{l^2 + b^2}}$

$$= \frac{W l b}{12(l^2 + b^2)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

B.M. per unit length along Y Y for beam $= \frac{Wb}{8l} = \frac{wb^2}{8}$

Diagonal slab coefficient

$$= \beta_{AC} = \frac{W l b}{12(l^2 + b^2)} \div \frac{Wb}{8l} = \frac{2l^2}{3(l^2 + b^2)}$$

$$= \frac{2}{3\left(1 + \left(\frac{b}{l}\right)^2\right)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

β has the following values—

$\frac{l}{b}$	Diagonal Slab Coefficient.
1	·333
1·25	·407
1·5	·461
1·75	·502
2	·533

To use these figures we find $\frac{W\,b^2}{8}$ and treat that when multiplied by β as the mean B.M. per unit length along the diagonal.

NUMERICAL EXAMPLE.—*Take the same case as worked out on p.* 498, *and adopting a stress of* 16,000 *lbs. per sq. in. find the necessary thickness of a rectangular metal slab, comparing the results by the two formulæ.*

On Bach's Theory—

$$\frac{w\,b^2}{8} = \frac{200 \times 144}{8} = 3{,}600 \text{ ft. lbs. per foot width}$$

$$= 3{,}600 \text{ in. lbs. per inch width.}$$

$$\therefore \text{ B.M. per inch length on diagonal} = \cdot 461 \times 3{,}600$$

$$= 1{,}660 \text{ in. lbs. nearly.}$$

$$M = f \,.\, \frac{b\,t^2}{6} = \frac{16{,}000 \times 1 \times t^2}{6}$$

$$\therefore\ t^2 = \frac{6 \times 1{,}660}{16{,}000} = \cdot 62$$

$$t = \sqrt{\cdot 62} = \underline{\cdot 79 \text{ in. nearly, say } \tfrac{7}{8}''.}$$

(2) SECTIONS PARALLEL TO SIDES.—The following modification of Bach's treatment is more suitable for reinforced concrete slabs where the reinforcement is parallel to the edges.

We can consider in a similar manner the strength of the section X X (Fig. 239).

The reactions on the sides will have reactions at the mid-points equal to $\frac{p\,l}{2}$ at D and E and $p\,b$ at Y. The load acts at the point F.

Therefore, taking moments about the line X X we have

$$M_x = 2 \cdot \frac{p\,l}{2} \cdot \frac{l}{4} + p\,b \cdot \frac{l}{2} - \frac{W}{2} \cdot \frac{l}{4}$$

$$= \frac{p\,l}{4}(2\,b + l) - \frac{W\,l}{8}$$

$$= \frac{W\,l\,(2\,b + l)}{8\,(l + b)} - \frac{W\,l}{8}$$

$$= \frac{W\,b\,l}{8\,(l + b)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

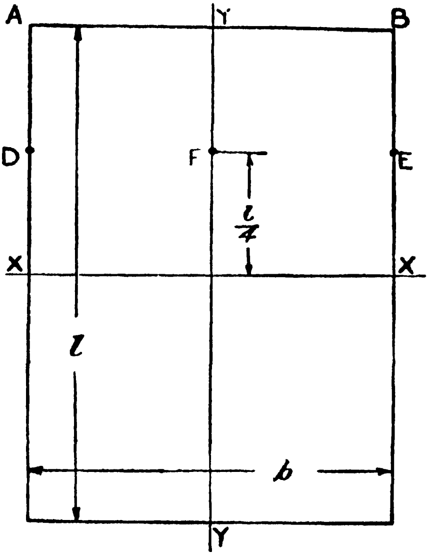


FIG. 239.—Rectangular Slab: Modified Bach Treatment.

Neglecting side support on long side, we should have

$$M_x = \frac{W\,l}{8}$$

$$\therefore \text{ Slab coefficient for X X} = \beta_l = \frac{b}{l + b} = \frac{1}{1 + \frac{l}{b}} \quad \ldots\ldots (8)$$

Similarly, if we consider the strength of the section Y Y we should get

$$\text{Slab coefficient for Y Y} = \beta_b = \frac{l}{l + b} = \frac{1}{1 + \frac{b}{l}} \quad \ldots\ldots (9)$$

These results can be tabulated as follows—

$\frac{l}{b}$	Slab Coefficients.	
	Short Section and Long Span β_l	Long Section and Short Span β_b
1	·500	·500
1·25	·444	·556
1·5	·400	·600
1·75	·364	·636
2	·333	·667

It follows from this that the B.M. comes the same on the two spans, so that the long span is the weaker as the breadth is less. Experiments do not bear this out.

NUMERICAL EXAMPLE.—Taking the previous case we shall have $\beta_l = {\cdot}400$, $\beta_b = {\cdot}600$

$$\therefore \text{ B.M. on long span} = {\cdot}4 \times 1,166,400$$
$$= 466,600 \text{ in. lbs.}$$
$$\text{B.M. on short span} = {\cdot}6 \times 777,600$$
$$= 466,600 \text{ in. lbs.}$$
$$\therefore \text{ on long span } 16,000 \times \frac{144\,t^2}{6} = 466,600$$
$$t^2 = \frac{466,600}{384,000} = 1{\cdot}22$$
$$t = \sqrt{1{\cdot}22} = 1\tfrac{1}{8} \text{ ins. nearly.}$$

This agrees fairly well with the Rankine value, although it is deduced from a different span. It is interesting to note that although Bach worked upon the diagonal as being the weakest section the above treatment requires a greater thickness.

Variation of Bending Moment.—To obtain an idea of the variation of the Bending moment in this case consider a portion A B U U (Fig. 240) of the slab. Then the load on the shaded area $= \frac{W x}{l}$

As before, we have $p = \frac{W}{2(l + b)}$

∴ We have at X a force $p\,b$ and at D and E a force $\frac{p\,x}{2}$, and at F a downward force $\frac{W\,x}{l}$

∴ Taking moments about U U we have

$$M_U = p\,b\,x + 2\,\frac{p\,x}{2}\cdot\frac{x}{2} - \frac{W\,x^2}{2\,l}$$

$$= W\,x\left\{\frac{b}{2\,(l+b)} + \frac{x}{2\,(l+b)} - \frac{x}{2\,l}\right\}$$

$$= \frac{W\,x}{2\,l\,(l+b)}\{b\,l + x\,l - x\,(l+b)\}$$

$$= \frac{W\,x}{2\,l\,(l+b)}\{b\,l - b\,x\} = \frac{W\,b\,x\,(l-x)}{2\,l\,(l+b)}\,\ldots\ldots\ldots(10)$$

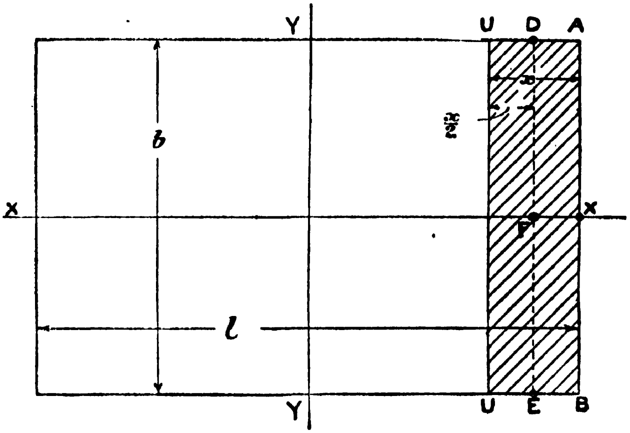


FIG. 240.

putting $x = \frac{l}{2}$ this gives

$$M = \frac{W\,b\,l^2}{8\,l\,(l+b)} = \frac{W\,b\,l}{8\,(l+b)}$$ as before, thus checking

$$M_U = \frac{W\,b}{2\,l\,(l+b)}\{l\,x - x^2\}\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

A diagram showing the variation of M_U with y would come a parabola with vertex at the centre, similar to the ordinary B.M. diagram for a simply supported beam. For a corresponding consideration for the other span we should get $M_U = \frac{W\,l}{2\,b\,(l+b)}\{b\,x - x^2\}$ which would also be a parabola.

We may therefore assume that the stress will vary across a section approximately in the form of a parabola, so that the maximum stress will be 1·5 *times the mean stress; this would require the thickness at the centre to be* $\sqrt{1{\cdot}5} = 1{\cdot}22$, *say* 1·25 *times the thickness given by the ordinary treatment of Bach's Theory.*

In our example this would make t for diagonal consideration $= {\cdot}96$, which agrees quite well with the Grashof-Rankine theory.

Variations of Bach's Theory.—There is reason to believe that the pressure in rectangular slabs is greater at the centres of the supporting edges than at the corners; we will therefore consider two variations in pressure.

VARIATION I.—*Pressure varies according to a Parabola.*—We will now therefore assume the pressure to vary in the form of a parabola as shown in Fig. 241. We will take, as before, the total pressure on each side proportional to its length, so that the total pressure on each long side

$$= \mathrm{P}_l = \frac{\mathrm{W}\,l}{2\,(l+b)}$$

and that on each short side

$$= \mathrm{P}_s = \frac{\mathrm{W}\,b}{2\,(l+b)}$$

The pressure at the centre of each side is therefore 1·5 p, p being the value given in equation (1). The resultant pressure along A B will act at the point Y, while that on the half sides A X, B X will be at the centroids of the parabolas, *i. e.* $\frac{3\,l}{16}$ from X.

Therefore, taking moments about X X we have

$$\begin{aligned}\mathrm{M}_x &= \mathrm{P}_s \cdot \frac{l}{2} + \frac{2\,\mathrm{P}_l}{2} \cdot \frac{3\,l}{16} - \frac{\mathrm{W}}{2} \cdot \frac{l}{4}\\ &= \frac{\mathrm{W}\,b\,l}{4\,(l+b)} + \frac{3\,\mathrm{W}\,l^2}{32\,(l+b)} - \frac{\mathrm{W}\,l}{8}\\ &= \frac{\mathrm{W}\,l}{8\,(l+b)}\left\{2\,b + \frac{3\,l}{4} - (l+b)\right\}\\ &= \frac{\mathrm{W}\,l}{8\,(l+b)}\left\{b - \frac{l}{4}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(12)\end{aligned}$$

Neglecting side support, we have as before

$$M_x = \frac{W\,l}{8}$$

$$\therefore \text{ Slab coefficient for } \mathrm{X\,X} = \beta_l = \frac{b - \frac{l}{4}}{(l + b)}$$

$$= \frac{1 - \frac{l}{4\,b}}{1 + \frac{l}{b}} \quad \ldots\ldots\ldots\ldots(13)$$

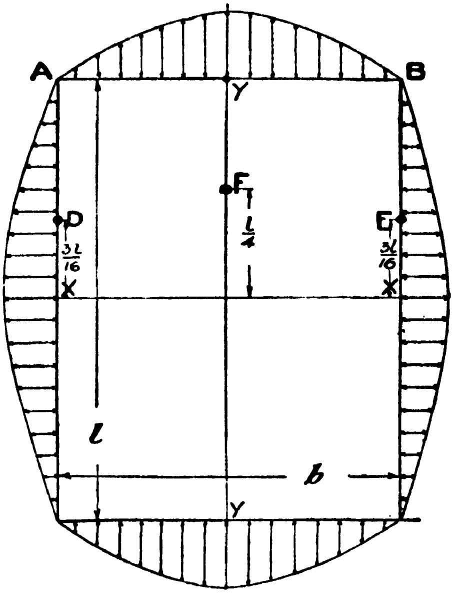


Fig. 241.—Rectangular Slab with Parabolic Distribution of Edge Pressure.

Similarly, we get for Y Y by reversing l and b

$$\text{Slab coefficient for } \mathrm{Y\,Y} = \beta_b = \frac{l - \frac{b}{4}}{l + b}$$

$$= \frac{\frac{l}{b} - \frac{1}{4}}{1 + \frac{l}{b}} \quad \ldots\ldots\ldots\ldots(14)$$

These results can be tabulated as follows—

$\frac{l}{b}$	Slab Coefficients.	
	Short Section and Long Span β_l	Long Section and Short Span β_b
1	·375	·375
1·25	·306	·444
1·5	·250	·500
1·75	·205	·545
2	·167	·583

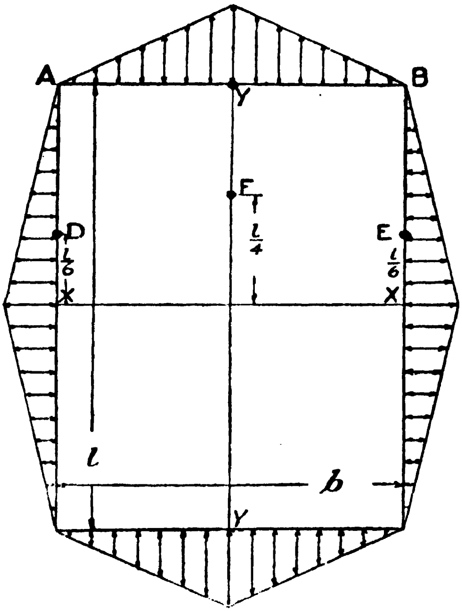


FIG. 242.—Rectangular Slab with Triangular Distribution of Edge Pressure.

VARIATION II.—*Pressure varies according to a Triangle.*—In this case we will assume the pressures to be even more concentrated at the centres than in the previous case, and assume the pressure distribution shown in Fig. 242.

As before, we take total pressure on each long side $= P_L = \frac{Wl}{2(l+b)}$ and that on each short side $= P_s = \frac{Wb}{2(l+b)}$

Taking the side pressures as acting as the centroids of the triangles, we get

$$
\begin{aligned}
M_x &= P_b \,.\, \frac{l}{2} + \frac{2 \,.\, P_l}{2} \,.\, \frac{l}{6} - \frac{W}{2} \,.\, \frac{l}{4} \\
&= \frac{W\,b\,l}{4\,(l+b)} + \frac{W\,l^2}{12\,(l+b)} - \frac{W\,l}{8} \\
&= \frac{W\,l}{8\,(l+b)} \left\{ 2\,b + \frac{2\,l}{3} - (l+b) \right\} \\
&= \frac{W\,l}{8\,(l+b)} \left(b - \frac{l}{3} \right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)
\end{aligned}
$$

$$
\therefore \text{ Slab coefficient for X X} = F_l = \frac{b - \frac{l}{3}}{l + b}
$$

$$
= \frac{1 - \frac{l}{3\,b}}{1 + \frac{l}{b}} \ldots\ldots\ldots\ldots (16)
$$

Similarly, by reversing l and b, slab coefficient for Y Y

$$
= F_b = \frac{\frac{l}{b} - \frac{1}{3}}{\frac{l}{b} + 1} \quad \ldots\ldots\ldots\ldots (17)
$$

These results can be tabulated as follows—

$\frac{l}{b}$	Slab Coefficients.	
	Short Section and Long Span β_l	Long Section and Short Span β_b
1	·333	·333
1·25	·259	·407
1·5	·200	·467
1·75	·151	·515
2	·111	·555

Numerical Example.—*Taking the parabolic variation, calculate the thickness of the slab previously considered.*

We have, neglecting slab action—

B.M. on long span = 1,166,400 in. lbs.

B.M. on short span = 777,600 in. lbs.

For $\frac{l}{b} = 1{\cdot}5, \beta_l = {\cdot}250, \text{B} = {\cdot}500$

$$\therefore \text{M}_y = 388{,}800 \text{ in. lbs.}$$

$$\text{M}_x = 291{,}600 \text{ in. lbs.}$$

$$\therefore \text{Short span } 16{,}000 \times \frac{b\,t^2}{6} = 388{,}800$$

$$\frac{16{,}000 \times 12 \times 12\,t^2}{6} = 388{,}800$$

$$t^2 = \frac{388{,}800}{16{,}000 \times 24} = 1{\cdot}01$$

$$t = \sqrt{1{\cdot}01} = \underline{1 \text{ in.}} \text{ nearly.}$$

Rectangular Slabs Clamped on their Edges.—An approximate treatment commonly adopted in practice for this case is to regard the B.M.s at the edges to be $\frac{\text{W}\,l}{12}$ and $\frac{\text{W}\,b}{12}$ respectively, multiplied by the slab coefficient derived for supported ends; those at the centre to be $\frac{\text{W}\,l}{24}$ and $\frac{\text{W}\,b}{24}$ reduced in the same ratio. In cases where the load may be on only a part of the slab the B.M.s at the centre are usually taken as the same as for the edges.

CHAPTER XVIII

* THICK PIPES

We have considered already the strength of a thin pipe and obtained simple formulæ by assuming that the stress was constant throughout the length and thickness of the pipe. When a pipe is not very thin compared with its diameter we have to allow for the variation of stress across the section.

Lamé's Theory.—Let a pipe be of internal radius r (Fig. 243) and external radius R and let it be under pressure either from the inside or from the outside.

Now consider an imaginary thin ring of thickness δx and internal radius x. This ring will be subjected to a radial pressure p on the inside which by considerations of symmetry must be the same all round, and on the outside it will be subjected to a radial pressure which will differ slightly from p and which we may call $p + \delta p$. This assumes that the tube is subjected to pressure on the outside; if it is on the inside the same formulæ hold with appropriate change of sign as explained later.

We may therefore apply to this imaginary hoop the same treatment as for a thin pipe, the circumferential stress, or *hoop stress*, being f.

Considering a unit length of pipe we have

Force tending to cause collapse of ring

$$= (p + \delta p) \times 2 (x + \delta x)$$

Force resisting collapse of ring $= 2 f \delta x + p . 2 x$

These must be equal

∴ dividing by 2 and neglecting the product, $\delta p \,.\, \delta x$, of two very small quantities we have

$$p \,.\, x + x \,.\, \delta p + p\,\delta x = f\,\delta x + p \,.\, x$$

$$\therefore (f - p)\,\delta x = x\,\delta p$$

$$(f - p) = \frac{x\,\delta p}{\delta x}$$

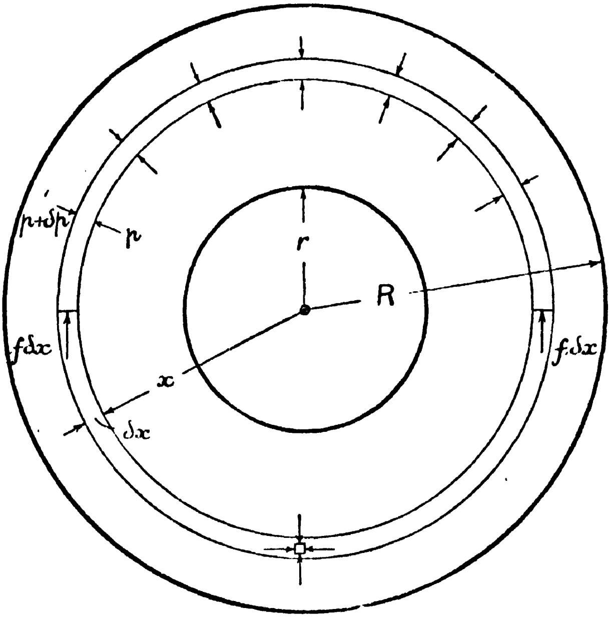


FIG. 243.—Stresses in Thick Pipes.

In the limit when the increments are infinitely small this gives

$$(f - p) = \frac{x\,d\,p}{d\,x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

This is one relation between f and p.

Now let us assume that the strains along the length of the pipe will be such that a plane section before subjection to pressure remains plane after subjection to pressure, *i.e.* that longitudinal strain is constant.

$$i.e.\ \frac{\eta f}{E} + \frac{\eta p}{E} = \text{constant} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

because both f and p will cause transverse strains in the direction of the length of the pipe, and they will have the same sign.

$\therefore$ Since η and E are constant, if our stresses are within the elastic limit we may write

$$f + p = \text{constant} = 2a \text{ (say)}$$

$$\therefore f = (2a - p) \qquad \text{.................. (3)}$$

Put this value in (1) and we get

$$2a - 2p = \frac{x\,dp}{dx}$$

$$2a = 2p + \frac{x\,dp}{dx} \qquad \text{................ (4)}$$

$$\text{but } \frac{d(px^2)}{dx} = 2px + \frac{x^2\,dp}{dx}$$

$$= x\left(2p + \frac{x\,dp}{dx}\right) = 2ax$$

$$\therefore d(px^2) = 2ax \,.\, dx \qquad \text{.................. (5)}$$

Integrating we get

$$px^2 = ax^2 + b$$

where b is a constant

$$\therefore p = a + \frac{b}{x^2} \qquad \text{...................... (6)}$$

$$\text{but } f = 2a - p \text{ (by 3)}$$

$$\text{i.e. } f = a - \frac{b}{x^2} \qquad \text{.................... (7)}$$

by calculating a and b in any particular case we can find formula for p and f.

Special Cases.—(1) PRESSURE INSIDE = p_i; PRESSURE OUTSIDE = 0

$$\text{i.e. } p = p_i \text{ for } x = r$$

$$p = 0 \text{ for } x = \text{R}$$

$$\therefore p_i = a + \frac{b}{r^2} \qquad \text{.................... (8)}$$

$$0 = a + \frac{b}{\text{R}^2}$$

$$\therefore a = -\frac{b}{\text{R}^2}$$

Put this value in (8), then

$$p_i = b\left(\frac{1}{r^2} - \frac{1}{R^2}\right) = \frac{b\,(R^2 - r^2)}{R^2\,r^2}$$

$$\therefore\ b = \frac{p_i\,R^2\,r^2}{R^2 - r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

$$\therefore\ a = -\frac{b}{R^2} = -\frac{p_i\,r^2}{R^2 - r^2} \quad \ldots\ldots(10)$$

$\therefore$ *Hoop stress at inside* $= f_i$ is obtained by putting $x = r$ in (7)

$$\text{i.e.}\quad f_i = a - \frac{b}{r^2}$$

$$= -\frac{p_i\,r^2}{R^2 - r^2} - \frac{p_i\,R^2}{R^2 - r^2}$$

$$= -\frac{p_i\,(R^2 + r^2)}{R^2 - r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

The negative sign indicates that the stress is a tension.

Hoop stress at outside $= f_o = a - \dfrac{b}{R^2}$

$$= -\frac{p_i\,r^2}{(R^2 - r^2)} - \frac{p_i\,r^2}{(R^2 - r^2)}$$

$$= -\frac{2\,p_i\,r^2}{(R^2 - r^2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

This is also a tensile stress and is clearly less than f_i, so that with internal pressure the maximum stress occurs on the inside.

At any intermediate radius x

$$p = a + \frac{b}{x^2}$$

$$= -\frac{p_i\,r^2}{(R^2 - r^2)} + \frac{p_i\,R^2\,r^2}{(R^2 - r^2)\,x^2}$$

$$= \frac{p_i\,r^2}{(R^2 - r^2)}\left\{\frac{R^2}{x^2} - 1\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

$$f = a - \frac{b}{x^2}$$

$$= -\frac{p_i\,r^2}{(R^2 - r^2)}\left\{\frac{R^2}{x^2} + 1\right\} \quad \ldots\ldots\ldots\ldots\ldots(14)$$

It should be noted from equation (11) that no matter how great the thickness of the tube may be the hoop stress is always greater than the internal pressure, so that for any given material there is a certain maximum pressure which must not be exceeded.

It should also be noted that the assumption of the constancy of longitudinal strain holds only while the stress is within the elastic limit.

✓Numerical Examples.—(1) *A cast-steel cylinder 2 ft. in external diameter and 3 inches thick is subjected to an internal pressure of 2 tons per sq. in. Where and of what magnitude is the maximum stress?*

The maximum stress is on the inside and is given by the formula

$$f_i = \frac{p\,(R^2 + r^2)}{(R^2 - r^2)}$$

$$= \frac{2\,(24^2 + 18^2)}{(24^2 - 18^2)} = \frac{2 \times 6^2\,(16 + 9)}{6^2\,(16 - 9)}$$

$$= \frac{2 \times 25}{7} = 7{\cdot}14 \text{ tons per sq. in.}$$

(2) *Plot a curve showing the maximum stress in terms of the internal pressure in a tube whose ratio of external to internal radius varies from* 1·10 *to* 4.

$$f_i = \frac{p_i\,(R^2 + r^2)}{(R^2 - r^2)}$$

$$\therefore \frac{f_i}{p_i} = \frac{R^2 + r^2}{R^2 - r^2}$$

$$= \frac{\left(\frac{R}{r}\right)^2 + 1}{\left(\frac{R}{r}\right)^2 - 1}$$

This gives the following values—

$\frac{R}{r}$	1·10	1·20	1·30	1·5	2·00	2·50	3·00	3·50	4·00
$\frac{f_i}{p_i}$	10·52	5·55	3·90	2·60	1·67	1·38	1·25	1·18	1·13

If $\frac{R}{r} = 1{\cdot}10, \frac{r+t}{r} = 1{\cdot}10, \frac{t}{r} = {\cdot}10.$

$\therefore$ The thin pipe formula $f = \frac{pr}{t}$ would give $\frac{f}{p} = 10$, so that the thin pipe formula would be about 5 % in error.

The above figures give the curve shown in Fig. 244. These results should be compared with Example 2, p. 521.

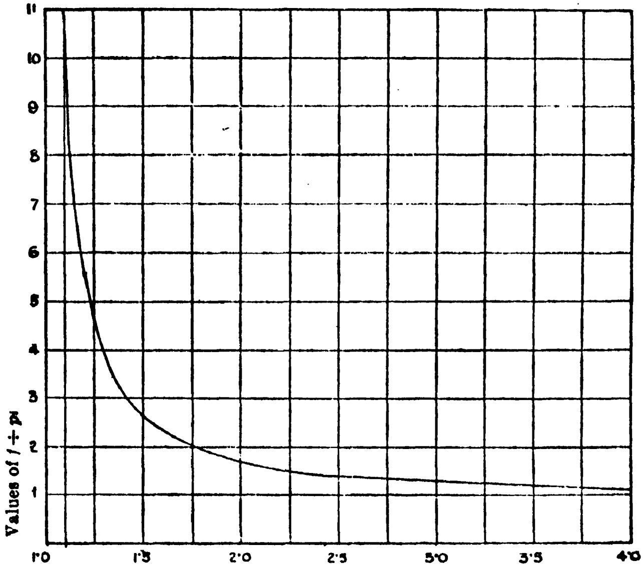


Fig. 244.—Variation of Hoop Stress for various ratios of External to Internal Radii of Pipes with Internal Pressure.

Curves of Variation of Radial and Hoop Stress for Internal Pressure for R = 2 r.

By equation (9) $\frac{b}{p_i} = \frac{R^2 r^2}{R^2 - r^2} = \frac{4\,r^4}{3\,r^2} = \frac{4\,r^2}{3}.$

By equation (10) $\frac{a}{p_i} = -\frac{r^2}{3\,r^2} = -\frac{1}{3}$

$$\therefore p = a + \frac{b}{x^2}$$

$$\frac{p}{p_i} = \frac{a}{p_i} + \frac{b}{p_i x^2}$$

$$= -\frac{1}{3} + \frac{4\,r^2}{3\,x^2}$$

$$= \frac{4}{3}\left(\frac{r}{x}\right)^2 - \frac{1}{3} = \frac{1}{3}\left\{4\left(\frac{r}{x}\right)^2 - 1\right\}$$

$$f = a - \frac{b}{x^2}$$

$$\frac{f}{p_i} = -\left\{\frac{1}{3} + \frac{4}{3}\left(\frac{r}{x}\right)^2\right\} = -\frac{1}{3}\left\{4\left(\frac{r}{x}\right)^2 + 1\right\}$$

These results are plotted in Fig. 245 and show clearly the variation of the stresses across the section.

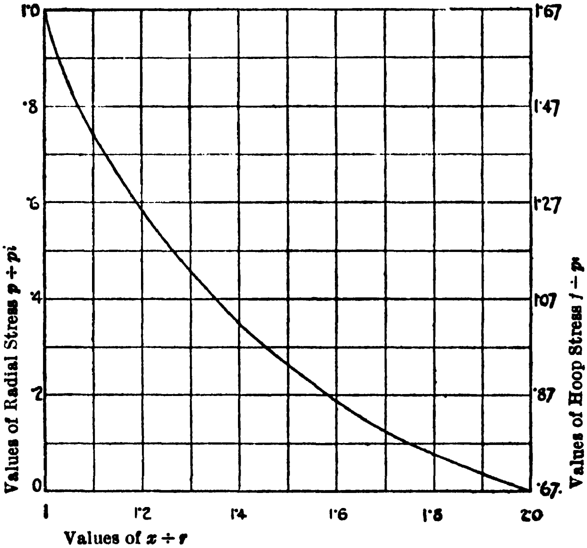


Fig. 245.—Variation of Stress in a Thick Pipe with Internal Pressure.

Maximum Shear Stress.—The stresses f and p are the principal stresses in the material and, as we showed on p. 19 the maximum shear stress will be equal to

$$s = \frac{f - p}{2} = \frac{b}{x^2} \text{ (from (6) and (7))}$$

$$= \frac{p_i\,R^2\,r^2}{(R^2 - r^2)\,x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

The maximum shear stress will occur when x has its least value, *i. e.* at the inside edge where $x = r$

$$\therefore \text{ Max. shear stress} = \frac{p_i R^2}{(R^2 - r^2)} \quad \ldots\ldots\ldots\ldots(16)$$

Maximum Stress equivalent to Strain.—There will be a strain in the circumferential direction equal to $-\frac{\eta p}{E}$

$$\therefore \text{ Total circumferential strain} = \frac{f}{E} - \frac{\eta p}{E}$$

$$\therefore \text{ Equivalent stress} = \text{Total strain} \times E$$

$$= f_e = f - \eta p$$

$$= a(1 - \eta) - \frac{b(1 + \eta)}{x^2} \quad \ldots\ldots\ldots(17)$$

This gives a maximum equivalent stress at the inside of

$$f_e = -\frac{p_i (R^2 + r^2)}{(R^2 - r^2)} - \eta p_i$$

$$= -p_i \left\{\frac{R^2 + r^2}{R^2 - r^2} + \eta\right\} \quad \ldots\ldots\ldots\ldots(18)$$

putting $\eta = \frac{1}{4}$ this gives

$$f_e = -p_i \left\{\frac{R^2 + r^2}{R^2 - r^2} + \frac{1}{4}\right\}$$

NUMERICAL EXAMPLE.—*Take the case of a tube with internal radius* 4 *in. and external radius* 12 *in. and take* $p = 10{,}000$.

This case was given by Professor C. A. M. Smith,* and we have added the equivalent stresses.

This gives $b = 180{,}000$, $a = -1{,}250$.

The results may then be tabulated as follows—

Radius.	Maximum Stresses.			
	Compression. p	Hoop Tension. f	Shear. s	Hoop Tension equivalent to Strain. f_e
4	10,000	12,500	11,250	15,000
5	5,950	8,450	7,200	10,950
6	3,750	6,250	5,000	8,750
7	2,420	4,920	3,670	7,420
8	1,560	4,060	2,810	6,560
9	970	3,470	2,220	5,970
10	550	3,050	1,800	5,550
11	240	2,740	1,490	5,240
12	0	2,500	1,250	5,000

---

* *Engineering*, September 2, 1910.

Formula for Thickness of Pipe in Terms of Internal Diameter and Pressure.—*On the maximum stress theory* we have, if f_t is the safe tensile stress,

$$f_t = \frac{p_i(R^2 + r^2)}{R^2 - r^2}$$

$$\therefore \frac{f_t}{p_i} = \frac{R^2 + r^2}{R^2 - r^2}$$

$$\frac{f_t + p_i}{f_t - p_i} = \frac{2R^2}{2r^2}$$

$$\therefore \frac{R}{r} = \sqrt{\frac{f_t + p_i}{f_t - p_i}}$$

$$\text{i. e. } \frac{r + t}{r} = 1 + \frac{t}{r} = \sqrt{\frac{f_t + p_i}{f_t - p_i}}$$

$$\therefore \frac{t}{r} = \sqrt{\frac{f_t + p_i}{f_t - p_i}} - 1$$

$$t = r\left\{\sqrt{\frac{f_t + p_i}{f_t - p_i}} - 1\right\}$$

$$= \frac{d}{2}\left\{\sqrt{\frac{f_t + p_i}{f_t - p_i}} - 1\right\} \quad \ldots\ldots\ldots\ldots(19)$$

ON THE MAXIMUM STRAIN THEORY we have

$$f_t = p_i\left(\frac{R^2 + r^2}{R^2 - r^2} + \eta\right)$$

$$\frac{f_t}{p_i} - \eta = \frac{R^2 + r^2}{R^2 - r^2}$$

$$\frac{f_t + (1 - \eta)\,p_i}{f_t - (1 + \eta)\,p_i} = \frac{R^2}{r^2}$$

$$\sqrt{\frac{f_t + (1 - \eta)\,p_i}{f_t - (1 + \eta)\,p_i}} = \frac{R}{r} = 1 + \frac{t}{r}$$

$$\therefore t = r\left\{\left(\sqrt{\frac{f_t + (1 - \eta)\,p_i}{f_t - (1 + \eta)\,p_i}}\right) - 1\right\}$$

If $\eta = \frac{1}{4}$

$$t = r\left\{\left(\sqrt{\frac{4f_t + 3p_i}{4f_t - 5p_i}}\right) - 1\right\} \quad \ldots.(19a)$$

If $\eta = \cdot 3$

$$t = r\left\{\left(\sqrt{\frac{f_t + \cdot 7p_i}{f_t - 1\cdot 3p_i}}\right) - 1\right\} \quad \ldots.(19b)$$

If $\eta = \frac{1}{3}$

$$t = r\left\{\left(\sqrt{\frac{3f_t + 2p_i}{3f_t - 4p_i}}\right) - 1\right\} \quad \ldots.(19c)$$

Fig. 246 shows a chart reproduced from *Machinery*, January 14, 1915, for determining the necessary thickness of a cylinder or pipe in accordance with Formula 19.

In using the chart to determine the constant, the horizontal line through the proper pressure value is located, and the curve starting from the desired value of the stress is next found; this curve is then followed to the point where it intersects the horizontal pressure line, after which the vertical line is followed to the bottom of the chart to determine the constant. This constant multiplied by the inside radius r of the cylinder gives the required thickness of the cylinder wall.

Case 2.—OUTSIDE PRESSURE $= p_o$, INSIDE PRESSURE $= 0$

In this case $p = p_o$ when $x = R$

and $p = 0$ when $x = r$

$$\therefore\ p_o = a + \frac{b}{R^2}$$

$$0 = a + \frac{b}{r^2}$$

$$\text{i.e. } a = -\frac{b}{r^2}$$

$$\therefore\ p_o = b\left(\frac{1}{R^2} - \frac{1}{r^2}\right)$$

$$\therefore\ b = \frac{p_o}{\left(\frac{1}{R^2} - \frac{1}{r^2}\right)} = -\frac{p_o R^2 r^2}{(R^2 - r^2)} \qquad \text{(20)}$$

$$\therefore\ a = \frac{p_o R^2}{(R^2 - r^2)} \qquad \text{(21)}$$

HOOP STRESS AT INSIDE.

At inside, where $x = r,\ f_i = a - \frac{b}{r^2}$

$$= \frac{p_o R^2}{(R^2 - r^2)} + \frac{p_o R^2}{(R^2 - r^2)}$$

$$= \frac{2\,p_o R^2}{(R^2 - r^2)} \qquad \text{(22)}$$

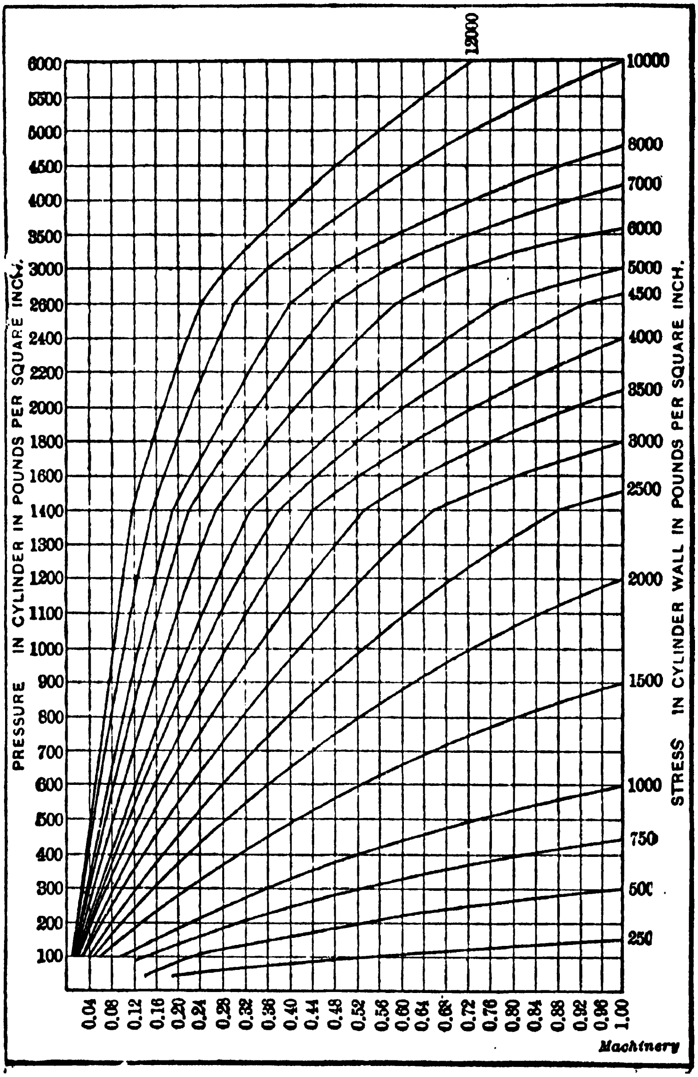


FIG. 246.—Diagram for Thick Tubes.

HOOP STRESS AT OUTSIDE.

At the outside, where $x = \mathrm{R}$, hoop stress $= f_o$

$$= a - \frac{b}{\mathrm{R}^2}$$

$$= \frac{p_o \mathrm{R}^2}{(\mathrm{R}^2 - r^2)} + \frac{p_o r^2}{(\mathrm{R}^2 - r^2)}$$

$$= p_o \frac{(\mathrm{R}^2 + r^2)}{(\mathrm{R}^2 - r^2)} \qquad \text{(23)}$$

In this case the maximum hoop stress, which is compressive throughout, also occurs on the inside and is greater than the radial pressure; irrespective of the thickness of the pipe, the external pressure may never be more than half the safe compressive stress on the material.

Putting in the values of a and b in the general formula for this case we have at a radius x

$$p = \frac{p_o \mathrm{R}^2}{(\mathrm{R}^2 - r^2)} - \frac{p_o \mathrm{R}^2 r^2}{(\mathrm{R}^2 - r^2)\, x^2}$$

$$= \frac{p_o \mathrm{R}^2}{(\mathrm{R}^2 - r^2)} \left\{1 - \left(\frac{r}{x}\right)^2\right\} \qquad \text{(24)}$$

$$f = \frac{p_o \mathrm{R}^2}{(\mathrm{R}^2 - r^2)} \left\{1 + \left(\frac{r}{x}\right)^2\right\} \qquad \text{(25)}$$

NUMERICAL EXAMPLES.—(1) *A cast-steel cylinder* 2 *ft. external diameter and* 3 *in. thick is subjected to an external pressure of* 2 *tons per sq. in. Where and of what magnitude is the maximum stress? (Compare Example* 1, *p.* 514.) *The maximum stress occurs on the inside and is given by the formula*

$$f_i = \frac{2\, p_o \mathrm{R}^2}{(\mathrm{R}^2 - r^2)} = \frac{4 \times 24^2}{24^2 - 18^2} = \frac{4 \times 4^2 \times 6^2}{6^2(16 - 9)}$$

$$= \frac{4 \times 16}{7} = 9{\cdot}14 \text{ tons per sq. in.}$$

(2) *Plot a curve showing the maximum stress in terms of the external pressure in a tube whose ratio of external to internal radius varies from* 1·10 *to* 4.

$$\frac{f_i}{p_o} = \frac{2\,\mathrm{R}^2}{\mathrm{R}^2 - r^2} = \frac{2\left(\frac{\mathrm{R}}{r}\right)^2}{\left(\frac{\mathrm{R}}{r}\right)^2 - 1}$$

This gives the following values—

$\frac{R}{r}$	1·10	1·20	1·30	1·50	2·00	2·50	3·00	3·50	4·00
$\frac{f_i}{p_o}$	11·52	6·55	4·90	3·60	2·67	2·38	2·25	2·18	2·13

For $\frac{R}{r} = 1{\cdot}10$ we have $\frac{R}{R - t} = 1{\cdot}10 \quad \therefore t = \frac{R}{11}$.

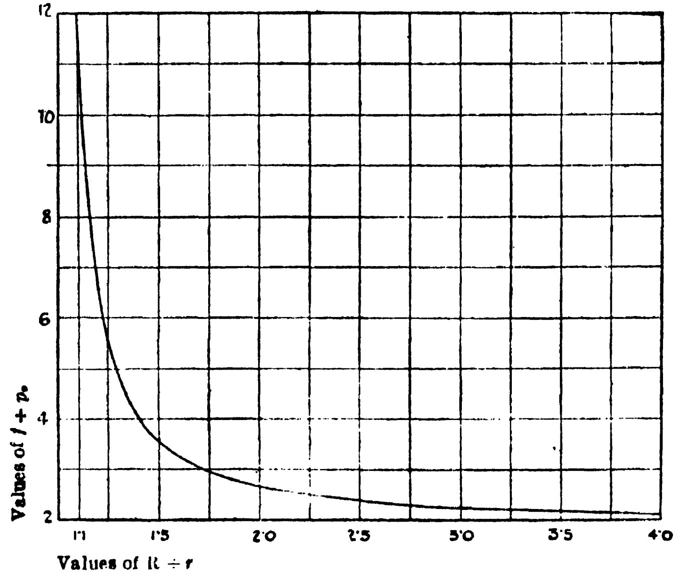


FIG. 247.—Variation of Hoop Stress for various ratios of External to Internal Radii of Pipes with External Pressure.

The thin pipe formula would give

$$f = \frac{p_o R}{t} = 11 p_o, \quad \therefore \frac{f}{p_o} = 11$$

so that the thin pipe formula would be about 5 % in error.

The above results gives the curve shown in Fig. 247, which should be compared with that shown in Fig. 244 for internal pressure.

Curves of Variation of Radial and Hoop Stress for External Pressure for R = 2 r.

By equation (20) $\dfrac{b}{p_o} = -\dfrac{R^2 r^2}{R^2 - r^2} = -\dfrac{4r^2}{3}$

" " (21) $\dfrac{a}{p_o} = \dfrac{R^2}{R^2 - r^2} = \dfrac{4}{3}$

$$\therefore \frac{p}{p_o} = \left(a + \frac{b}{x^2}\right) \div p_o$$

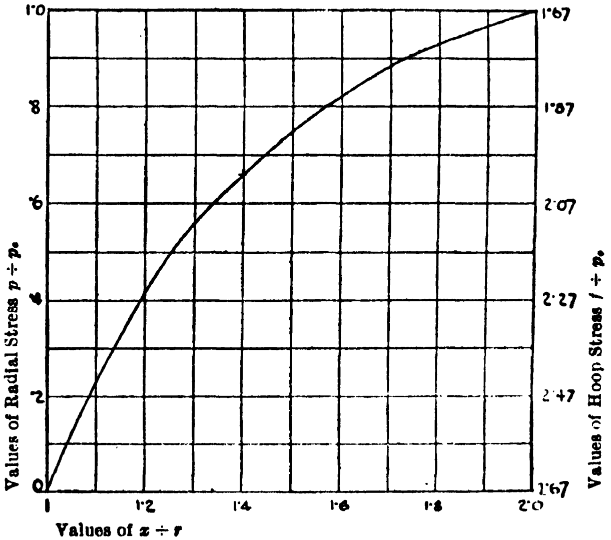


FIG. 248.—Variation of Stress in a Thick Pipe with External Pressure.

$$= \frac{4}{3} - \frac{4r^2}{3x^2}$$

$$= \frac{4}{3}\left\{1 - \left(\frac{r}{x}\right)^2\right\} \quad \text{.................. (26)}$$

$$\frac{f}{p_o} = \left(a - \frac{b}{x^2}\right) \div p_o$$

$$= \frac{4}{3} + \frac{4r^2}{3x^2}$$

$$= \frac{4}{3}\left\{1 + \left(\frac{r}{x}\right)^2\right\} \quad \text{.................. (27)}$$

These give the curves shown in Fig. 248, which should be compared with Fig. 245.

Strengthening Thick Pipes for Internal Pressure by Initial Compression.

We have seen that the maximum hoop stress is always greater than the internal pressure, so that if we are to be able to make pipes sustain very high pressures we must devise some method of reducing the hoop stresses. This may be effected by bringing the metal of the tube into a state of initial compression.

In the early days guns were cast around chills to cause the metal to solidify immediately on the inside and so come into compression when the remainder of the metal contracted upon cooling.

Another method, now commonly adopted, is to wind strong steel wire under heavy tension on the outside of a tube, thus bringing it into compression which will balance to some extent the pressure caused inside the gun by the explosion.

A further method is to shrink an outer tube on to an inner one; this puts initial tension stresses into the outer tube and initial compression stresses into the inner tube. The effect of the shrinking is shown in Fig. 249. The top diagram indicates the distribution of the hoop stresses across the section for a solid tube; the shrinkage stresses are shown in the central diagram; these being obtained by applying the condition that the radial pressures of the junction are equal and opposite.

The combined stresses are shown in the bottom diagram, from which it is seen that the maximum tensile stress is very much reduced and that the tensile stresses are more nearly constant.

Necessary Difference in Radius for Shrinkage.— For the outer and inner tubes, the same general formulæ will hold, but the constants will be different.

i. e. For the outer tube we have

$$p = a_o + \frac{b_o}{x^2} \quad \dots\dots\dots\dots\dots\dots\dots (1)$$

$$f = a_o - \frac{b_o}{x^2} \quad \dots\dots\dots\dots\dots\dots\dots (2)$$

For the inner tube

$$p = a_i + \frac{b_i}{x^2} \quad \dots\dots\dots\dots\dots\dots\dots(3)$$

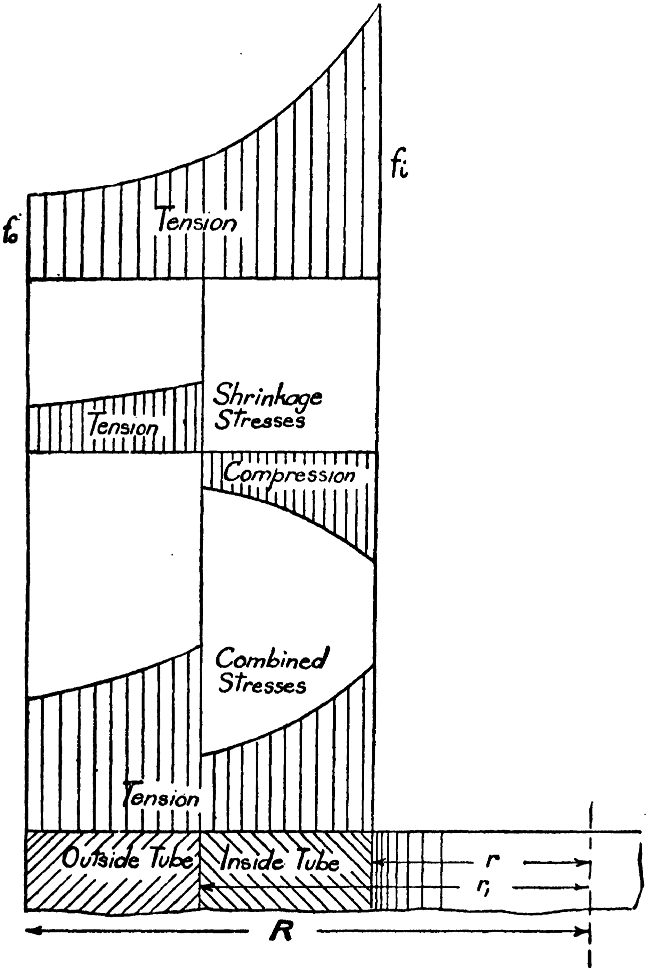


FIG. 249.—Shrinkage Stresses in Compound Tube.

$$f = a_i - \frac{b_i}{x^2} \quad \dots\dots\dots\dots\dots\dots\dots(4)$$

We must have the same value of p for the junction where $x = r_1$

$$\therefore\ a_o + \frac{b_o}{r_1^2} = a_i + \frac{b_i}{r_1^2}$$

$$\text{i.e. } (a_o - a_i)\, r_1^2 = (b_i - b_o) \quad \ldots\ldots\ldots\ldots(5)$$

Next consider the circumferential strains at the junction.

For the outer tube we have

$$\text{Unital circumferential strain} = \frac{f}{E} + \frac{\eta\, p}{E} \quad \ldots\ldots\ldots(6)$$

Similarly for inner tube

$$\text{Unital circumferential strain} = -\left(\frac{f}{E} + \frac{\eta\, p}{E}\right) \ldots(7)$$

The value of p is the same in each case, and in (6) f is as in (2) and in (7) as in (4).

$\therefore$ Increase in circumference of outer tube

$$= \frac{2\,\pi\, r_1}{E}\left\{\left(a_o - \frac{b_o}{r_1^2}\right) + \eta\, p\right\}$$

Decrease in circumference of inner tube

$$= \frac{2\,\pi\, r_1}{E}\left\{\left(a_i - \frac{b_i}{r_1^2}\right) + \eta\, p\right\}$$

$\therefore$ Difference in circumference of two tubes before heating and shrinking on

$$= \frac{2\,\pi\, r_1}{E}\left\{a_o - \frac{b_o}{r_1^2} + \eta\, p - \left(a_i - \frac{b_i}{r_1^2}\right) - \eta\, p\right\}$$

$$= \frac{2\,\pi\, r_1}{E}\left\{(a_o - a_i) - \frac{(b_o - b_i)}{r_1^2}\right\}$$

$\therefore$ Corresponding difference in radius

$$= \frac{\text{difference in circumference}}{2\,\pi}$$

$$= \frac{r_1}{E}\left\{(a_o - a_i) - \frac{(b_o - b\ )}{r_1^2}\right\}$$

$$= (\text{from } (5))\ \frac{2\, r_1}{E}\,(a_o - a_i)$$

i.e. Proportional difference in radius $= \dfrac{2}{E}(a_o - a_i)\ldots.(8)$

This will probably be made more clear by the following numerical example.

Numerical Example.—*A compound mild-steel cylinder consists of a tube 6 ins. in external radius and 4·5 internal radius shrunk on to another tube which has an internal radius of 3 ins. If the radial compression at the common surface is 4000 lbs. per sq. in. after shrinking find the circumferential stress at the inner and outer surfaces and at the common surface, and find also the original difference in radius necessary to effect the given radial pressure at the junction.*

Outer tube.

$$p = 0 \text{ for } x = 6$$

$$= 4000 \text{ for } x = 4{\cdot}5$$

$$\therefore\ 0 = a_o + \frac{b_o}{36}$$

$$4000 = a_o + \frac{4\,b_o}{81} = \left(\frac{4}{81} - \frac{1}{36}\right) b_o$$

$$b_o = \frac{324 \times 4000}{7}$$

$$a_o = -\frac{36{,}000}{7}$$

Interior hoop stress ($x = 4{\cdot}5$)

$$= -\frac{36{,}000}{7} - \frac{324 \times 4000}{7} \times \frac{4}{81}$$

$$= \underline{14{,}286 \text{ lbs. per sq. in. (tension).}}$$

Exterior hoop stress ($x = 6$)

$$= -\frac{36{,}000}{7} - \frac{324 \times 4000}{7} \times \frac{1}{36}$$

$$= \underline{10{,}286 \text{ lbs. per sq. in. (tension).}}$$

Inner tube.

$$p = 0 \text{ for } x = 3$$

$$= 4000 \text{ for } x = 4{\cdot}5$$

$$\therefore\ 0 = a_i + \frac{b_i}{9} \qquad \therefore\ a_i = -\frac{b_i}{9}$$

$$4000 = a_i + \frac{4\,b_i}{81}$$

$$= \frac{4\,b_i}{81} - \frac{b_i}{9} = -\frac{5\,b_i}{81}$$

$$b_i = -\frac{4000 \times 81}{5} = -\,64{,}800$$

$$a_i = 7{,}200$$

$\therefore$ Interior hoop stress ($x = 3$)

$$= 7{,}200 - \left(-\frac{64{,}800}{9}\right)$$

$$= \underline{\underline{14{,}400 \text{ lbs. per sq. in. (compression).}}}$$

Exterior hoop stress ($x = 4{\cdot}5$)

$$= 7{,}200 - \left(-\frac{64{,}800 \times 4}{81}\right)$$

$$= \underline{\underline{10{,}400 \text{ lbs. per sq. in. (compression).}}}$$

Taking $E = 30 \times 10^6$ lbs. per sq. in. and $\eta = \frac{1}{4}$, the circumferential strain at $4\frac{1}{2}$ ins. radius in the outer cylinder

$$= \frac{14{,}286}{30 \times 10^6} + \frac{4000}{4 \times 30 \times 10^6}$$

$$= {\cdot}0004762 + \frac{1000}{30 \times 10^6}$$

Circumferential compressive strain at $4\frac{1}{2}$ ins. radius in the inner cylinder

$$= \frac{10{,}400}{30 \times 10^6} - \frac{4000}{4 \times 30 \times 10^6}$$

$$= {\cdot}0003467 - \frac{1000}{30 \times 10^6}$$

$\therefore$ original difference in diameter $= 9\,({\cdot}0004762 + {\cdot}0003467)$

$$= \underline{\underline{{\cdot}00741 \text{ in. nearly.}}}$$

CHAPTER XIX

* CURVED BEAMS

WE have seen in Chapter VII that the ordinary formulæ for beams hold only for cases in which the beam is initially practically straight. To obtain relations between the stresses and the bending moment in the general case we may proceed as follows—

Let A B D E (Fig. 250) represent a short piece of a curved beam, O being the centre of curvature and A E and B D being sections normal to the centre line C C′. Then, obviously, the material at E D will not require the same *total* strain to produce a given unital strain and thus stress as the material in A B will, because its original length is less, and, as a result, the neutral axis will not pass through the centroid.

While still making the assumption that stress and strain are proportional, and also Bernoulli's assumption that a section originally plane remains plane after bending, we can find a more accurate theory of bending of curved beams, as follows—

Let the portion A B D E take up the position $A_1\ B_1\ D_1\ E_1$ after bending. Consider an element of area a situated at a point P at distance y from the centroid line C C′ and consider a fibre P Q of the material enclosing the area a.

After strain the fibre P Q takes up the position $P_1\ Q_1$ at distance y_1 from the strained centroid line $C_1\ C_1'$

Then unital strain in P Q $= \dfrac{P_1 Q_1 - P Q}{P Q}$

And if f_y is the stress at the point P

$$\frac{f_y}{E} = \frac{P_1Q_1 - PQ}{PQ} = \frac{P_1Q_1}{PQ} - 1$$

$$\therefore \frac{P_1Q_1}{PQ} = 1 + \frac{f_y}{E} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Similarly unital strain along C C′ $= \dfrac{C_1C_1' - CC'}{CC'}$

and if f_o is the stress at the centroid, we get similarly

$$\frac{C_1C_1'}{CC'} = 1 + \frac{f_o}{E} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

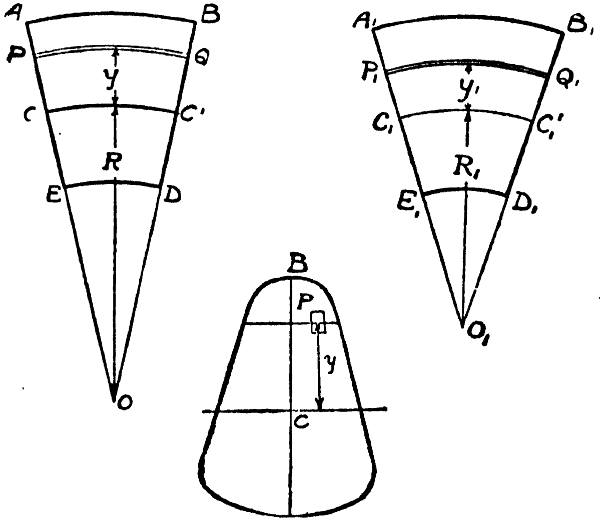


Fig. 250.—Stresses in Curved Beams.

Dividing (1) by (2), we get

$$\frac{P_1Q_1 \times CC'}{C_1C_1' \times PQ} = \frac{1 + \dfrac{f_y}{E}}{1 + \dfrac{f_o}{E}}$$

But $$\frac{P_1Q_1}{C_1C_1'} = \frac{y_1 + R_1}{R_1} = 1 + \frac{y_1}{R_1}$$

$$\frac{CC_1}{PQ} = \frac{R}{y + R} = \frac{1}{1 + \dfrac{y}{R}}$$

Also since $\frac{f_v}{E}$ and $\frac{f_o}{E}$ are extremely small, we may write

$$\frac{1+\frac{f_v}{E}}{1+\frac{f_o}{E}}=\left(1-\frac{f_o}{E}+\frac{f_v}{E}\right)$$

$$\therefore \text{ We get } \frac{1+\frac{y_1}{R_1}}{1+\frac{y}{R}}=1-\frac{f_o}{E}+\frac{f_v}{E}\text{ .... }(3)$$

$$\therefore \frac{f_o}{E}=\frac{f_v}{E}+1-\frac{1+\frac{y_1}{R_1}}{1+\frac{y}{R}}$$

$$=\frac{f_v}{E}+\frac{\frac{y}{R}-\frac{y_1}{R_1}}{1+\frac{y}{R}}$$

$$=\frac{f_v}{E}-\frac{\frac{y_1}{R_1}-\frac{y}{R}}{1+\frac{y}{R}}\text{ .... }(4)$$

$$\therefore f_v=f_o+\frac{E\left(\frac{y_1}{R_1}-\frac{y}{R}\right)}{1+\frac{y}{R}}\text{ .... }(5)$$

Then the load across the whole cross section is $\Sigma f_v \,.\, a$ and in the case of pure bending this is zero.

$\therefore$ We have $\Sigma f_v \,.\, a=0$

$$=\Sigma f_o \,.\, a+\Sigma\frac{E\left(\frac{y_1}{R_1}-\frac{y}{R}\right)}{\left(1+\frac{y}{R}\right)}\,.\,a$$

But $\Sigma f_o a=f_o\Sigma a=f_o A$

$$\therefore f_o=-\frac{E}{A}\Sigma\frac{\left(\frac{y_1}{R_1}-\frac{y}{R}\right)}{\left(1+\frac{y}{R}\right)}\,.\,a\text{ .... }(6)$$

The moment of the force on the given element about O O′ $= f_y \,.\, a \,.\, y$ and the sum of these moments is equal to the moment of resistance and thus equal to the bending moment M

∴ We have $M = \Sigma f_y \,.\, y \,.\, a$

$$= \Sigma f_o a \,.\, y + \Sigma \frac{E\left(\frac{y_1}{R_1} - \frac{y}{R}\right)}{\left(1 + \frac{y}{R}\right)} \,.\, a\, y$$

But $\Sigma f_o a\, y = f_o \Sigma a \,.\, y = f_o \times$ first moment of area about centroid $= f_o \times 0 = 0$

∴ We have

$$M = \Sigma \frac{E\left(\frac{y_1}{R_1} - \frac{y}{R}\right)}{\left(1 + \frac{y}{R}\right)} \,.\, a\, y \quad \ldots\ldots\ldots\ldots\ldots(7)$$

This is the most general case and is true for the assumption given.

Now consider the following special cases—

(1) **Ordinary Straight Beam**; R INFINITE, R_1 VERY GREAT.

$$\text{In this case } f_o = \frac{E}{A} \Sigma \frac{\frac{y_1}{R_1}}{1} \,.\, a$$

$$= \frac{E}{R_1 A} \Sigma y_1 a = 0$$

$$\text{Then } M = \Sigma \frac{E}{R_1} \cdot y_1 y\, a$$

y_1 is practically equal to y

$$\therefore M = \frac{E}{R_1} \Sigma y^2 a$$

$$= \frac{E I}{R_1}$$

and from equation (5) $f_y = 0 + \frac{E \,.\, y_1}{R_1}$

$$= \frac{E \times y}{R_1}$$

$$\therefore \frac{E}{R_1} = \frac{f_y}{y}$$

$$\therefore M = \frac{f_y I}{y}$$

This is the result we have previously obtained.

(2) **Winkler's Formula.**—Winkler drew attention to the error of applying the ordinary bending formulæ to chain links, etc., where the original curvature is appreciable, and improved such formulæ as follows—

He takes $y_1 = y$.

Then equation (5) becomes

$$f_y = f_o + \frac{E \cdot y\left(\frac{1}{R_1} - \frac{1}{R}\right)}{1 + \frac{y}{R}}$$

$$= f_o + E\left(\frac{1}{R_1} - \frac{1}{R}\right) \cdot \frac{y\,R}{y + R} \quad \ldots\ldots(8)$$

Then from equation (6)

$$f_o = -\frac{E}{A}\left(\frac{1}{R_1} - \frac{1}{R}\right)\Sigma\left(\frac{y\,R}{y + R}\right) \cdot a$$

$$\text{Now } \Sigma\left(\frac{y\,R}{y + R}\right) \cdot a = \Sigma\, y\, a - \Sigma\left(\frac{y^2}{y + R}\right) \cdot a$$

$$= 0 - \Sigma\left(\frac{y^2}{y + R}\right) \cdot a$$

$$\text{Now let } A\,h^2 = \Sigma\left(\frac{R \cdot y^2}{y + R}\right) \cdot a$$

where h is defined by the above relation, and may be called the *link radius.* It corresponds to the radius of gyration in the ordinary case.

$$\therefore \text{We see } \frac{A\,h^2}{R} = \Sigma\left(\frac{y^2}{y + R}\right) \cdot a$$

$$= -\Sigma \frac{y\,R}{y + R} \cdot a$$

$$\therefore \Sigma\left(\frac{y}{y + R}\right) \cdot a = -\frac{A\,h^2}{R^2}$$

$$\text{Then } f_o = \frac{E}{A} \cdot \left(\frac{1}{R_1} - \frac{1}{R}\right)\frac{A\,h^2}{R}$$

$$\therefore \text{ We have } f_o = \frac{E\,h^2}{R}\left(\frac{1}{R_1} - \frac{1}{R}\right) \quad \ldots\ldots\ldots\ldots\ldots (9)$$

From equation (7)

$$M = \Sigma \frac{E\left(\frac{1}{R_1} - \frac{1}{R}\right)}{1 + \frac{y}{R}} \cdot y^2 a$$

$$= E\,R\left(\frac{1}{R_1} - \frac{1}{R}\right)\Sigma \frac{y^2 a}{R + y}$$

$$= E\left(\frac{1}{R_1} - \frac{1}{R}\right)\Sigma \frac{R\,y^2}{R + y} \cdot a$$

$$= E\,A\,h^2\left(\frac{1}{R_1} - \frac{1}{R}\right) \quad \ldots\ldots\ldots\ldots (10)$$

∴ returning to equation (8) we see

$$f_y = \frac{E\,h^2}{R}\left(\frac{1}{R_1} - \frac{1}{R}\right) + E\left(\frac{1}{R} - \frac{1}{R_1}\right)\frac{y\,R}{y + R}$$

$$= \frac{M}{R\,.\,A} + \frac{M\,y}{A\,h^2} \cdot \left(\frac{R}{R + y}\right)^{*} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

General Graphical Solution.—Let Fig. 251 represent the section A D B E of a beam, and O the centre of curvature of the centre line D E, the beam being, of course, curved in the plane of bending.

Now consider a very narrow strip P Q of the half section at distance y from C D. Join P O, cutting C D in S and draw S R parallel to Q O to cut P Q in R.

$$\text{Then } \frac{PR}{PQ} = \frac{RS}{QO} = \frac{y}{R + y}$$

Repeating this construction for a number of strips such as P Q, and joining up the points obtained, we get a curve A R D R_1 B which is one-half of a curve called the *link rigidity curve.* In a symmetrical section, which is the most common, the two halves will be identical.

---

[*] This is the stress due to bending only; in the case of hooks we have to add the direct stress over the whole section.

Then the area of this link rigidity curve

$$= -\mathrm{A}_{\mathrm{L}}{}^{*} = -\Sigma\left(\frac{y}{\mathrm{R}+y}\right).a$$

But $$\frac{\mathrm{A}h^2}{\mathrm{R}^2} = \Sigma\left(\frac{y}{\mathrm{R}+y}\right)a$$

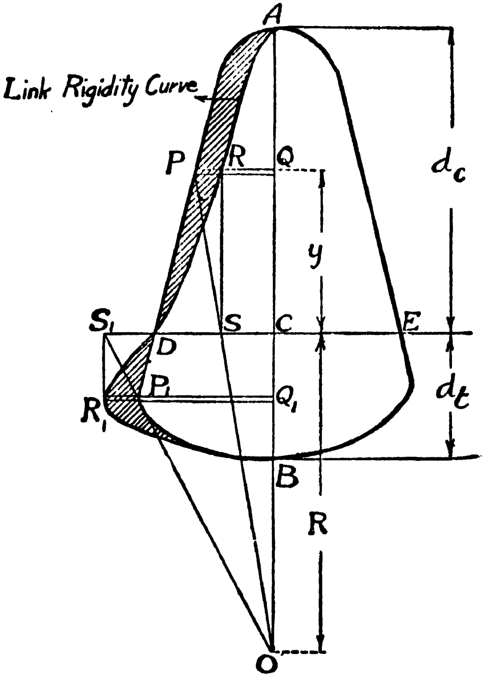


FIG. 251.—Curved Beams, etc.

$$\therefore \mathrm{A}_{\mathrm{L}} = \frac{\mathrm{A}h^2}{\mathrm{R}^2}$$

Now let $\frac{\mathrm{A}_{\mathrm{L}}}{\mathrm{A}} = \frac{\text{area of half link rigidity curve}}{\text{area of half section}}$

$$= \mathrm{L} \quad \textit{i.e.}\ \mathrm{A}h^2 = \mathrm{A}\times\mathrm{L}\times\mathrm{R}^2$$

---

* The area is negative because the area of the portion D R_1 B is greater than that of A P D. A_{L} represents the excess of the former area over the latter, *i. e.* area A R D R_1 B – area A P D P_1 D.

Now put these values in the equation (11) for stress. Then we have

$$f_y = \frac{M}{R . A} + \frac{M y . R}{A . L . R^2 (R + y)}$$

$$= \frac{M}{A}\left\{\frac{1}{R} + \frac{y}{R L (R + y)}\right\}$$

$$= \frac{M}{A R}\left\{1 + \frac{y}{L (R + y)}\right\}$$

Then if d_c and d_t are the distances from the line D E to the extreme compression and tension fibres respectively, assuming the inside to be in tension and the outside to be in compression, we have

Maximum compressive stress

$$= f_c = \frac{M}{A R}\left\{1 + \frac{d_c}{L (R + d_c)}\right\} \quad ..(12)$$

Maximum tensile stress $= f_t = \dfrac{M}{A R}\left\{\dfrac{d_t}{L (R - d_t)} - 1\right\}$ ..(13)

Position of Neutral Axis.—The value of y to make $f_y = 0$ gives the distance y_o of the neutral axis from D E.

$$i.e. \ \frac{y_o}{L (R + y_o)} = -1$$

$$y_o = - L R - L y_o$$

$$y_o = - \frac{L R}{1 + L} = - \frac{R}{1 + \frac{R^2}{h^2}}$$

This enables us to find the position of the neutral axis.

Alternative Formulæ.—The stress formula on Winkler's assumption that $y_1 = y$ can be put in a number of alternative forms.

Suppose, for instance, that the neutral axis is at distance y_o below the centroid line O O_1 (Fig. 250). Then total strain at P Q $=$ $P_1 Q_1$ $-$ P Q will be proportional to $(y + y_o)$ the distance from the N.A.

∴ We write $P_1 Q_1 - P Q = m (y + y_o)$.

Moreover, P Q $= (y + R) \times \angle E O D$

$$\therefore \frac{f_y}{\mathrm{E}} = \frac{\mathrm{P_1 Q_1 - P Q}}{\mathrm{P Q}} = \frac{(y + y_o)}{(y + \mathrm{R})} \cdot \frac{m}{\underline{\angle \mathrm{EOD}}}$$

$$= n \cdot \frac{(y + y_o)}{(y + \mathrm{R})} \text{ where } n \text{ is a constant}$$

$$\therefore f_y = \frac{\mathrm{E}\, n\, (y + y_o)}{(y + \mathrm{R})} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

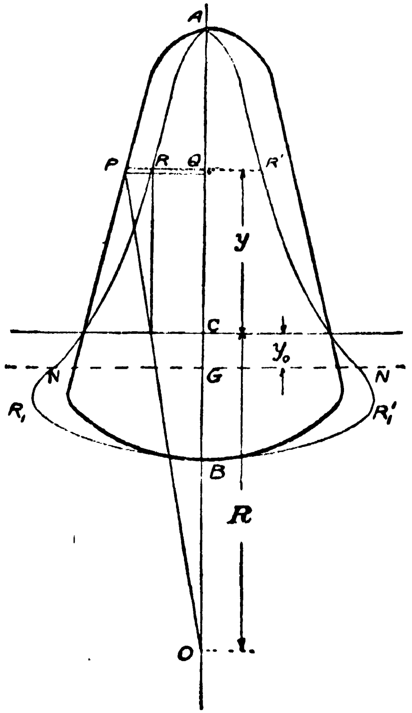


FIG. 252.

Since $\Sigma f_y \cdot a = 0$

$$\mathrm{E}\, n \sum \left(\frac{y + y_o}{y + \mathrm{R}}\right) a = 0$$

$$\text{i.e. } y_o = -\frac{\sum \frac{y\, a}{y + \mathrm{R}}}{\sum \frac{a}{y + \mathrm{R}}} \quad \ldots\ldots\ldots\ldots (14)$$

Again, taking moments about the neutral axis we have $\Sigma f_y \cdot a\,(y + y_o) = M$

$$i.e.\ E\,n \sum \left\{ \frac{(y + y_o)^2}{y + R} \cdot a \right\} = M$$

$$\therefore n = \frac{M}{E \sum \frac{(y + y_o)^2}{y + R} \cdot a} \quad \ldots\ldots\ldots\ldots (15)$$

$$\therefore f_y = \frac{M\,(y + y_o)}{(y + R) \left\{ \sum \frac{(y + y_o)^2}{y + R} \right\} a} \quad \ldots\ldots\ldots\ldots (16)$$

RÉSAL'S CONSTRUCTION.—According to this construction we proceed as before and find the link rigidity curve. The line N N passing through the centroid G of the curve A R R_1 B R_1' R′ is then found by graphical or other methods, and the moment of inertia I_1 of the link rigidity curve is then found about the line N N.

In practice it is sufficient to apply the construction to one half only.

Then N N is the neutral axis line and

$$f_y = \frac{M R}{I_1} \cdot \frac{(y + y_o)}{(y + R)} \quad \ldots\ldots\ldots\ldots (17)$$

The rule for the line N N passing through the centroid of the link rigidity curve is

$$y_o = -\frac{\Sigma a_1 y}{\Sigma a_1}$$

$$\text{Now } a_1 = \frac{a \times \text{RQ}}{\text{PQ}} = \frac{a.\,R}{R + y}$$

$$\therefore y_o = \frac{\sum \frac{a\,y}{R + y}}{\sum \frac{a}{R + y}}$$

This is the relation required by equation (14)

$$I_{NN} = \Sigma a_1 (y + y_o)^2$$

$$= R \sum \frac{(y + y_o)^2}{R + y} \cdot a$$

$$\therefore \text{In equation (16) } f_y = \frac{M\,R\,(y + y_o)}{I_1 \cdot (y + R)}$$

This formula looks simpler, but it involves the determination of y_0 and I_1, which are rather more troublesome than the calculations in the previous method.

SPECIAL CASES

(1) RECTANGULAR SECTION.—If the section is a rectangular one of breadth b and depth d, we may proceed by mathematical analysis as follows—

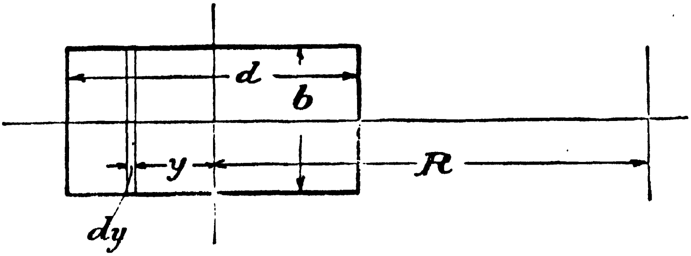


FIG. 253.

$$A_L = -\int_{-\frac{d}{2}}^{+\frac{d}{2}} \frac{b\,y}{R+y} \cdot d\,y$$

$$= -b\left(\int_{-\frac{d}{2}}^{+\frac{d}{2}}\left(1-\frac{R}{R+y}\right)d\,y\right)$$

$$= -b\Big[y - R\log_e (y+R)\Big]_{-\frac{d}{2}}^{+\frac{d}{2}}$$

$$= -b\left(d - R\log_e \frac{2R+d}{2R-d}\right)$$

$$= b\left(-d + R\log_e \frac{2R+d}{2R-d}\right)$$

$$\therefore\ L = \frac{A_L}{b\,d} = -1 + \frac{R}{d}\log_e \cdot \frac{2R+d}{2R-d} = \frac{R}{d}\log_e \frac{1+\dfrac{d}{2R}}{1-\dfrac{d}{2R}} - 1$$

$$= \frac{R}{d}\left\{\log_e\left(1 + \frac{d}{2R}\right) - \log_e\left(1 - \frac{d}{2R}\right)\right\} - 1$$

$$= \frac{R}{d}\left\{\left(\frac{d}{2R} - \frac{d^2}{2(2R)^2} + \frac{d^3}{3(2R)^3} - \frac{d^4}{4(2R)^4} + \frac{d^5}{5(2R)^5} + \frac{d^6}{6(2R)^6} - \frac{d^7}{7(2R)^7}\cdots\right)\right.$$
$$\left. - \left(- \frac{d}{2R} - \frac{d^2}{2(2R)^2} - \frac{d^3}{3(2R)^3} - \frac{d^4}{4(2R)^4} - \frac{d^5}{5(2R)^5} - \frac{d^6}{6(2R)^6} - \frac{d^7}{7(2R)^7}\right)\cdots\right\}^{*} - 1$$

$$= \frac{R}{d}\left\{\frac{d}{R} + \frac{1}{12}\left(\frac{d}{R}\right)^3 + \frac{1}{80}\left(\frac{d}{R}\right)^5 + \frac{1}{448}\left(\frac{d}{R}\right)^7\right\} - 1$$

$$= \frac{1}{12}\left(\frac{d}{R}\right)^2 + \frac{1}{80}\left(\frac{d}{R}\right)^4 + \frac{1}{448}\left(\frac{d}{R}\right)^6 + \cdot\cdot$$

(2) CIRCULAR SECTION.

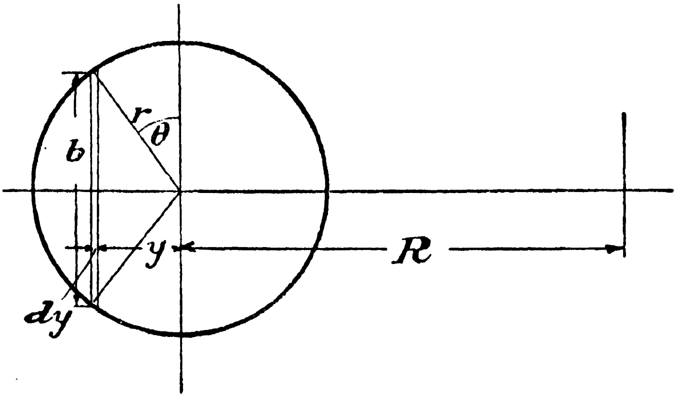


FIG. 254.

$$A_L = -\int_{-r}^{+r} \frac{b\,y\,dy}{R + y}$$

$$= -\left(\int_{-r}^{+r} b\,dy - \int_{r}^{+r} \frac{R\,b\,dy}{R + y}\right)$$

$$= \int_{-r}^{+r} \frac{R\,b\,dy}{R + y} - \int_{-r}^{+r} b\,dy \quad \ldots\ldots\ldots\ldots\ldots(1)$$

The second term $\int_{-r}^{+r} b\,dy$ = area of circle = πr^2

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* See, for instance, C. Smith's *Treatise on Algebra* (Macmillan), p. 383.

$$R\int_{-r}^{+r}\frac{b\,d\,y}{R+y}=R\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{2\,r\cos\theta\,d\,(r\sin\theta)}{R+r\sin\theta}$$

$$=2\,r^2\,R\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\cos^2\theta\,d\,\theta}{R+r\sin\theta}\quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\frac{\cos^2\theta}{R+r\sin\theta}=\frac{(1-\sin^2\theta)}{R+r\sin\theta}=\frac{1}{R+r\sin\theta}-\frac{1}{r}\left(\sin\theta-\frac{R\sin\theta}{R+r\sin\theta}\right)$$

$$=\frac{1}{R+r\sin\theta}-\frac{\sin\theta}{r}+\frac{R}{r}\left(\frac{\sin\theta}{R+r\sin\theta}\right)$$

$$=\frac{1}{R+r\sin\theta}-\frac{\sin\theta}{r}+\frac{R}{r^2}\left(1-\frac{R}{R+r\sin\theta}\right)$$

$$=\left(1-\frac{R^2}{r^2}\right)\frac{1}{R+r\sin\theta}-\frac{\sin\theta}{r}+\frac{R^2}{r^2}\quad\ldots\ldots\ldots\ldots\ldots(3)$$

$$\therefore\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\cos^2\theta\,d\,\theta}{R+r\sin\theta}=\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\left(1-\frac{R^2}{r^2}\right)\frac{d\,\theta}{R+r\sin\theta}-\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\sin\theta\,d\,\theta}{r}$$

$$+\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{R\,d\,\theta}{r^2}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\text{Now }-\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\sin\theta\,d\,\theta}{r}+\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{R\,d\,\theta}{r^2}=\left[\frac{\cos\theta}{r}+\frac{R\,\theta}{r^2}\right]_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}=0+\frac{R\,\pi}{r^2}\,..(4a)$$

$$\text{Again }\int\frac{d\,\theta}{R+r\sin\theta}=\int\frac{d\,\theta}{R+2\,r\sin\frac{\theta}{2}\,.\cos\frac{\theta}{2}}$$

$$=\int\frac{d\,\theta}{R+2\,r\tan\frac{\theta}{2}\,.\cos^2\frac{\theta}{2}}=\int\frac{\sec^2\frac{\theta}{2}\,d\,\theta}{R\sec^2\frac{\theta}{2}+2\,r\tan\frac{\theta}{2}}$$

$$=\int\frac{\sec^2\frac{\theta}{2}\,.\,d\,\theta}{R\left(1+\tan^2\frac{\theta}{2}\right)+2\,r\tan\frac{\theta}{2}}\qquad\left(\text{put }\tan\frac{\theta}{2}=u,\text{ then}\right.$$

$$d\,u=\tfrac{1}{2}\sec^2\frac{\theta}{2}\,d\,\theta$$

$$= \int \frac{2\,d\,u}{\mathrm{R}\,(1 + u^2) + 2\,r\,u} = 2\int \frac{d\,u}{\mathrm{R} + 2\,r\,u + \mathrm{R}\,u^2}$$

$$= \frac{2}{\mathrm{R}}\int \frac{d\,u}{\left(u^2 + \frac{2\,r}{\mathrm{R}}\,.\,u + 1\right)}$$

$$= \frac{2}{\mathrm{R}}\int \frac{d\,u}{\left(u + \frac{r}{\mathrm{R}}\right)^2 + \left(1 - \frac{r^2}{\mathrm{R}^2}\right)}$$

$$= \frac{2}{\mathrm{R}}\int \frac{d\,u}{\left(u + \frac{r}{\mathrm{R}}\right)^2 + \left(\sqrt{1 - \frac{r^2}{\mathrm{R}^2}}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

This is of the form $\int \frac{d\,x}{(x - a)^2 + \beta^2} = \frac{1}{\beta}\tan^{-1}\left(\frac{x - a}{\beta}\right)$*

$$\therefore \int \frac{d\,\theta}{\mathrm{R} + r\sin\theta} = \frac{2}{\mathrm{R}}\frac{1}{\sqrt{1 - \frac{r^2}{\mathrm{R}^2}}}\tan^{-1}\left(\frac{u + \frac{r}{\mathrm{R}}}{\sqrt{1 - \frac{r^2}{\mathrm{R}^2}}}\right)$$

When $\theta = \frac{\pi}{2}, \tan\frac{\theta}{2} = 1 = u$

" $\theta = -\frac{\pi}{2}, \tan\frac{\theta}{2} = -1 = u$

$$\therefore \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{d\,\theta}{\mathrm{R} + r\sin\theta}$$

$$= \frac{2}{\sqrt{\mathrm{R}^2 - r^2}}\left\{\tan^{-1}\left(\frac{\left(1 + \frac{r}{\mathrm{R}}\right)}{\sqrt{1 - \frac{r^2}{\mathrm{R}^2}}}\right) - \tan^{-1}\left(\frac{-1 + \frac{r}{\mathrm{R}}}{\sqrt{1 - \frac{r^2}{\mathrm{R}^2}}}\right)\right\}$$

$$= \frac{2}{\sqrt{\mathrm{R}^2 - r^2}}\left\{\tan^{-1}\frac{\mathrm{R} + r}{\sqrt{\mathrm{R}^2 - r^2}} + \tan^{-1}\frac{\mathrm{R} - r}{\sqrt{\mathrm{R}^2 - r^2}}\right\}$$

$$= \frac{2}{\sqrt{\mathrm{R}^2 - r^2}}\left\{\tan^{-1}\sqrt{\frac{\mathrm{R} + r}{\mathrm{R} - r}} + \tan^{-1}\sqrt{\frac{\mathrm{R} - r}{\mathrm{R} + r}}\right\} \quad \ldots.(6)$$

The portion inside the bracket is of the form

$$\tan^{-1} y + \tan^{-1}\frac{1}{y}$$

---

* See Lamb's *Infinitesimal Calculus* (Camb. Univ. Press), p. 172.

$$\text{If } \tan a = \frac{1}{y},\ \cot a = y = \tan\left(\frac{\pi}{2} - a\right)$$

$$\therefore \tan^{-1} y = \frac{\pi}{2} - a = \frac{\pi}{2} - \tan^{-1}\frac{1}{y}$$

$$\therefore \tan^{-1} y + \tan^{-1}\frac{1}{y} = \frac{\pi}{2}$$

$$\therefore \text{(6) becomes} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{d\,\theta}{\mathrm{R} + r\sin\theta} = \frac{\pi}{\sqrt{\mathrm{R}^2 - r^2}}$$

$$\therefore \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \left(1 - \frac{\mathrm{R}^2}{r^2}\right)\frac{d\,\theta}{\mathrm{R} + r\sin\theta} = -\frac{\mathrm{R}^2 - r^2}{r^2} \cdot \frac{\pi}{\sqrt{\mathrm{R}^2 - r^2}}$$

$$= -\frac{\pi\sqrt{\mathrm{R}^2 - r^2}}{r^2}$$

$$\therefore \text{from (4) and (4}a\text{)} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\cos^2\theta\,d\,\theta}{\mathrm{R} + r\sin\theta} = \frac{\mathrm{R}\,\pi}{r^2} - \frac{\pi\sqrt{\mathrm{R}^2 - r^2}}{r^2}$$

$$\therefore \text{in (2) R} \int_{-r}^{+r} \frac{b\,d\,y}{\mathrm{R} + y} = 2\,\pi\,\mathrm{R}^2 - 2\,\pi\,\mathrm{R}\sqrt{\mathrm{R}^2 - r^2}$$

$$\therefore \text{In (1) } \mathrm{A_L} = 2\,\pi\,\mathrm{R}^2 - 2\,\pi\,\mathrm{R}\sqrt{\mathrm{R}^2 - r^2} - \pi\,r^2$$

$$= \pi\,r^2\left\{\frac{2\,\mathrm{R}^2}{r^2} - \frac{2\,\mathrm{R}}{r}\sqrt{\left(\frac{\mathrm{R}}{r}\right)^2 - 1} - 1\right\} \quad \ldots\ldots(7)$$

Correction Coefficients for Ordinary Beam Formulæ.—If the bending moment on an ordinary beam is M, we have for a rectangular section the bending stress

$$f = \frac{6\,\mathrm{M}}{b\,d^2}$$

and for a circular section of diameter d

$$f = \frac{32\,\mathrm{M}}{\pi\,d^3}$$

Let f_i, f_o be the correct stresses at inside and outside respectively of the curve, then we may write

$$f_i = \alpha f$$
$$f_o = \beta f$$

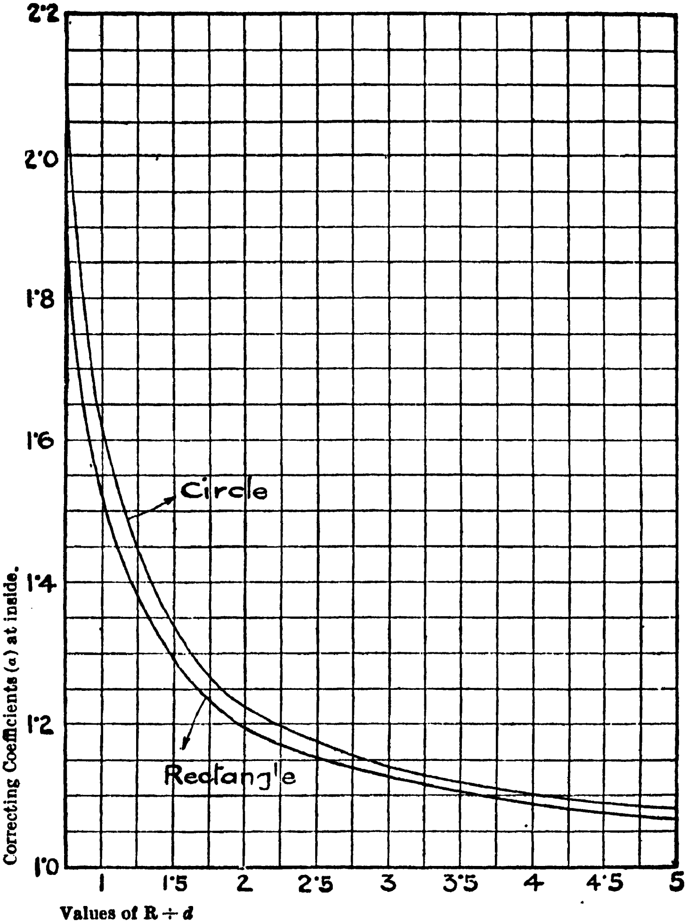

FIG. 255.—Correcting Coefficients at Inside.

where α and β are correcting coefficients by which the stresses calculated in the ordinary manner should be multiplied to give the corrected values.

The following values of α and β have been calculated—

$\frac{R}{d}$	Circular Section.		Rectangular Section.	
	α	β	α	β
·75	2·11	·62	1·92	·65
1	1·62	·70	1·52	·73
2	1·23	·84	1·20	·85
3	1·14	·89	1·12	·90
4	1·10	·91	1·09	·92
5	1·08	·93	1·07	·95

These figures are plotted in Figs. 255, 256.

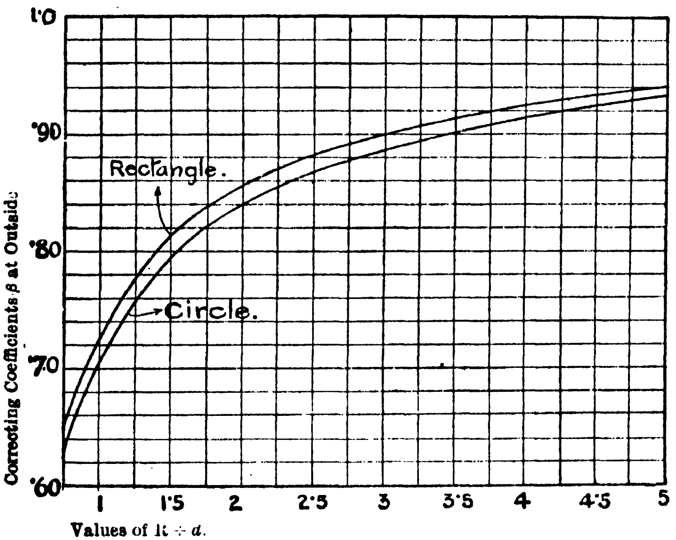

FIG. 256.—Correcting Coefficients at Outside.

The nearness of the curves for the two sections shows that the same coefficients may be used as a first approximation for other sections.

Andrews-Pearson Formula.—In this theory,* published in 1904 by the author and Professor Karl Pearson, F.R.S., a correction is made for the fact that owing to lateral strain it is not quite correct to say that $y = y_1$

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* *A Theory of the Stresses in Crane and Coupling Hooks*, Draper's Company Research Memoirs, Technical Series 1 (Dulan & Co., London). For other experimental investigations see "Maximum Stresses in Crane Hooks," by Professor Goodman, *Proc. Inst. C. E.*, Vol. CLXVII (1906-7); "An Investigation of Strength of Crane Hooks," *American Machinist*, Vol. 32, October 30, 1909.

The resulting formula is

$$f_v = \frac{M}{A R \gamma_2}\left\{\frac{1}{\left(1 + \frac{y}{R}\right)^{1+\eta}} - \gamma_1\right\}$$

where

$$A \gamma_1 = \sum \frac{a}{\left(1 + \frac{y}{R}\right)^{1+\eta}}$$

$$A \gamma_2 = -\sum \frac{\frac{y\,a}{R}}{\left(1 + \frac{y}{R}\right)^{1+\eta}}$$

For simplification of results we write

$$\gamma_3 = \gamma_1 - \gamma_2$$

For a rectangular section we have

$$\gamma_1 = \frac{R}{\eta\, d}\left\{\left(\frac{2\,R}{2\,R - d}\right)^{\eta} - \left(\frac{2\,R}{2\,R + d}\right)^{\eta}\right\}$$

$$\gamma_3 = \frac{R}{(1 - \eta)\, d}\left\{\left(\frac{2\,R + d}{2\,R}\right)^{1-\eta} - \left(\frac{2\,R - d}{2\,R}\right)^{1-\eta}\right\}$$

$$\text{Then } A \gamma_3 = \sum \frac{a}{\left(1 + \frac{y}{R}\right)^{\eta}}$$

A somewhat similar graphical construction can be employed to that in the Winkler formula, but still greater care has to be taken to ensure accuracy.

These formulæ are extremely troublesome to use and require the utmost care to avoid arithmetical error, and the additional accuracy over the Winkler formula is so small that it is doubtful if the revised formula is worth the additional trouble and risk of error. This point is dealt with very fully by Professor Morley in *Engineering*, September 11 and 25, 1914.

The Strength of Rings and Chain Links.—In determining the stresses in a ring or link we have, in addition to the problem of finding the stresses in a curved member, that of finding the bending moment at any point.

Approximate Theory of Circular Ring. — We will calculate the manner in which the bending moment varies in a ring in which the effect of the curvature on such variation is negligible. By designing the ring for the bending moment thus obtained, and making allowance for the curvature effect upon the stress by the correction coefficients given in Figs. 255, 256, we shall obtain a very fair approximation to the stresses.

Consider a radial section intersecting the centre line at C

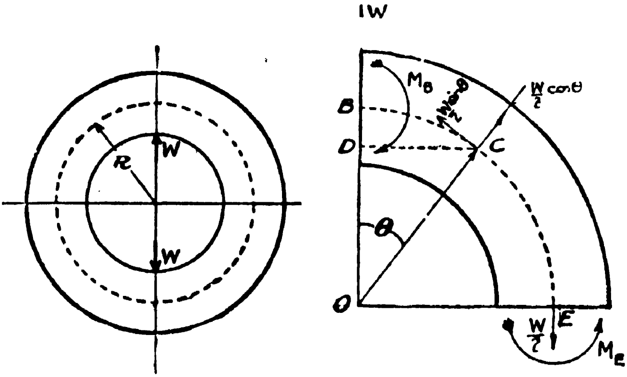

Fig. 257.—Circular Rings.

(Fig. 257). The stresses in each quadrant will be similar by symmetry.

The bending of the link must be such that the sections O E, O B remain parallel to their original positions.

But the angular change due to bending is given by the formula $\theta = \int \frac{M}{EI} ds$ (see p. 360)

$$\therefore \text{ we have } \int_B^E \frac{M\,ds}{EI} = 0 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Now the B.M. on the section

$$O\,C = M_c = \text{B.M. at } B - \frac{W}{2} . DC$$

$$= M_B - \frac{W}{2} . R \sin\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\therefore \int_{B}^{E} \frac{M\,ds}{EI} = \int_{0}^{\frac{\pi}{2}} \frac{\left(M_B - \frac{W}{2} R \sin\theta\right) . R\,d\theta}{EI}$$

$$= \frac{1}{EI}\left[M_B . R\theta + \frac{W}{2} R^2 \cos\theta\right]_{0}^{\frac{\pi}{2}}$$

$$= \left\{M_B R\left(\frac{\pi}{2} - 0\right) + \frac{W R^2}{2}(0 - 1)\right\} \times \frac{1}{EI}$$

$$= \left(M_B R \frac{\pi}{2} - \frac{W R^2}{2}\right) \frac{1}{EI}$$

$$\text{If this} = 0,\ M_B = \frac{W R}{\pi} = \cdot 318\ W R \ldots\ldots\ldots\ldots (3)$$

$$\therefore M_E = M_B - \frac{W R}{2} = W R\,(\cdot 318 - \cdot 5)$$

$$= -\cdot 182\ W R \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The point of zero bending moment is given by putting $M_O = 0$

$\therefore$ in equation (2) $\cdot 318\ W R - \cdot 5\ W R \sin\theta = 0$

$$\sin\theta = \frac{\cdot 318}{\cdot 5} = \cdot 636$$

i.e. $\theta = 39{\cdot}5$ degrees approximately.

The direct stress at B $= 0$; at O $= \dfrac{W \sin\theta}{2A}$; and at E $= \dfrac{W}{2A}$

The mean shear stress at B

$$= \frac{W}{2};\ \text{at O} = \frac{W\cos\theta}{2A} \text{ and at E} = 0$$

NUMERICAL EXAMPLE.—*Find the safe load upon a mild steel ring of* 4 *inches mean diameter formed of round rod* 1 *inch in diameter. We will take the safe tensile stress in the material as* 7 *tons per sq. in.*

Let the safe load be W

Then $M_B = \cdot 318\ W R = \cdot 636\ W$

$\therefore$ By ordinary bending formula

$$\frac{7 \times \pi \times 1^3}{32} = \cdot 636\ W$$

In our case $\frac{R}{d} = 2$

$\therefore$ from the table on p. 545 $a = 1{\cdot}23$

$$\therefore \cdot 636\,W \times 1{\cdot}23 = \frac{7\,\pi}{32}$$

$$\therefore W = \frac{7\,\pi}{32 \times \cdot 636 \times 1{\cdot}23} = \underline{\cdot 88 \text{ ton}} \text{ nearly.}$$

Considering the section at E and taking this value of W we have

$$\text{Direct stress} = \frac{W}{2\,A} = \frac{\cdot 88}{1{\cdot}57} = \cdot 56 \text{ ton per sq. in. approx.}$$

$$M_E = \cdot 182\,W\,R = \cdot 364\,W$$

$$\text{bending stress} = \frac{7 \times M_E}{M_B} = \frac{7 \times \cdot 364}{\cdot 636} = 4{\cdot}00 \text{ approx.}$$

$$\therefore \text{total stress} = 4{\cdot}00 + \cdot 56 = 4{\cdot}56 \text{ tons per sq. in.}$$

Chain Links with Straight Sides.—We can apply as follows the same approximate theory to the determination of the stresses in a chain link composed of semi-circular ends and straight sides. Considering one quarter of the link as before, we have that the angular change due to bending between the points B and E (Fig. 258) must be zero.

Between F and B we have as before

$$\text{Angular change} = \int_B^F \frac{M\,ds}{E\,I} = \left(M_B \cdot \frac{R\,\pi}{2} - \frac{W\,R^2}{2}\right)\frac{1}{E\,I}$$

The bending moment will be constant over the straight part of the link.

$$\therefore \text{Angular change between E and F} = \frac{M_E}{E\,I} \cdot \frac{L}{2}$$

$$\text{and } M_E = M_F = M_B - \frac{W\,R}{2} \text{ as before}$$

$$\therefore \text{we have} \left(M_B \cdot \frac{R\,\pi}{2} - \frac{W\,R^2}{2} + \frac{M_E\,L}{2}\right) \cdot \frac{1}{E\,I}$$

$$= \left(M_B \frac{R\,\pi}{2} - \frac{W\,R^2}{2} + \frac{M_B\,L}{2} - \frac{W\,R\,L}{4}\right) \cdot \frac{1}{E\,I}$$

If this is zero,

$$\frac{M_B}{2}(\pi R + L) = \frac{W R}{2}\left(R + \frac{L}{2}\right)$$

$$\therefore M_B = \frac{W R\left(R + \frac{L}{2}\right)}{\pi R + L}$$

$$= \frac{W R}{2}\left(\frac{2 R + L}{\pi R + L}\right) \quad \ldots\ldots\ldots\ldots (5)$$

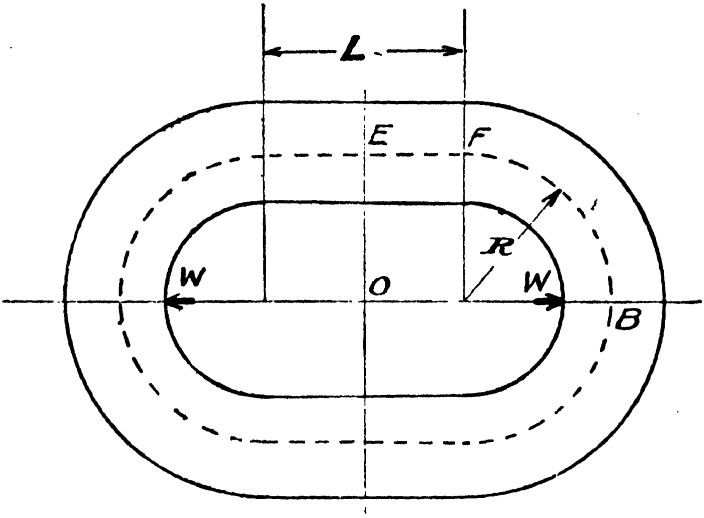

FIG. 258.—Oval Chain Links.

$$\therefore M_E = M_B - \frac{W R}{2}$$

$$= \frac{W R}{2}\left\{\frac{2 R + L}{\pi R + L} - 1\right\}$$

$$= \frac{W R}{2}\left(\frac{2 R - \pi R}{\pi R + L}\right) = \frac{W R^2}{2}\left(\frac{2 - \pi}{\pi R + L}\right) = -\frac{W R^2}{2}\left(\frac{1 \cdot 1416}{\pi R + L}\right) \ldots (6)$$

If L = 0 these formulæ reduce to the same result as in the previous case.

Experiments on Chain Links.—The strength of chain links has been investigated very thoroughly by Professors G. A. Goodenough and L. E. Moore,* who give a very complete theoretical treatment of the subject.

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* *University of Illinois Bulletin*, No. 18.

Their summary and conclusions are as follows—

1. The experiments on steel rings confirm the theoretical analysis employed in the calculation of stresses.

2. The experiments on various chain links confirm the analysis and show that the pressure may be taken as uniform over the arc of contact.

3. A load on the link produces an average intensity of stress $\frac{Q}{2A}$ in the cross section of the link containing the minor axis, and with an open link of usual proportions the maximum tensile stress is four times this value.

4. The introduction of a stud in the link equalises the stresses throughout the link, reduces the maximum tensile stresses about 20 per cent. and reduces the excessive compression stress at the end of the link about 50 per cent.

5. The stud-link chain of equal dimensions will, within the elastic limit, bear from 20 to 25 per cent. more load than the open-link chain, but the ultimate strength of the stud link is probably less than that of the open link.

6. In the formulæ for the safe loading of chains given by the leading authorities on machine design, the maximum stress to which the link is subjected seems to be underestimated and the constants are such as to give stresses from 30,000 to 40,000 lbs. per sq. in. for full load.

7. The following formulæ are applicable to chains of the usual form $\mathrm{P} = 0{\cdot}4\, d^2 s$ for open links

$\mathrm{P} = 0{\cdot}5\, d^2 s$,, stud ,,

where P = safe load, d = diameter of stock and s the maximum permissible tensile stress.

CHAPTER XX

* ROTATING DRUMS, DISKS AND SHAFTS

Thin Rotating Drum or Ring.—If a thin drum or ring of radius r (Fig. 259) rotates with a velocity v, there will be acting on each unit length of the ring a centripetal pressure p equal to $\frac{w\,v^2}{g\,r}$, where w is weight of unit length of the ring.

Thus pressure p will cause a hoop stress f and on any diametral section the resulting force tending to cause bursting of the ring will be equal to $p\,d$, as in the case of thin pipes dealt with on p. 115. The force resisting bursting will be equal to $f \times 2\,A$ where A is the cross-sectional area of the ring.

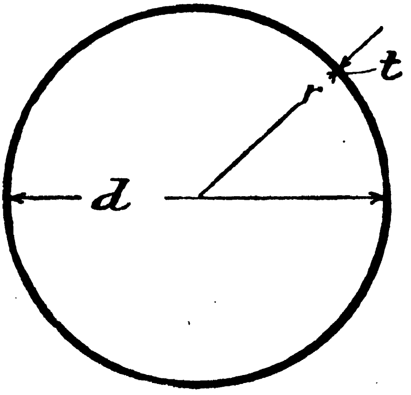

FIG. 259.—Thin Rotating Cylinder.

We have, therefore, $f \times 2\,A = p\,d = \frac{w\,v^2\,.\,d}{g\,r} = \frac{2\,w\,v^2}{g}$

$$\therefore f = \frac{2\,w\,v^2}{2\,g\,A} = \frac{w\,v^2}{g\,A}$$

Since w is the weight per unit length of the ring, we have, if ρ is the weight per unit volume of the material, $w = \rho A$, since the volume of a unit length of the ring is $1 \times A$

$$\therefore f = \frac{\rho v^2}{g}$$

If ρ is the weight in lbs. per cu. in. of the material, v is in feet per sec. and $g = 32{\cdot}2$ feet per second per second, we have, bringing everything to inch units

$$f = \frac{\rho v^2 \times 12 \times 12}{32{\cdot}2 \times 12} = \frac{12 \rho v^2}{32{\cdot}2} \text{ lbs. per sq. in.}$$

$$= \frac{v^2}{10} \text{ approx. for cast iron.}$$

$$= \frac{v^2}{9{\cdot}5} \text{ approx. for mild steel.}$$

This stress is often called the centrifugal stress.

Numerical Example.—*At what peripheral speed may a thin mild-steel ring be rotated if the centrifugal stress is not to exceed* 16,000 *lbs. per sq. in.?*

We then have

$$16{,}000 = \frac{v^2}{9{\cdot}5}$$

$$v^2 = 9{\cdot}5 \times 16{,}000$$

$$v = \sqrt{9{\cdot}5 \times 16{,}000} = 390 \text{ feet per sec. approx.}$$

Revolving Disk.—The consideration of the stresses in a revolving disk bears considerable resemblance to that of the stresses in a thick pipe, but it presents greater difficulty. If the disk is uniform in breadth and such breadth is comparatively small we may proceed as follows—

Considering, as in the case of Lamé's theory (p. 510), an elemental ring at radius x of thickness δx (Fig. 260) and of unit breadth, we have a resultant centrifugal tension on the section given by

$$F_c = \frac{w v^2}{g r} \cdot d = \frac{2 w v^2}{g}$$

If the angular velocity $= \omega$, $v = \omega x$

$$F_c = \frac{2\rho\omega^2 x^2}{g} . \delta x \text{ since } w = \rho \delta x$$

For convenience we will write $\frac{\rho}{g} = q$

$$\therefore F_c = 2 q \omega^2 x^2 \delta x \text{................} (1)$$

Then by the same reasoning as in Lamé's theory we shall have

Force tending to cause bursting of ring

$$= F_c + (p + \delta p) . 2 (x + \delta x)$$

Force resisting bursting of ring $= 2 f \delta x + p . 2 x$

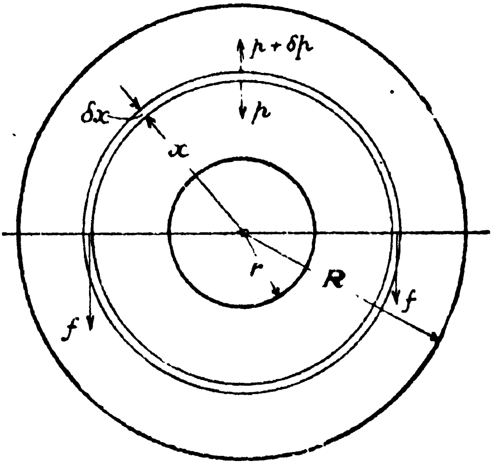

FIG. 260.

These must be equal

$$\therefore \frac{F_c}{2} + (p + \delta p)(x + \delta x) = f \delta x + p x$$

∴ neglecting products of small quantities

$$\frac{F_c}{2} + p x + p \delta x + x \delta p = f \delta x + p x$$

$$\therefore (f - p) \delta x = x \delta p + \frac{F_c}{2}$$

$$\therefore (f - p) = \frac{x \delta p}{\delta x} + \frac{F_c}{2 \delta x}$$

$$= \frac{x \delta p}{\delta x} + q \omega^2 x^2 \text{ (by (1))}$$

$$\therefore \; f = p + \frac{x\,\delta p}{\delta x} + q\,\omega^2 x^2 \quad \ldots\ldots\ldots\ldots(2)$$

$$\therefore \text{ In the limit } f = p + \frac{x\,d p}{d x} + q\,\omega^2 x^2 \quad \ldots\ldots\ldots\ldots(3)$$

Next consider the strains. If the radius x increases to $(x + u)$, the circumference increases from $2\,\pi\,x$ to $2\,\pi\,(x + u)$

$$\therefore \text{ increase in circumference } = 2\,\pi\,u$$

$$\therefore \text{ Unital circumferential strain } = \frac{2\,\pi\,u}{2\,\pi\,x} = \frac{u}{x}$$

Also the thickness of the ring increases from δx to $\delta x + \delta u$

$$\therefore \text{ Unital radial strain } = \frac{\delta u}{\delta x} = \frac{d u}{d x} \text{ in the limit.}$$

Now the principal stresses acting in an element of this ring are f and p

$\therefore$ (as shown on p. 25)

$$\text{Unital circumferential strain } = \frac{1}{\text{E}}\,(f - \eta\,p)$$

$$\text{Unital radial strain } = \frac{1}{\text{E}}\,(p - \eta\,f)$$

$$\text{i.e. } \frac{u}{x} = \frac{1}{\text{E}}\,(f - \eta\,p) \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\frac{d u}{d x} = \frac{1}{\text{E}}\,(p - \eta\,f) \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$\therefore$ solving these two simultaneous equations we have

$$f = \frac{\text{E}}{(1 - \eta^2)}\left(\frac{u}{x} + \frac{\eta\,d u}{d x}\right) \quad \ldots\ldots\ldots(6)$$

$$p = \frac{\text{E}}{(1 - \eta^2)}\left(\frac{\eta\,u}{x} + \frac{d u}{d x}\right) \quad \ldots\ldots\ldots(7)$$

Putting these results in (3) we have

$$\frac{\text{E}}{(1 - \eta^2)}\left(\frac{u}{x} + \frac{\eta\,d u}{d x}\right)$$

$$= \frac{\text{E}}{(1 - \eta^2)}\left(\frac{\eta\,u}{x} + \frac{d u}{d x}\right) + \frac{x\,\text{E}}{1 - \eta^2}\left(-\frac{\eta\,u}{x^2} + \frac{\eta}{x}\cdot\frac{d u}{d x} + \frac{d^2 u}{d x^2}\right)$$

$$+ q\,\omega^2 x^2$$

i.e. multiplying through by $\dfrac{\text{E}}{(1 - \eta^2)}$

$$\frac{u}{x} + \frac{\eta\,d u}{d x} = \frac{\eta\,u}{x} + \frac{d u}{d x} - \frac{\eta\,u}{x} + \frac{\eta\,d u}{d x} + \frac{x\,d^2 u}{d x^2} + \frac{(1 - \eta^2)}{\text{E}}\cdot q\,\omega^2 x^2$$

$$\therefore \frac{x\, d^2 u}{d x^2} + \frac{d u}{d x} - \frac{u}{x} + \frac{(1 - \eta^2)}{\text{E}} . q\, \omega^2 x^2 = 0 \quad \ldots\ldots (8)$$

To solve this differential equation we write it

$$\frac{x^2\, d^2 u}{d x^2} + \frac{x\, d u}{d x} - u + \frac{(1 - \eta^2)}{\text{E}} . q\, \omega^2 x^3 = 0 \quad \ldots . (9)$$

Now assume $u = \text{C}\, x^3$

Then $\frac{x^2\, d^2 u}{d x^2} + \frac{x\, d u}{d x} - u + \frac{(1 - \eta^2)}{\text{E}} . q\, \omega^2 x^3$

$$= x^2 . 6\, \text{C}\, x + x . 3\, \text{C}\, x^2 - \text{C}\, x^3 + \frac{(1 - \eta^2)}{\text{E}} . q\, \omega^2 x^3$$

$$= 8\, \text{C}\, x^3 + \frac{(1 - \eta^2)}{\text{E}} . q\, \omega^2 x^3$$

Equation (9) will be solved if $\text{C} = - \frac{(1 - \eta^2)\, q\, \omega^2}{8\, \text{E}}$ ....(10)

This equation is of the kind dealt with in Forsyth's *Differential Equations* (Macmillan), §§ 38, 39.

The "complementary function" is

$$\frac{x^2\, d^2 u}{d x^2} + \frac{x\, d u}{d x} - u = 0$$

$$\text{i.e.}\ \frac{d^2 u}{d x^2} + \frac{1}{x}\frac{d u}{d x} - \frac{u}{x^2} = 0$$

$$\text{i.e.}\ \frac{d^2 u}{d x^2} + \frac{d}{d x}\left(\frac{u}{x}\right) = 0$$

Integrating, we have

$$\frac{d u}{d x} + \frac{u}{x} = \text{constant} = \text{C}_1 \quad \ldots\ldots\ldots (11)$$

To integrate again, write it

$$\frac{x\, d u}{d x} + u = \text{C}_1 x$$

$$\text{i.e.}\ \frac{d\,(u\, x)}{d x} = \text{C}_1 x$$

$$\therefore \text{integrating}\ u\, x = \frac{\text{C}_1 x^2}{2} + \text{C}_2$$

$$\therefore \frac{u}{x} = \frac{\text{C}_1}{2} + \frac{\text{C}_2}{x^2} \quad \ldots\ldots\ldots\ldots\ldots (12)$$

$\therefore$ putting this in (11)

$$\frac{d u}{d x} = \frac{\text{C}_1}{2} - \frac{\text{C}_2}{x^2} \quad \ldots\ldots\ldots\ldots\ldots (13)$$

We also have from "particular integral" $u = \mathrm{C}\, x^3$

$$\frac{u}{x} = \mathrm{C}\, x^2$$

$$\frac{d\,u}{d\,x} = 3\,\mathrm{C}\, x^2$$

$\therefore$ adding the complementary and particular integral we have

$$\frac{u}{x} = \mathrm{C}\, x^2 + \frac{\mathrm{C}_1}{2} + \frac{\mathrm{C}_2}{x^2}$$

$$= -\frac{(1-\eta^2)\, q\, \omega^2 x^2}{8\,\mathrm{E}} + \frac{\mathrm{C}_1}{2} + \frac{\mathrm{C}_2}{x^2} \quad \ldots\ldots\ldots(14)$$

$$\frac{d\,u}{d\,x} = -\frac{3\,(1-\eta^2)\, q\, \omega^2 x^2}{8\,\mathrm{E}} + \frac{\mathrm{C}_1}{2} - \frac{\mathrm{C}_2}{x^2} \quad \ldots\ldots(15)$$

SPECIAL CASES.—(1) *External radius* R, *internal radius r*. We must have $p = 0$ at the inside and outside

$$\therefore\ p = 0 \text{ for } x = \mathrm{R} \text{ and } x = r$$

$\therefore$ at inside in equation (7)

$$0 = \frac{\mathrm{E}}{1-\eta^2}\left(\frac{\eta\, u}{r} + \frac{d\,u}{d\,r}\right)$$

$$i.e.\ 0 = \frac{\mathrm{E}}{(1-\eta^2)}\left\{-\frac{(1-\eta^2)\, q\, \omega^2 r^2}{8\,\mathrm{E}}\,.\,\eta + \frac{\mathrm{C}_1\,\eta}{2} + \frac{\mathrm{C}_2\,\eta}{r^2} - \frac{3\,(1-\eta^2)\, q\, \omega^2 r^2}{8\,\mathrm{E}} + \frac{\mathrm{C}_1}{2} - \frac{\mathrm{C}_2}{r^2}\right\}$$

$$i.e.\ \frac{\mathrm{C}_1}{2}(1+\eta) - \frac{\mathrm{C}_2}{r^2}(1-\eta) = \frac{(1-\eta^2)\, q\, \omega^2 r^2}{8\,\mathrm{E}}\,(3+\eta) \quad \ldots(16)$$

Similarly at outside where $x = \mathrm{R}$ we shall have

$$\frac{\mathrm{C}_1}{2}(1+\eta) - \frac{\mathrm{C}_2}{\mathrm{R}^2}(1-\eta) = \frac{(1-\eta^2)\, q\, \omega^2 \mathrm{R}^2}{8\,\mathrm{E}}\,(3+\eta)$$

$$\therefore\ \mathrm{C}_2\,(1-\eta)\left(\frac{1}{\mathrm{R}_2} - \frac{1}{r^2}\right) = \frac{(1-\eta^2)\, q\, \omega^2}{8\,\mathrm{E}}\,(3+\eta)\,\{r^2 - \mathrm{R}^2\}$$

$$\mathrm{C}_2 = \frac{(1-\eta^2)\, q^2\, \omega^2\,(3+\eta)\,(r^2-\mathrm{R}^2)\,\mathrm{R}^2 r^2}{(1-\eta)\,(r^2-\mathrm{R}^2)\,.\,8\,\mathrm{E}}$$

$$= \frac{(1+\eta)\,(3+\eta)\, q\, \omega^2\, \mathrm{R}^2 r^2}{8\,\mathrm{E}} \quad \ldots\ldots\ldots\ldots\ldots(17)$$

Putting this in (16) and simplifying

$$\mathrm{C}_1 = \frac{(1-\eta)\,(3+\eta)\, q\, \omega^2\,(\mathrm{R}^2 + r^2)}{4\,\mathrm{E}} \quad \ldots\ldots\ldots(18)$$

Now put these values into the equations (6), (14) and (15) for f

$$f = \frac{E}{(1-\eta^2)}\left(\frac{u}{x} + \frac{\eta\, d\, u}{d\, x}\right)$$

$$= \frac{E}{(1-\eta^2)}\left\{-\frac{(1-\eta^2)\, q\, \omega^2 x^2}{8\,E} + \frac{C_1}{2}(1+\eta) + \frac{C_2}{x^2}(1-\eta) - \frac{3\,\eta\,(1-\eta^2)\, q\, \omega^2 x^2}{8\,E}\right\}$$

$$= \frac{E}{(1-\eta^2)}\left\{\frac{(1-\eta^2)\, q\, \omega^2}{8\,E}\right\}\left\{-x^2 + (3+\eta)(R^2+r^2) + (3+\eta)\frac{R^2 r^2}{x^2} - 3\,\eta\, x^2\right\}$$

$$= \frac{q\,\omega^2}{8}\left\{(3+\eta)\left(R^2 + r^2 + \frac{R^2 r^2}{x^2}\right) - x^2(1+3\,\eta)\right\}$$

$$= \frac{\rho\,\omega^2}{8\,g}\left\{(3+\eta)\left(R^2 + r^2 + \frac{R^2 r^2}{x^2}\right) - x^2(1+3\,\eta)\right\} \;..(19)$$

Similarly to obtain p we use equations (7), (14) and (15) and take the given values of the constants.

$$p = \frac{E}{(1-\eta^2)}\left\{\frac{\eta\, u}{x} + \frac{d\, u}{d\, x}\right\}$$

$$= \frac{E}{(1-\eta^2)}\left\{-\frac{\eta\,(1-\eta^2)\, q\, \omega^2 x^2}{8\,E} + \frac{C_1}{2}(1+\eta) - \frac{C_2}{x^2}(1-\eta) - \frac{3\,(1-\eta^2)\; q\, \omega^2 x^2}{8\,E}\right\}$$

$$= \frac{E}{(1-n^2)}\left\{\frac{(1-\eta^3)\,.\, q\, \omega^2}{8\,E}\right\} - \eta\, x^2 + (3+\eta)(R^2+r^2) - (3+\eta)\frac{R^2 r^2}{x^2} - 3\,x^2$$

$$= \frac{\rho\,\omega^2(3+\eta)}{8\,g}\left\{R^2 + r^2 - \frac{R^2 r^2}{x^2} - x^2\right\} \;............(20)$$

It is clear from equation (19) that the greatest hoop tension occurs at the inside where $x = r$

$$\therefore\; f_{\text{max.}} = \frac{\rho\,\omega^2}{4\,g}\{(3+\eta)\,R^2 + (1-\eta)\,r^2\} \;......(21)$$

and if r is very small

$$f_{\text{max.}} = \frac{\rho\,\omega^2 R^2\,(3+\eta)}{4\,g} \;....................(22)$$

(2) *Disk without central hole.*—If the disk is a solid one of radius R, $p = 0$ for $x = \mathrm{R}$ and u the circumferential strain must be 0 for $x = 0$

$\therefore$ from equation (14) $\mathrm{C}_2 = 0$

$\therefore$ Equations (14) and (15) become

$$\frac{u}{x} = -\frac{(1-\eta^2)\,q\,\omega^2 x^2}{8\,\mathrm{E}} + \frac{\mathrm{C}_1}{2} \quad \ldots\ldots\ldots (23,$$

$$\frac{d\,u}{d\,x} = -\frac{3\,(1-\eta^2)\,q\,\omega^2 x^2}{8\,\mathrm{E}} + \frac{\mathrm{C}_1}{2} \quad \ldots\ldots\ldots\ldots (24)$$

$\therefore$ Since $p = 0$ for $x = \mathrm{R}$, equation (7) becomes

$$0 = \frac{\mathrm{E}}{(1-\eta^2)}\left\{\frac{\eta\,\mathrm{C}_1}{2} - \frac{\eta\,(1-\eta^2)\,q\,\omega^2\,\mathrm{R}^2}{8\,\mathrm{E}} + \frac{\mathrm{C}_1}{2} - \frac{3\,(1-\eta^2)\,q\,\omega^2\,\mathrm{R}^2}{8\,\mathrm{E}}\right\}$$

$$i.e.\ \frac{\mathrm{C}_1}{2}(1+\eta) = \frac{(1-\eta^2)}{8\,\mathrm{E}}\,q\,\omega^2\,\mathrm{R}^2\,(3+\eta)$$

$$\mathrm{C}_1 = \frac{(1-\eta)\,(3+\eta)\,q\,\omega^2\,\mathrm{R}^2}{4\,\mathrm{E}} \quad \ldots\ldots (25)$$

$\therefore$ Equation (6) becomes

$$f = \frac{\mathrm{E}}{(1-\eta^2)}\left\{-\frac{(1-\eta^2)\,q\,\omega^2 x^2}{8\,\mathrm{E}} + \frac{(1-\eta)\,(3+\eta)\,q\,\omega^2\,\mathrm{R}^2}{8\,\mathrm{E}} - \frac{3\,\eta\,(1-\eta^2)\,q\,\omega^2 x^2}{8\,\mathrm{E}} + \eta\,\frac{(1-\eta)\,(3+\eta)\,q\,\omega^2\,\mathrm{R}^2}{8\,\mathrm{E}}\right\}$$

$$= \frac{\mathrm{E}}{(1-\eta^2)}\cdot\frac{q\,\omega^2}{8\,\mathrm{E}}\,\{(1-\eta)\,(3+\eta)\,\mathrm{R}^2\,(1+\eta) - (1+3\eta)\,(1-\eta^2)\,x^2\}$$

$$= \frac{q\,\omega^2}{8}\,\{(3+\eta)\,\mathrm{R}^2 - (1+3\,\eta)\,x^2\}$$

$$= \frac{\rho\,\omega^2}{8\,g}\,\{(3+\eta)\,\mathrm{R}^2 - (1+3\,\eta)\,x^2\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

Similarly equation (7) becomes

$$p = \frac{\rho\,\omega^2}{8\,g}\,(3+\eta)\,(\mathrm{R}^2 - x^2) \quad \ldots\ldots\ldots\ldots\ldots (26)$$

Each of these is a maximum at the centre where $x = 0$, where we have

$$f_{\max.} = p_{\max.} = \frac{\rho\,\omega^2\,\mathrm{R}^2}{8\,g}\,(3+\eta) \quad \ldots\ldots\ldots\ldots\ldots (27)$$

This is exactly one half the value given by equation (22) for a disk with a very small hole, so that on this theory a disk has its strength reduced by one half by having a hole made through it. For this reason the De Laval turbine drums are made without a central hole.

If v is the peripheral speed at the outside we have $v = \omega R$

$\therefore$ Equation (27) gives $f = \frac{\rho v^2}{8g}(3 + \eta)$ which is $\frac{(3+\eta)}{8}$ of the stress in a thin drum of the same external radius (cf. p. 553).

$$\text{Taking } \eta = \tfrac{1}{4},\ \frac{(3+\eta)}{8} = \frac{13}{32}$$

NUMERICAL EXAMPLE.—*At what peripheral speed may a narrow mild-steel disk be rotated so that the maximum tensile stress shall not exceed* 16,000 *lbs. per sq. in.* (*a*) *if it has a small hole in the centre,* (*b*) *if it is quite solid?*

(*a*) By equation (22)

$$16{,}000 = \frac{\rho v^2}{4g} \,.\, (3{\cdot}25)$$

$$= \frac{v^2}{9{\cdot}5} \times \frac{3{\cdot}25}{4}$$

$$\therefore v^2 = \frac{9{\cdot}5 \times 16{,}000 \times 4}{3{\cdot}25}$$

$$v = \sqrt{\frac{9{\cdot}5 \times 16{,}000 \times 4}{3{\cdot}25}} = 432 \text{ feet per sec. approx.}$$

(*b*) by equation (27)

$$16{,}000 = \frac{\rho v^2}{8g} \,.\, 3{\cdot}25$$

$$\therefore v^2 = 432\sqrt{2} = 612 \text{ feet per sec. approx.}$$

Rotating Disk of Uniform Strength.—A problem which is of interest in turbine design is that of finding the shape of a disk which will have the same stress throughout. We have seen already that the maximum stress occurs at the centre so that the disk ought to be broader at the centre than at the edge.

Referring to Fig. 261 and considering the elemental ring of breadth b at radius x increasing to $(b + \delta b)$ at radius $(x + \delta x)$ we have the principal stresses each equal to f which does not vary radially.

Force tending to burst ring $= F_c + 2f\,(x + \delta x)\,(b + \delta b)$

Force resisting bursting $= 2f\,x\,b + 2f\,\delta x\left(b + \frac{\delta b}{2}\right)$

∴ If these are equal, neglecting products of small quantities we have

$$F_c + 2f\,x\,b + 2f\,b\,\delta x + 2f\,x\,\delta b = 2f\,b\,x + 2f\,b\,\delta x$$

$$F_c + 2f\,x\,\delta b = 0$$

$$\text{Now } F_c = \frac{w\,v^2}{g\,x}\,.\,2x = \frac{2\rho\left(b + \frac{\delta b}{2}\right)\delta x\,.\,\omega^2 x^2}{g}$$

$$= \frac{2\rho\,b\,\delta x\,.\,\omega^2 x^2}{g}$$

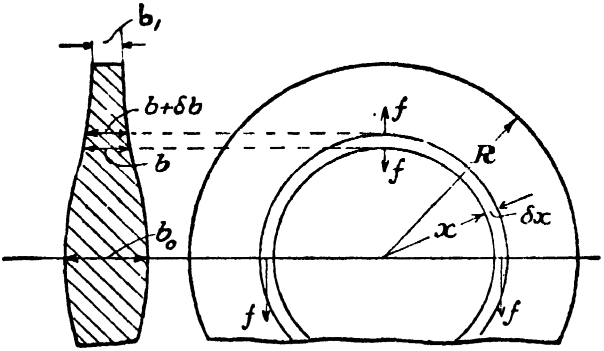

FIG. 261.—Disks of Uniform Strength.

∴ Dividing by $2\,x\,f$

$$\frac{\rho\,b\,\omega^2}{f\,g}\,.\,x\,\delta x + \delta b = 0$$

∴ in the limit

$$\frac{\rho\,b\,\omega^2}{f\,g}\,.\,x + \frac{d\,b}{d\,x} = 0$$

The solution of this is $b = C\,e^{-\frac{\rho}{g}\cdot\frac{\omega^2}{2f}\cdot x^2}$

At the centre where $x = 0$

$$b = C = b_0$$

$$\therefore b = b_0\,e^{-\frac{\rho}{g}\cdot\frac{\omega^2}{2f}\cdot x^2}$$

The thickness at the outside is given by

$$b_1 = b_0\, e^{-\frac{\rho}{g}\cdot\frac{\omega^2}{2f}\cdot R^2}$$

If therefore b_1 is given we can calculate b_0 by

$$b_0 = b_1\, e^{+\frac{\rho\,.\,\omega^2}{g\,2f}R^2}$$

$$\therefore \text{At any radius}\,{}^{r}_{b} = b_1\, e^{\frac{\rho\,\omega^2}{g\,.\,2f}(R^2 - x^2)}$$

Whirling of Rotating Shafts; Critical Speeds.—If a shaft rotates at a high speed, the lack of mathematically exact balancing results in an eccentricity of load which causes centrifugal forces to be induced and these centrifugal forces will cause deflections which increase the eccentricity; this increased eccentricity causes further deflection and so on, the deflection increasing indefinitely and giving rise to *whirling* at certain speeds called *critical speeds.*

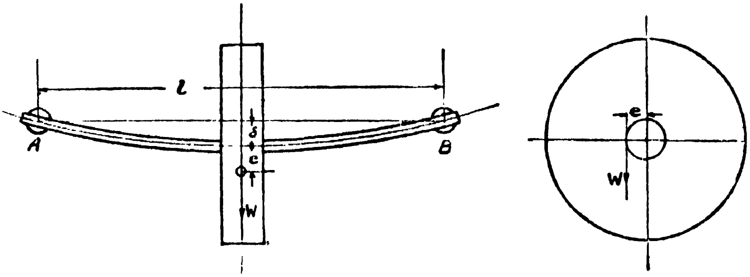

FIG. 262.—Whirling of Centrally Loaded Shaft.

In certain cases the whirling speed is the same as the natural frequency of transverse vibration of the shaft.

The centrifugal forces may be regarded as having a neutralising effect upon the elastic forces tending to return the shaft to its natural shape, so that when whirling occurs the effective stiffness of the shaft is reduced to zero.

FLEXIBLE SHAFT LOADED AT CENTRE.—Referring to Fig. 262 let a shaft A B of length l be loaded at the centre with a disk of weight W and let the shaft be provided with flexible bearings which do not interfere with the natural deflection of the shaft. Then if δ is the deflection caused by the centrifugal force and e is the eccentricity of the load, *i.e.* the distance from the centre of the shaft to the centre of gravity of the load, the effective eccentricity is $(\delta + e)$. If the angular velocity is ω

$$\therefore \text{Centrifugal force} = \text{F} = \frac{\text{W}\,\omega^2\,(\delta + e)}{g}$$

$$\text{Also } \delta = \frac{\text{F}\,l^3}{48\,\text{E}\,\text{I}} \quad i.e.\ \text{F} = \frac{48\,\text{E}\,\text{I}\,\delta}{l^3}$$

$$\therefore \frac{48\,\text{E}\,\text{I}\,\delta}{l^3} = \frac{\text{W}\,\omega^2}{g}(\delta + e)$$

$$\therefore \delta\left(\frac{48\,\text{E}\,\text{I}}{l^3} - \frac{\text{W}\,\omega^2}{g}\right) = \frac{\text{W}\,\omega^2\,e}{g}$$

$$\therefore \delta = \frac{\text{W}\,\omega^2\,e}{g} \div \frac{48\,\text{E}\,\text{I}}{l^3} - \frac{\text{W}\,\omega^2}{g}$$

$$= \frac{\text{W}\,\omega^2\,l^3\,e}{48\,\text{E}\,\text{I}\,g - \omega^2\,\text{W}\,l^3} \quad \ldots\ldots\ldots\ldots(1)$$

From this equation it is clear that δ will become indefinitely great if $48\,\text{E}\,\text{I}\,g - \omega^2\,\text{W}\,l^3 = 0$

$$i.e.\ \text{if } \omega = \sqrt{\frac{48\,\text{E}\,\text{I}\,.\,g}{\text{W}\,l^3}} \quad \ldots\ldots\ldots\ldots(2)$$

This value of ω gives the critical speed.

For mild steel $\text{E} = 30 \times 10^6$ lbs. per sq. in. and $g = 32{\cdot}2 \times 12$ ins. per sec. per sec.; if therefore W is in lbs. and I and l in inch units

$$\omega = \sqrt{\frac{48 \times 30 \times 10^6 \times 32{\cdot}2 \times 12\,.\,\text{I}}{\text{W}\,l^3}}$$

$$= 746{,}000\sqrt{\frac{\text{I}}{\text{W}\,l^3}} \text{ radians per sec} \ldots\ldots\ldots\ldots(3)$$

$$n = \frac{746{,}000 \times 60}{2\,\pi}\sqrt{\frac{\text{I}}{\text{W}\,l^3}}$$

$$= 7{\cdot}11 \times 10^6\sqrt{\frac{\text{I}}{\text{W}\,l^3}} \text{ revolutions per minute} \ldots\ldots(4)$$

For a round shaft of diameter d inches we have $\text{I} = \frac{\pi\,d^4}{64}$

$$\therefore n = \frac{1{\cdot}58 \times 10^6\,d^2}{\sqrt{\text{W}\,l^3}}$$

Working from the transverse vibration we have, as on p. 337,

$$t = 2\pi\sqrt{\frac{\text{Weight}}{g \times \text{force to cause unit displacement}}}$$

$$\delta = \frac{W l^3}{48 E I}$$

$$\therefore \text{Force} = W = \frac{48 E I}{l^3} \text{ for unit deflection}$$

$$\therefore t = 2\pi\sqrt{\frac{W l^3}{48 E I g}}$$

Frequency

$$= \frac{1}{t} = \frac{1}{2\pi}\sqrt{\frac{48 E I g}{W l^3}} \text{ per second.}$$

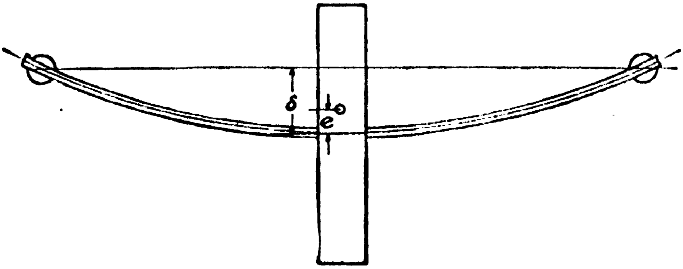

FIG. 263.

This, for the given value of E, is exactly the same result as is obtained in equation (4) above.

If the critical speed is exceeded either by providing guides which prevent the excessive deflection or by speeding up so quickly that the inertia of the shaft prevents the dangerous deflections from developing, the shaft will "settle down" and run smoothly in a deflected form (Fig. 263), the weight rotating about an axis which gradually approaches its centre of gravity as the speed increases. This fact is made use of in the flexible shaft of the De Laval turbine.

If ω_c is the critical velocity we may put in equation (1) $\omega_c^2 W l^3 = 48 E I g$

$$i.\,e.\ \delta = \frac{W\,\omega^2\,l^3\,e}{W\,l^3\,\omega_c^2 - W\,l^3\,\omega^2} = \frac{\omega^2\,e}{\omega_c^2 - \omega^2}$$

$$= -\frac{\omega^2}{\omega^2 - \omega_c^2}\,.\,e \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

This gets numerically less as ω increases, so that as the speed increases more and more the shaft tends to straighten out.

If the shaft is horizontal and the weight is perfectly balanced, and there is an initial deflection $\delta_o = \frac{W\,l^3}{48\,E\,I}$, this will give rise to a centrifugal force which causes an additional deflection δ_1

$$\therefore F = \frac{W\,\omega^2}{g}\,(\delta_1 + \delta_o)$$

$$\text{but } \delta_1 = \frac{F\,l^3}{48\,E\,I}$$

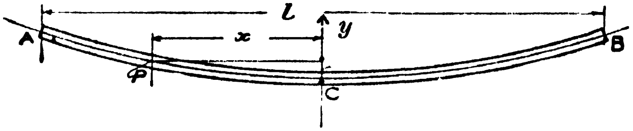

FIG. 264.—Whirling of Unloaded Shaft.

$$\therefore \frac{48\,E\,I\,\delta_1}{l^3} = \frac{W\,\omega^2}{g}\,.\,\delta_1 + \frac{W\,\omega^2}{g}\,.\,\delta_o$$

$$i.\,e.\ \delta_1\left(\frac{48\,E\,I}{l^3} - \frac{W\,\omega^2}{g}\right) = \frac{W\,\omega^2}{g}\,.\,\delta_o$$

$$\text{or } \delta_1 = \frac{W\,\omega^2}{g}\,.\,\delta_o \div \frac{48\,E\,I}{l^3} - \frac{W\,\omega^2}{g}$$

$$= \frac{W\,\omega^2\,l^3\,\delta_o}{48\,E\,I\,g - \omega^2\,W\,l^3} \quad \ldots\ldots(6)$$

This, as one would expect, gives the same result as before with $e = \delta_o$.

UNLOADED SHAFT.—In this case we obtain at certain critical speeds a condition of instability which is very similar to that which occurs in a loaded column.

Suppose that the shaft of length l is initially straight and that due to some cause it becomes deflected so that at some point P (Fig. 264) at distance x from a convenient origin, say the centre point O, the deflection is y.

If w is the weight per unit length of the shaft this deflection will cause at the point P a load equal to $\frac{w\,\omega^2\,y}{g}$ per unit length.

But since the B.M. diagram is the second integral of the load diagram (p. 149) we shall have

$$\frac{d^2\,\mathrm{M}}{d\,x^2} = \frac{w\,\omega^2\,y}{g}$$

Moreover, $\frac{\mathrm{M}}{\mathrm{E\,I}} = \frac{d^2\,y}{d\,x^2}$

$$\therefore \frac{w\,\omega^2\,y}{\mathrm{E\,I}\,g} = \frac{d^4\,y}{d\,x^4}$$

putting $\frac{w\,\omega^2}{\mathrm{E\,I}\,g} = m^4$, we get the differential equation

$$m^4\,y = \frac{d^4\,y}{d\,x^4} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The general solution of this is

$$y = \mathrm{A}\cosh m\,x + \mathrm{B}\sinh m\,x + \mathrm{C}\cos m\,x + \mathrm{D}\sin m\,x \quad ..(2)$$

Ends freely supported.—If the ends are freely supported, the deflection and B.M. are each zero at each end.

$\therefore y = 0$ when $x = -\frac{l}{2}$ and $x = +\frac{l}{2}$

$\frac{d^2\,y}{d\,x^2} = 0$ when $x = -\frac{l}{2}$ and $x = \frac{l}{2}$

also the slope $\frac{d\,y}{d\,x} = 0$ when $x = 0$

$$\therefore 0 = \mathrm{A}\cosh -\frac{m\,l}{2} + \mathrm{B}\sinh -\frac{m\,l}{2} + \mathrm{C}\cos -\frac{m\,l}{2} + \mathrm{D}\sin -\frac{m}{2}$$

$$0 = \mathrm{A}\cosh\frac{m\,l}{2} + \mathrm{B}\sinh\frac{m\,l}{2} + \mathrm{C}\cos\frac{m\,l}{2} + \mathrm{D}\sin\frac{m\,l}{2}$$

Now $\cosh\frac{m\,l}{2} = \cosh -\frac{m\,l}{2}$; $\cos\frac{m\,l}{2} = \cos -\frac{m\,l}{2}$

$\sinh -\frac{m\,l}{2} = -\sinh\frac{m\,l}{2}$; $\sin -\frac{m\,l}{2} = -\sin\frac{m\,l}{2}$

D and B each $= 0$

$$\therefore \mathrm{A}\cosh\frac{m\,l}{2} + \mathrm{C}\cos\frac{m\,l}{2} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$\therefore$ our equation becomes

$$y = \text{A} \cosh m x + \text{C} \cos m x$$

$$\text{Now } \frac{d \cosh x}{d x} = \sinh x \, ; \quad \frac{d \cos x}{d x} = -\sin x$$

$$\frac{d \sinh x}{d x} = \cosh x \, ; \quad \frac{d \sin x}{d x} = \cos x$$

$$\therefore \frac{d^2 y}{d x^2} = \text{A} \, m^2 \cosh m x - \text{C} \, m^2 \cos m x$$

$\therefore$ putting $x = +\frac{l}{2}$ or $-\frac{l}{2}$

$$\therefore 0 = \text{A} \, m^2 \cosh \frac{m l}{2} - \text{C} \, m^2 \cos \frac{m l}{2}$$

$$i.e. \ \text{A} \cosh \frac{m l}{2} - \text{C} \cos \frac{m l}{2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$\therefore$ Comparing this with (3) we see that A must $= 0$

$$\therefore y = \text{C} \cos m x$$

$$\text{Further, C} \cos \frac{m l}{2} = 0$$

$$\therefore \frac{m l}{2} = \frac{\pi}{2} \text{ etc.}$$

Taking the lowest value

$$l = \frac{\pi}{m} \cdot = \frac{\pi}{\left(\frac{w \, \omega^2}{\text{E I} g}\right)^{\frac{1}{4}}}$$

$$\therefore \omega^2 = \frac{\pi^4}{l^4} \cdot \frac{\text{E I} g}{w}$$

$$\omega = \frac{\pi^2}{l^2} \sqrt{\frac{\text{E I} g}{w}} \text{ radians per second}$$

$\therefore$ If n is the number of revolutions per minute

$$\omega = \frac{2 \pi n}{60} = \frac{\pi n}{30} \quad \text{i. e. } n = \frac{30 \, \omega}{\pi}$$

$$\therefore n = \frac{30 \, \pi}{l^2} \sqrt{\frac{\text{E I} g}{w}} \text{ revolutions per minute} \ldots\ldots (5)$$

If the shaft is of diameter d inches and l is in inches

Taking $\text{E} = 30 \times 10^6$ lbs. per sq. in.

$g = 32{\cdot}2 \times 12$ ins. per sec. per sec.

$$I = \frac{\pi d^4}{64}$$

$$w = \frac{\pi d^2}{4} \times \cdot 28 \text{ lb.}$$

$$\text{we have } n = \frac{4\cdot 8 \times 10^6 d}{l^2} \quad \dots\dots\dots\dots\dots\dots (6)$$

The higher critical speeds will occur for $m = \frac{2\pi}{2}, \frac{3\pi}{2}, \frac{4\pi}{2}$ etc., giving values of n multiplied by 4, 9, 16, etc.

Both ends fixed.—In this case $\frac{dy}{dx} = 0$ for $x = \pm \frac{l}{2}$ in addition to the condition that $y = 0$ for $x = \frac{l}{2}$

$\therefore$ B and D are each equal to 0 as before

$$\text{and } A \cosh \frac{ml}{2} + C \cos \frac{ml}{2} = 0$$

$$\therefore \text{ when } x = \frac{l}{2}, \frac{dy}{dx} = m A \sinh \frac{ml}{2} - m C \sin \frac{ml}{2} = 0$$

$$\text{i.e. } A \sinh \frac{ml}{2} - C \sin \frac{ml}{2} = 0$$

$$\therefore \frac{A}{C} = \frac{\sin \frac{ml}{2}}{\sinh \frac{ml}{2}}$$

$$\text{Also from (3) } \frac{A}{C} = - \frac{\cos \frac{ml}{2}}{\cosh \frac{ml}{2}}$$

$$\therefore - \tan \frac{ml}{2} = \tanh \frac{ml}{2}$$

The solution of this gives $ml = 4\cdot 74 = \frac{3\pi}{2}$ approx.

Taking the approximate value, this will give the first critical speed about nine times that in the case of the freely supported shaft.

Dunkerley's Empirical Formulæ.—Professor Dunkerley,[*] who was one of the first investigators of the theoretical and actual whirling speeds of shafts, has given the following empirical formulæ which agreed very well with his experiments. Let ω_1 be the critical angular velocity for a given unloaded shaft; let ω_2 be the critical angular velocity of the same shaft carrying a wheel at any position, neglecting the mass of the shaft. Then the critical velocity w_o of the loaded shaft will be given by

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[*] *Phil. Trans. Roy. Soc.* 1895, Liverpool Engineering Society, 1894-5.

$$\omega_o = \frac{\omega_1 \, \omega_2}{\sqrt{\omega_1^2 + \omega_2^2}}$$

or if n_o, n_1, n_2 are the corresponding number of revolutions per minute

$$\frac{1}{n_o^2} = \frac{1}{n_1^2} + \frac{1}{n_2^2}$$

If a second load be keyed at another position, the critical angular velocity of which is ω_3 with the first load removed neglecting the weight of the shaft, then

$$\omega_o = \frac{\omega_1 \, \omega_2 \, \omega_3}{\sqrt{\omega_1^2 \, \omega_2^2 + \omega_1^2 \, \omega_3^2 + \omega_2^2 \, \omega_3^2}}$$

$$\text{or in general } \frac{1}{n_o^2} = \sum \frac{1}{n^2}$$

For further information on this subject, the reader may refer to Professor Dunkerley's paper and to Stodola's *Steam Turbines* (Constable).

EXERCISES

CHAPTER I

1. A tie rod in a roof structure has to stand a total pull of 40 tons. If the stress in the material is to be not greater than 5 tons per sq. in., find a suitable diameter. (A.M.I.C.E.) *Ans.* 3¼ *ins. diam.*

2. Taking the shearing strength of mild steel to be 20 tons per sq. in., calculate the force necessary to punch a ¾ in. hole in a ⅝ in. plate. Find also the stress in the punch. *Ans.* 29·4 *tons ;* 66·7 *tons per sq. in.*

3. A bar of mild steel ¾ in. diam. and 10 ins. long stretches ·00816 in. when carrying a load of 5 tons. Calculate Young's modulus (E) in lbs. per sq. in. *Ans.* 30×10^6 *lbs. per sq. in.*

4. If E is 29,000,000 lbs. per sq. in. for wrought iron, what decrease in length of a column 20 ft. high and 12 sq. ins. sectional area takes place when carrying a load of 36 tons? *Ans.* ·0556 *in.*

5. What load in lbs. is hung on an iron wire 50 ft. long and ·1 in. diameter to make it stretch $\frac{1}{5000}$ in.? *Ans.* ·076 *lb.*

6. Plot a stress-strain diagram for the following test of a specimen from a mild-steel boiler plate—

Load lbs... ..	4,000	8,000	12,000	16,000	20,000	24,000	28,000
Extension ins...	·0009	·0020	·0033	·0044	·0056	·0070	·0082

Load lbs... ..	30,000	34,000	36,000	40,000	44,000	48,000	52,000
Extension ins...	·0103	·016	·7	·19	·30	·47	·75

Load lbs... ..	56,000	59,780	54,900
Extension ins...	1·3	2·5	2·9

Scales { Loads—1″ = 10,000 lbs.
Extensions—up to yield point 500 times full size.
Beyond = 4 times do.

Orig. dimens. Length = 10 ins., width = 1·753 ins., thickness = ·64 in.
Final „ „ = 12·9 ins. „ = 1·472 ins. „ = ·482 in.
Find stress at elastic limit, maximum stress, Young's modulus, and percentage extension and reduction of area.

7. In a plate girder the maximum intensity of stress at right angles to the vertical cross section of the web is 5 tons per sq. in., and the intensity of shearing stress is 2 tons per sq. in. Find the position of the planes of principal stress at that point and their intensities. (A.M.I.C.E.)

Ans. 19° 20′ *and* 70° 40′ *to vertical;* 5·7 *and* 0·7 *tons per sq. in.*

8. The limit of elasticity of a W.I. bar was found to be 20,000 lbs. per sq. in., the strain at that point being 0·0006; what was the resilience of the material? (A.M.I.C.E.) *Ans.* 6 *in. lbs.*

9. Two rods, one of copper and the other of steel, are fixed at their top ends, 24 ins. from one another, and hang vertically downwards. They are connected at their bottom ends by a horizontal cross-bar, and on this bar is to be placed a weight of 2000 lbs. If each rod is 18 ins. long, and if the diameter of the copper rod is 1 in. and of the steel rod ¾ in., find where the weight must be placed so that the cross-bar may remain horizontal. E for copper $= 16 \times 10^6$ lbs. per sq. in.; for steel $= 29 \times 10^6$ lbs. per sq. in. (B.Sc. Lond.) *Ans.* 11·9 *ins. from the steel rod.*

10. A load of 560 lbs. falls through ½ in. on to a stop at the lower end of a vertical bar 10 ft. long and 1 sq. in. in section. If E = 13,000 tons per sq. in., find the stresses produced in the bar.

Ans. 5·45 *tons per sq. in.*

11. A bar of iron is at the same time under a direct pull of 5000 lbs. per sq. in., and a shearing stress of 3,500 lbs. per sq. in. What will be the resultant tensile stress in the material? *Ans.* 6,800 *lbs. per sq. in.*

12. In Question 11, find the resultant tensile stress from the strain consideration. *Ans.* 7,250 *lbs. per sq. in.*

13. Find whether, in the problem of Questions 11 and 12, on the assumption that the shear strength of the material is ⅘ of the tensile strength, the resultant shear stress is more serious than the resultant tensile stress or strain.

Ans. Res. shear stress = 4,300 *lbs. per sq. in. Not so serious.*

14. Steel rails are welded together and are unstressed at a temperature of 60° F. They are prevented from buckling and cannot expand or contract. Find the stresses when the temperature is: (1) 20° F., (2) 120° F., taking steel as expanding ·0012 of its length for a temperature change of 180° F. $E = 30 \times 10^6$ lbs. per sq. in. If the elastic limit is 40,000 lbs./in.^{2}, at what temperature would it be reached? (A.M.I.C.E.)

Ans. 8000, 12,000 *lbs. per sq. in.;* 260° F.

15. If the stress p at a point on one plane is inclined at an angle of 60° to that plane and on a plane at right angles to the former the stress is a simple shear, find the principal stresses at the point and their direction.

$$Ans.\ \frac{\sqrt{3}p}{4}\left(1 \pm \sqrt{\frac{7}{3}}\right);\ \tan 2\theta = \frac{2}{\sqrt{3}}$$

CHAPTER III

1. In a roof truss a certain tie has in it a pull of 3·05 tons due to the dead weight *alone*. When the wind is on the left of the truss it *alone* causes a pull of 5·5 tons in the same tie, and when it is on the right side it causes a compression of 1·2 tons. Work out what you would consider a satisfactory section for the tie if it is made of mild steel.

Ans. 3 *ins.* × ¾ *in. flat.*

2. Estimate the dead load equivalent to a tensile dead load of 15 tons and a live load of 20 tons; if the strain is not to exceed ·001, find the area of section required, E being 13,500 tons per sq. in.

Ans. 55 *tons;* 4·07 *sq. ins.*

3. A 3-girder bridge to carry a double line of rails has an effective span of 38 ft. 6 ins. Find a suitable working stress assuming that the weight of the girders is $\frac{1}{10}$ of the weight to be carried; that the flooring weighs 7 cwt. per ft. run of the whole width of the bridge; that the permanent way, etc., weighs 160 lbs. per foot run for each line of rails; and the live load is 40 cwt. per foot run per line of rail. *Ans.* 5 *tons per sq. in.*

4. What load, suddenly applied, will produce in a mild steel bar an extension of $\frac{1}{40}$ of an inch? The bar is 5 ft. long and 1½ sq. ins. in section. Take E = 13,000 tons per sq. in. *Ans.* 4·06 *tons.*

CHAPTER IV

1. Two lengths of a flat steel tie bar, which has to carry a load of 50 tons, are connected together by a double butt joint. The thickness of the plate is ¾ in. Find the diameter and the number of rivets required, and the necessary width of the bar for both chain and zigzag riveting. What is the efficiency of each and the working bearing pressure? Make a dimensioned sketch of the joint.

Ans. ⅞ *in. rivets,* 12 *and* 10½ *ins. wide;* 75 *per cent. and* 87 *per cent.;* 9·2 *tons per sq. in.*

2. A diagonal tie in a lattice girder has to carry a load of 15½ tons and is ½ in. thick. Using ¾ in. rivets, find the necessary width of tie and calculate the number of rivets required (in single shear) and sketch the arrangement. *Ans.* 5¼ *ins. wide,* 7 *rivets.*

3. Plates 1 in. thick are connected by a treble riveted butt joint, the pitch in outside rows being twice that in the others, and $d = 1$ in. Taking shear resistance in double shear = 1·75 times that in single shear, determine p for equal shear and tearing resistance. Find also the efficiency.

Ans. 6⅝ *ins.;* 85 *per cent.*

4. For equal strengths in tension and shear calculate the pitch for a

butt joint, given the following data: Plates 1 in. thick; rivets 1¼ ins. diam.; two rows of rivets on each side of joint; $f_s = 54{,}000$; $f_t = 65{,}000$ lbs. per sq. in. *Ans.* 5⅜ *ins.*

5. A steel boiler 4 ft. in diameter, and subject to a pressure of 200 lbs. per sq. in., is ½ in. thick. Find the intensity of the circumferential and longitudinal stresses, the efficiency of the joints being 75 per cent. *Ans.* 12,800; 6,400 *lbs. per sq. in.*

6. Find a suitable thickness of plate and design a double riveted lap joint (longitudinal) for a cylindrical drum 5 ft. in diameter, subjected to an internal gauge pressure of 250 lbs. per sq. in. Take a working stress of 5 tons per sq. in. (A.M.I.C.E.)
Ans. Plates 1 *in. thick; rivets* 1¼ *ins. diameter;* 3½ *ins. pitch.*

7. Calculate the thickness of shell of a boiler 4 ft. 6 ins. in diameter to resist a pressure of 150 lbs. per sq. in. Assume an efficiency of riveted joints of 70 per cent. and take the working stress as 6 tons per sq. in. (A.M.I.C.E.) *Ans.* $\frac{7}{16}$ *in.*

8. Determine the stresses across the longitudinal and transverse sections of the plates of a boiler drum 3 ft. in diameter and ½ in. thick, subject to a steam pressure of 200 lbs. per sq. in., assuming that the drum is long and that it has no longitudinal seam. *Ans.* 7,200; 3,600 *lbs. per sq. in.*

CHAPTER V

1. A cantilever whose weight may be neglected carries isolated loads of 2 tons and ½ ton at distances of 5 ft. and 8 ft. respectively from its built-in end, the cantilever being 10 ft. long. Sketch shear and B.M. diagrams.
Ans. Max. B.M. = 14 *ft. tons; shear* = 2½ *tons.*

2. A certain joist used as a cantilever weighs 18 lbs. per foot, and the max. B.M. which it can carry is 63·56 in. tons. Find how long the span may be for the cantilever to be able to safely sustain its own weight.
Ans. 36·3 *ft.*

3. A beam of 12 ft. span carries loads of 3 and 4 tons at distances of 5 and 8 ft. from the left-hand support. Draw the shear and B.M. curves.
Ans. Max. B.M. = 15·66 *ft. tons; reaction,* 3·91 *and* 3·09 *tons.*

4. A beam of 25 ft. span carries a load of ½ ton per foot run, and an isolated load of 6 tons at a distance of 4 ft. from the left-hand support. Find the maximum bending moment, and sketch the shear and B.M. curves. *Ans. Max B.M.* = 52·2 *ft. tons.*

5. A beam of 40 ft. span carries a uniformly distributed load of 20 tons; at points 11 ft. 3 ins. from each end isolated loads of 11 tons are carried, and between these points and each end additional loads of 4·5 tons are uniformly distributed. Draw the B.M. diagram.
Ans. Max. B.M. = 250 *ft. tons nearly.*

6. A beam 25 ft. long is anchored down at one end and rests over a support 6 ft. from the other end. It carries a load of 15 tons at the free end, and a uniform load of 5 cwt. per foot run. Sketch the shear and bending moment curves. *Ans. Max. B.M.* = 94·5 *ft. tons.*

7. A beam 34 ft. long overhangs one support by 6 ft. and carries a load of 10 tons uniformly distributed. In addition it carries a load of 3 tons at the overhanging end and a load of 12 tons uniformly distributed along a length of 12 ft. commencing from the other end. Find the maximum bending moment on the beam. *Ans.* 62·2 *ft. tons.*

8. A beam is laid horizontally upon two supports which are 12 ft. apart, and projects at each end 6 ft. beyond the support. A load of 2 tons is carried upon each of the projecting ends, and 1 ton at the centre of the span. What is the B.M. at the centre and at each support? Sketch the B.M. diagram. (A.M.I.C.E.) *Ans.* 9 *ft. tons;* 12 *ft. tons.*

9. A plate girder is built of depth = $\frac{1}{12}$ span. The maximum permissible B.M. in ft. tons in such girder is roughly given by formula: B.M. = 7 × area of flange in inches × depth in feet. Find the maximum span for such a girder to carry its own weight: (*a*) neglecting its web altogether; (*b*) taking its web as half the sectional area of one flange. Neglect all angles, rivets, and stiffeners. Take steel as weighing 490 lbs. per cub. ft. *Ans.* (*a*) 1,536 *ft.;* (*b*) 1,229 *ft.*

10. A beam of 20 ft. span carries a uniform load of 5 cwt. per ft. run and an additional load of 9 tons spread over 12 ft. starting from the right-hand end. Draw the B.M. and shear diagrams.

Ans. Max. B.M. 38·7; *Reactions* 8·8 *and* 5·2 *tons.*

CHAPTER VI

1. Find the moment of inertia about the centroid of an **I** beam 8 ins. deep, the width of flanges being 5 ins. The flanges are ·575 in. and the web ·35 in. thick. *Ans.* 89·1 *in. units.*

2. A stanchion section consists of two standard channels 11 ins. × 3½ ins. placed back to back at 6½ ins. apart and two 14 ins. × ½ in. plates riveted to each flange. Find the least radius of gyration. *Ans.* 4·12 *ins.*

3. Find the radius of gyration of a hollow cylindrical column with an external diameter of 12 ins. and a thickness of 1 in.; also of a solid square column 4 ins. by 4 ins. *Ans.* 3·90 *ins.;* 1·15 *ins.*

4. A cast-iron girder has an upper flange 4 ins. by 1 in.; a lower flange 8 ins. by 1½ ins. and a web 6 ins. by 1 in. Find its moment of inertia and radius of gyration about an axis through the centroid parallel to the flanges. *Ans.* 195 *ins.*^{4}; 2·98 *ins.*

5. A channel section has a base of 10 ins.; sides 3 ins.; the thickness of

metal being $\frac{3}{4}$ in. Find the position of the centroid and the moment of inertia about a line through the centroid parallel to the base.
Ans. ·726 *in. from base ;* 6·62 *ins.*^{4}.

6. A column is built up of two **I** beams 10 ins. deep and with flanges 5 ins. wide, the centres of the beams being 10 ins. apart. The area of each is 8·82 sq. ins., and the greatest and least moments of inertia are 145·7 and 9·78 in. units respectively. Riveted at the top of each pair is a plate 12 ins. wide. Neglecting the rivets, find the thickness of the plate if the greatest and least moments of inertia are the same. *Ans.* $\frac{7}{16}$ *in.*

7. A column is built up of two channel sections 12 × 3½ × ½ in., with a plate ½ in. thick riveted to the flanges at top and bottom. Find the distance x apart that the channels must be for the moments of inertia to be equal about the two axes of symmetry, the width of the plates being $x + 7\frac{1}{2}$ ins. *Ans.* 9⅝ *ins.*

CHAPTER VII

1. A 20 in. × 7½ in. joist is supported at both ends. The weight per foot of this section is 89 lbs., and the moment of inertia = 1,646 ins.^{4}. Find the distributed load in a 25 ft. span which will cause a max. flange stress of 7 tons per sq. in. *Ans.* 29·7 *tons net.*

2. The moment of inertia of a 12 in. × 5 in. × 32 lb. joist is 221 ins.^{4}. Two such joists are placed side by side, and support a water-tank which weighs 1 ton when empty. Effective span = 15 ft. What is the weight of the water in the tank when the stress in the extreme fibres of the joist is 6·5 tons per sq. in.? *Ans.* 20·3 *tons.*

3. Two 6 × 3 × ½ in. Ts are used back to back as a girder on which a light crane runs. Compare the safe load which such a beam would carry with that of a joist of same span, depth, width, and thickness of metal.
Ans. Joist 5·36 *times as good.*

4. Find the bending moment which may be resisted by a cast-iron pipe 6 ins. external and 4½ ins. internal diameter when the greatest intensity of stress due to bending is 1,500 lbs. per sq. in. *Ans.* 21,750 *in. lbs.*

5. A rolled-steel joist 16 ins. deep, with flanges 6 ins. wide and 1 in. thick (the web being ¾ in. thick), is used to support a uniformly distributed load of 2 tons per ft. run. If the span is 12 ft. 6 ins., what is the maximum stress in the lower flange? (A.M.I.C.E.) *Ans.* 4·42 *tons per sq. in.*

6. Find what diameter of axle should be employed if the wheels are 4 feet 9 ins. apart and the loads on them are 7 and 3 tons respectively, the axle boxes projecting 9 ins. beyond the wheels. Draw the B.M. diagram.
Ans. Max. B.M. = 58·68 *in. tons ; diameter* = 5 *ins.*

7. A gallery is carried by two 9 in. × 3 in. timber cantilevers, each 5 ft. long. What distributed load may the gallery carry if the safe stress is 10 cwt. per sq. in.? *Ans.* 27 *cwt.*

8. Either of the following sections is available for a beam which is required to be as strong as possible: (*a*) Circular, 2 ins. diam.; (*b*) rectangular, 2 ins. deep, 1·178 ins. wide. Which would you use? (A.M.I.C.E.)
Ans. Circular.

9. A cast-iron beam section is 20 ins. deep; top flange 4 ins. × 1 in.; bottom flange, 16 ins. × 1½ ins.; web, 1 in. Find the safe distributed load which a cast-iron girder of the above section, and of 20 ft. span, could safely carry. Take the safe stresses as 1 ton/in.^{2} in tension, and 4 tons/in.^{2} in compression. *Ans.* 10·6 *tons net;* 11·9 *tons gross.*

10. Find the bending stress in a locomotive coupling-rod 8 ft. long, 2 ins. broad and 4½ ins. deep. It runs at 200 revolutions per minute, the crank radius being 11 ins. *Ans.* 2·6 *tons per sq. in.*

CHAPTER VIII

1. A tie bar 9 ins. wide and 1½ ins. thick is curved in the plane of its width. If there is a total tensile load on the bar of 30 tons, and if the mean line of pull passes 3 ins. to one side of the geometrical axis at the middle of the bar, find the maximum and minimum stresses at the centre section of the bar. (A.M.I.C.E.)
Ans. 6⅔ *tons per sq. in. tension;* 2⅔ *tons per sq. in. compression.*

2. An upright timber post 12 ins. in diameter supports a vertical load of 18 tons, 3 ins. from the vertical axis of the post. Determine the maximum and minimum stresses on a normal cross section and show by a diagram how the intensity of stress varies across the section.
Ans. ·477 *and* ·159 *ton per sq. in.*

3. A cast-iron post 12 ins. in external diameter and 10 ins. internal diameter carries an axial load of 40 tons and also an eccentric load of 5 tons, parallel to the axis at an eccentricity of 12 ins. Find the maximum stress.
Ans. 1·98 *tons per sq. in.*

4. A short wooden pillar is 20 ins. high, and rectangular in cross section, the thickness of the section is 6 ins., and the width 12 ins. Two vertical loads act on the top of the pillar, both loads act in the middle of the thickness, one of them, W_1, acts at a point 1½ ins. on one side of the centre, and the other, W_2, acts at a point 2½ ins. on the other side of the centre. If the stress over the base of the pillar is everywhere compressive and varies uniformly, its intensity being twice as great at the 6 in. edge near the line of action of W_2 as it is at the 6 in. edge near the line of action of W_1, what is the ratio of W_2 to W_1? (B.Sc. Lond.) *Ans.* 13 : 11.

5. A reinforced concrete beam, 8 ins. × 11 ins. deep, has four ½ in. bars, with centres at 1 in. from the bottom. Calculate for a span of 12 ft. the safe load (*a*) on the modified beam formulæ; (*b*) on the no-tension, straight-line formulæ. Take $t = 15{,}000$, $c = 500$, $t_c = 100$, $m = 15$.
Ans. (*a*) 1,205 *lbs.;* (*b*) 3,920 *lbs., including weight of beam.*

6. Find the average compressive stress in a reinforced concrete T beam with a 7 in. slab if the depth of the neutral axis is 10 ins. and the maximum compressive stress is 600 lbs. per sq. in. *Ans.* 391 *lbs. per sq. in.*

7. Find the relation between the depth of slab and effective depth of a T beam in terms of the stresses and reinforcement for the neutral axis to curve at the bottom of the slab. *Ans.* $\frac{d_1}{d} > \frac{2rt}{c}$

8. A T beam is required to carry a B.M. of 320,000 in. lbs. The depth to centre of reinforcement is 16 ins., and the depth of slab is 4 ins. If $c = 600$ and $t = 16{,}000$, what area of reinforcement and effective breadth of slab would you use? *Ans.* 1·39 *sq. ins.*; 12¾ *ins.*

9. A reinforced concrete floor is 9 ins. thick, the centre of the reinforcement being 2 ins. from the bottom edge. If $c = 600$, $t = 15{,}000$, and $m = 15$, calculate the reinforcement necessary, and the load that can be safely carried. *Ans.* ·63 *sq. in. per ft. width*; 386 *lbs. per sq. ft.*

10. A beam of rectangular section of breadth one half the depth is bent by a couple in a plane at 45° to the axes of the section. Find the safe B.M. in terms of those about the principal axes. *Ans.* $2\sqrt{\frac{2}{3}}$ *and* $\sqrt{\frac{2}{3}}$

CHAPTER IX

1. If two precisely similar beams of rectangular section, one of cast iron and the other of wrought iron, were laid across the same span and loaded with the same load (within the elastic limit), what would be the relative deflections of the two beams? (A.M.I.C.E.)

Ans. As $E_c : E_w$ = *about* 8 : 13.

2. A beam is of 20 ft. span and the movement of inertia of its section is 300 in. units; what will be the central deflection for a uniformly distributed load of 16 tons? (A.M.I.C.E.) *Ans.* ·72 *in.*

3. A beam of cast iron, 1 in. broad and 2 ins. deep, is tested upon supports 3 ft. apart, and shows a deflection of ¼ in. under a central load of 1 ton. Calculate the modulus E. (A.M.I.C.E.) *Ans.* 5,832 *tons per sq. in.*

4. Suppose that three beams or planks, A, B, and C, of the same material are laid side by side across a span $L = 100$ ins., and a load $W = 600$ lbs. is laid across them at the centre of the span so that they all bend together. The beams are all 6 ins. wide, but two are 3 ins. and one 6 ins. deep. What will be the load carried by each beam, and what will be the extreme fibre stress in each? (A.M.I.C.E.)

Ans. 480 *lbs.*, 60 *lbs.*; 333 *lbs. per sq. in.*, 167 *lbs. per sq. in.*

5. Calculate the least radius to which a 1 in. round bar of wrought iron [$E = 28 \times 10^6$ lbs. per sq. in.] may be bent, in order that the skin stress

may not exceed 15 tons per sq. in. What is then the moment of resistance of the section? (A.M.I.C.E.) *Ans.* 34·7 *ft.*; 1·47 *in. tons.*

6. A beam of uniformly rectangular section is supported freely at the ends and carries a uniformly distributed load. Find the ratio of depth to span so that when the maximum stress at the centre section due to bending is 4 tons per sq. in., the deflection at the centre is $\frac{1}{600}$ of the span. E = 12,000 tons per sq. in. (B.Sc. Lond.) *Ans. Span* = 24 × *depth.*

7. Find the greatest deflection in inches of a rectangular wooden beam carrying a load of 2 tons at the centre of a span of 20 ft., with a limiting intensity of stress of 1000 lbs. per sq. in. The depth of the beam is 14 ins. Calculate the breadth. E = 6000 tons per sq. in. (A.M.I.C.E.)
Ans. ½ *in. nearly*; 8·2 *ins. wide.*

8. A 16 in. × 6 in. × 62 lb. R.S.J. carries a load of 12 tons at quarter span, the span being 24 ft. Find graphically the maximum deflection and compare that calculated for the same beam with the load at the centre. (I for this section = 725·7 in. units, E = 12,500 tons/in.^{2})
Ans. ·46 *in.*; ·66 *in. at centre.*

9. A simply-supported beam of uniform section and 30 ft. span is found to deflect 6 ins. under its own weight. Find the slope of the beam at the supports and also the slope which would arise if the same deflection were caused by a central load instead of a uniform one. *Ans.* ·0533; ·05.

10. A vertical post, 24 ft. in height, supports at its upper end a horizontal arm projecting 6 ft. from the post. Find the horizontal and vertical displacements of the free end of the horizontal arm when a load of 6000 lbs. is suspended from it. E for post and arm = 28 × 10^6 lbs. per sq. in.; I for post = 412, for arm = 360 (inch units). Neglect direct compression of the post. (B.Sc. Lond.) *Ans. Horizontal* 1·55; *vertical* ·85 *in.*

11. A cantilever of circular section is of constant diameter from the fixed end to the middle, and of half that diameter from the middle to the free end. Estimate the deflection at the free end due to a weight W there.
Ans. $\frac{23\,\mathrm{W}\,l^3}{24\,\mathrm{E\,I}}$, *where I is that at fixed end.*

12. A timber beam 30 ft. long and 12 ins. square in cross section rests on a support at each end. If a load of 1 ton is placed in the centre of the beam, find the work done in deflecting it. *Ans.* 705·6 *in. lbs.*

CHAPTER X

1. A cast-iron column has its ends securely built in. It is 12 ins. in external diameter, and 18 ft. long. What total load could you place on it if the factor of safety is 10, and the thickness of metal 1⅜ ins.? The constant for the Gordon formula is $\frac{1}{800}$. (B.Sc. Lond.) *Ans.* 163 *tons.*

2. A mild steel strut, rectangular in cross section, the breadth being

four times its thickness, is 9 ft. long, and has pin ends. Determine the cross section for 24 tons, and a factor of safety of 5. Use Rankine's formula, and take f_c = 67,000 lbs. per sq. in., and the constant $\frac{1}{5000}$. (B.Sc. Lond.)

3. Which would carry the heavier load for fixed ends: (*a*) a solid mild-steel column 9 ins. diam.; (*b*) a built-up mild-steel stanchion consisting of two 14 × 6 **I** beams, at 8½ ins. centres, with two 16 × ½ in. plates each side? Length in each case 14 ft. *Ans. The built-up one.*

4. Discuss the formula of Gordon and Rankine in connection with the buckling of struts of moderate lengths, and state its limiting conditions. Four wrought-iron struts, rigidly held at the ends, all of section 1 in. × 1 in., and of lengths 15·0, 30·0, 60·0, and 90·0 ins. respectively, are found to buckle under loads of 15·9, 11·3, 7·7, and 4·35 tons. Test whether these satisfy the formula quoted, and, if so, find average values of the two empirical constants involved. (B.Sc. Lond.)

5. A stanchion for a workshop has to carry a small stanchion 10 ft. long from the roof which carries 5 tons, and also the girder for a 15-ton crane. If the centre line of the roof load and crane girder are 13 ins. apart, design a suitable section for the stanchion.

Ans. Two 10 *ins.* × 5 *ins.* × 30 **I** *beams* 13 *ins. apart.*

6. A hollow cylindrical steel strut has to be designed for the following conditions. Length 6 ft., axial load 12 tons, ratio of internal to external diameter = ·8, factor of safety, 10. Determine the necessary external diameter of the strut and thickness of the metal if the ends are securely fixed in. Use Rankine's formula, taking f = 21 tons per sq. in., constant for rounded ends = $\frac{1}{7500}$ *Ans.* 4½ *ins. diam.*; ½ *in. thick.*

7. A steel column is built up of two 10 × 3½ ins. × 28·21 lbs. channel sections placed 4½ ins. apart, and two 12 × ½ in. plates at each end. If the ends are pin-jointed, what would you consider a safe load on a length of 22 ft.? *Ans.* 122 *tons.*

8. What would be the safe load on the column of Question 7 if the load were 3 ins. out of centre? *Ans.* 48·8 *tons.*

9. Find what thickness a hollow circular cast-iron column should have for an axial load of 60 tons, the factor of safety being 8. The column is 20 ft. long and is securely fixed at each end. *Ans.* 2 *ins.*

CHAPTER XI

1. A shaft 3 ins. in diameter, running at 250 revolutions per minute, transmits 50 H.P. Find the maximum stress and the twist of the shaft in degrees in a length of 100 ft. Take G = 12 × 10^6 lbs. per sq. in.

Ans. 2,380 *lbs. per sq. in.*; 28·5°.

2. A turbine is connected to a dynamo placed vertically above it by a shaft, 2 ft. in diameter, made of steel plate ⅜ in. thick. Calculate the diameter of solid shaft required to transmit the same power at the same speed with the same maximum stress due to twist. Find the relative weights. (A.M.I.C.E.) *Ans.* 13·86 *ins. diameter;* ·304 : 1.

3. If a rod ¼ in. in diameter and 20 ins. long is fixed at one end and the other end is twisted through an angle of 15° relatively to it, what is the unital strain in the outer fibres of the rod? *Ans.* ·00164.

4. A bar of iron, ½ in. diameter, is twisted to destruction. Calculate what twisting movement is required for this purpose assuming that the shearing stress becomes uniform over the whole section and equals in the limit 19 tons per sq. in. (A.M.I.C.E.) *Ans.* ·622 *in. tons.*

5. Through what angle will a 2½ inch steel shaft be twisted if it is 80 ft. long and the twisting movement is 19,000 in. lbs.? *Ans.* 23° *nearly*.

6. If the end of a rod 1 in. in diameter is twisted by the turning effort of a force of 80 lbs. acting at the end of a 12 in. lever, find the force which, when applied to the end of the same lever, would twist equally a rod of the same material, but of 1½ ins. diameter and half the length.

Ans. 810 *lbs.*

7. A bar of mild steel 1 in. in diameter twists through an angle o. 2·2 degrees in a 20 in. length when subjected to a torque of 2,200 in. lbs. An exactly similar bar of the same material deflects ·03 in. when loaded at the centre of a 20 in. simply-supported span with a load of 264 lbs. Calculate the value of Young's Modulus, Rigidity Modulus, Bulk Modulus and Poisson's Ratio. (B.Sc. Lond.)

Ans. $E = 29·88$, $G = 11·67$, $K = 32·62$ *million pounds per sq. in.* $\eta = ·28$.

8. A shaft which runs at 135 revolutions per minute transmits 50 H.P. and is subjected to a bending moment equal to ·75 of the twisting moment. What diameter of shaft is necessary on the principal stress theory?

Ans. 2⅞ *ins.*

CHAPTER XII

1. A steel wire ½ in. in diameter is coiled into a spiral spring 5 ins. mean diameter. What weight could such a spring carry to produce a maximum stress in the wire of 5 tons per sq. in.? (A.M.I.C.E.)

Ans. 110 *lbs.*

2. Find the pull required to cause a deflection of 1 in. in a closely-wound helical spring of 2·5 in. mean diameter made of 120 turns of ¼ inch round wire, taking $G = 12 \times 10^6$ lbs. per sq. in. *Ans.* 3⅛ *lbs.*

3. A helical spring of 3 ins. diameter is composed of 20 turns of steel wire ·258 in. diameter. If a load of 25 lbs. is hung on it what will be the deflection and maximum stress? *Ans.* 2·03 *ins.;* 11,120 *lbs. per sq. in.*

4. A laminated spring composed of 20 plates each $\frac{3}{8}$ in. thick and 2·95 ins. wide has a span of 3 ft. Find the deflection under a load of 5 tons if E is 12,000 tons per sq. in. *Ans.* 2·37 *ins.*

5. How many plates $\frac{3}{8}$ inch thick and 3 ins. broad, the largest being 30 ins. long, are required in a leaf spring whose maximum stress is to be 30,000 lbs. per sq. in. with a load of 1 ton? *Ans.* 8.

6. A steel clock spring $\frac{3}{4}$ in. wide and $\frac{1}{64}$ in. thick is wound on a spindle $\frac{3}{16}$ in. in diameter. If the safe stress is 48,000 lbs. per sq. in. what is the maximum moment available for driving the clock?

Ans. 2 *in. lbs. approx.*

CHAPTER XV

1. A girder 100 ft. long is supported at each end and in the middle, and carries a uniform load of 2 tons per ft. run. Draw the B.M. and shear diagrams, and find the pressure on each support. (A.M.I.C.E.)

Ans. Max. B.M. 625 *ft. tons.; reaction* 37·5; 125; 37·5 *tons.*

2. A continuous girder consists of four spans, the two outer spans are each 20 ft. long, and the two inner spans are each 40 ft. long; the girder carries a uniformly distributed load of 1½ tons per ft. run. Find (*a*) The reactions at each of the piers; (*b*) The bending moment and shear at each of the piers; (*c*) The position of the points of zero bending moment. Sketch complete bending moment and shear diagrams for the girder. (B.Sc. Lond.)

3. A balk of timber, 30 ft. long, rests on two end supports, and is supported also by a prop which acts at a point 12 ft. from the left-hand end. If the balk of timber carries (including its own weight) a load of 2 cwt. per ft. run, and if the tops of the three supports are level, determine the reactions at the three supports, and the bending moment at the point at which the prop is applied. Draw complete bending moment and shear diagram. (B.Sc. Lond.)

4. A horizontal girder of uniform section 25 ft. long is firmly fixed at one end, and supported by a column at 18 ft. from the fixed end. The girder carries a uniform load of 2 tons per ft. run of its length, and, in addition, a concentrated load of 30 tons at 14 ft. from the fixed end. When unloaded, the girder just touches, but does not exert any pressure on the supporting column. Find the pressure on the column, and draw bending moment and shearing force diagrams for the girder. (B.Sc. Lond.)

5. A beam of 20 ft. span is built in at one end A, and is freely supported at other end B. It carries a uniform load of ½ ton per ft. run, and a central isolated load of 10 tons. Draw the bending moment diagram, first finding the bending at end A, and show where the maximum intermediate bending moment occurs. Draw also the shear diagram.

6. A continued girder of 2 spans, 20 ft. and 10 ft., has an overhang of 5 ft. from the smaller span. It carries a uniformly distributed load of ½ ton per ft. run, and an isolated load of 1½ tons at the free end (D) Find the support moments, and draw the shear and B.M. diagrams. Determine whether this arrangement is stronger than that in which the support C comes below the point D.

Ans. Max. B.M. 16·77 *ft. tons ; not so strong.*

7. A beam of span l is fixed horizontally at both ends. Two equal loads W are placed at equal distance h from the ends of the beam. Prove that the greatest deflection of the beam is equal to $\frac{W h^2}{24 E I}(3l - 4h)$, and that the bending moment at the centre of the beam is equal to $\frac{W h^2}{l}$. (B.Sc. Lond.)

8. A beam of 20 ft. span is fixed at the end and carries a uniformly-distributed load of 1 ton per ft. run from one abutment to the centre. Find the end B.M.s. *Ans.* 10·4, 22·9 *ft. tons.*

9. In a continuous beam of three spans, the centre span is 72 ft. and the end spans 36 ft. each. A dead load of ½ ton per ft. run covers the whole span. Determine the support moments when a live load of 1 ton per ft. run covers (*a*) the first span; (*b*) the first two spans; (*c*) the whole beam. *Ans.* (*a*) 243, 162; (*b*) 567, 486; (*c*) 547 *ft. tons.*

CHAPTER XVI

1. A C.I. beam has the following section: top flange, 4 × 1½ ins.; web, 12 × 1¾ ins.; bottom flange, 12 × 2 ins. The centroid of the section is 5·5 ins. from the base of the bottom flange, and the moment of inertia of the section about a line through its centroid, at right angles to the depth, is 1200 ins.^{4}. Draw a curve showing the intensity of shear at all points of the section, and find the ratio of maximum to mean shear stress. What proportion of shearing force is carried by the web? (B.Sc. Lond.)

2. Find the greatest intensity of shear stress at a section of an **I** beam at which the total shear is 15 tons; the overall depth is 8 ins.; flanges, 6 ins. × ·61 in.; web, ·44 in. thick; I = 111·6 in.^{4}. (B.Sc. Lond.)

3. Find the ratio of the maximum to the mean shear stress on the section of a cast-iron beam of the following dimensions: Top flange, 2 × 1½ ins.; bottom flange, 6 × 1½ ins.; web, 7 × 1 ins. *Ans.* 2·46.

4. A beam of uniform rectangular section, 6 ins. broad by 12 ins. deep, is supported at the ends, and has a span of 12 feet. It carries a uniformly distributed load of 20 tons. At a point in the cross section, 4 feet from

one end and 3 ins. vertically above the neutral axis, calculate the maximum intensity of compressive stress. *Ans.* 1·11 *tons per sq. in.*

5. An **I** beam, 20 ins. deep, has flanges 7½ ins. wide and 1 in. thick, and a web $\frac{17}{32}$ in. thick. The greatest moment of inertia is 1,650 in. units and the total shear over the section is 80 tons. Show by a diagram the intensity of shear stress at all points of the section. (A.M.I.C.E.) *Ans. Stress at N.A.* = 8·44 *tons per sq. in.*

6. Allowing a bending stress of 1,500 lbs. per sq. in. and a shearing stress with the grain of 120 lbs. per sq. in., what uniform load can be carried by a timber beam 12 ins. deep, 3 ins. wide and of 12 ft. span? *Ans.* 5,760 *lbs.*

7. A plate girder 4 ft. deep over the 3½ ins. × 3½ ins. × ½ in. angles has at each flange one plate 16 ins. × ½ in., the web being ⅜ in. thick. Find the distribution of shear stress on a section at which the shearing force is 44 tons.

Ans. Stress at N.A. = 1·67; *at junction of angle and web* = 1·25; *just above* = ·481; *at bottom of flange of angle* = ·348; *just above* = ·074; *junction of angle and plate* = ·049 *ton per sq. in.*

CHAPTER XVII

1. Show on the Bach Theory that the maximum central concentrated load that can be carried by a circular plate of given thickness is independent of the radius of the plate.

2. What must be the thickness of a mild steel plate covering an opening 4 ft. square if the load is 200 lbs. per sq. ft. and the safe stress is 16,000 lbs. per sq. in.? *Ans.* ·70 *on the Bach Theory.*

3. Prove that on the Bach Theory the uniform load that a square plate of given thickness can carry is independent of the size of the plate.

4. A rectangular reinforced concrete slab 10 ft. by 15 ft. has to carry a load of 300 lbs. per sq. ft. including its own weight. For what bending moment would you design the reinforcement in each direction? *Ans.* 560,000 *and* 162,000 *in. lbs.* (*Rankine*).

5. A cylinder end is 12 ins. in diameter and ¾ in. thick. Compare the maximum stresses on the Bach and Grashof theories if the steam pressure is 100 lbs. per sq. in., the end being taken as freely supported.
Ans. 6,400 *lbs. per sq. in.* (*Bach*), 7,800 *lbs. per sq. in.* (*Grashof*).

CHAPTER XVIII

1. A solid steel gun has an inside diameter of 7·5 ins. and a thickness of 1·75 ins. What is the greatest tensile stress carried by an explosion pressure of 10,000 lbs. per sq. in.? *Ans.* 27,300 *lbs. per sq. in.*

2. What should be the external diameter of a gun whose internal diameter is $3\frac{1}{2}$ ins., if the explosion causes a pressure of 15,000 lbs. per sq. in. and the allowable stress is 30,000 lbs. per sq. in.? *Ans.* 5·63 *ins.*

3. A solid gun has an external diameter of 12 ins. and an internal diameter of 6 ins. What inside pressure will cause a hoop tension of 30,000 lbs. per sq. in.? *Ans.* 18,000 *lbs. per sq. in.*

4. Find the safe internal pressure for an hydraulic press cylinder of external diameter 7 ins. and internal diameter 5 ins., the maximum safe stress being 3,000 lbs. per sq. in. *Ans.* 973 *lbs. per sq. in.*

5. An hydraulic press has an external diameter of 16 ins. and an internal diameter of 8 ins. If the pressure is 3 tons per sq. in., find the principal stresses at the external and internal circumference.

Ans. External 0, 2 (*tens.*); *internal* 3 (*comp.*) 5 (*tens.*) *tons per sq. in.*

6. Find the equivalent tensile hoop stresses in the problem of the above question if Poisson's ratio is $\frac{1}{3 \cdot 5}$ *Ans.* 5·86; 2 *tons per sq. in.*

7. Find the necessary thickness of a pipe of 8 ins. internal diameter subjected to an internal pressure of 520 lbs. per sq. in. Adopt the maximum strain theory taking $\eta = \frac{1}{4}$ and maximum tensile stress 10,000 lbs. per sq. in. *Ans.* ·22 *in.*

8. A tube whose internal and external radii are 2 and 3 ins. is hooped so as to cause an initial hoop compression on the inside of 18,000 lbs. per sq. in. What will be the tensile stress at the inside if the explosion causes a pressure of 25,000 lbs. per sq. in.? *Ans.* 29,000 *lbs. per sq. in.*

CHAPTERS XIX AND XX

1. Show that for a curved beam of rectangular section of depth d the deviation of the neutral axis from the centre is given approximately by $\frac{d^2}{12R}$, R being the radius of curvature of the centre line.

2. A rectangular bar 1 in. wide and 2 ins. deep is bent to a radius of 4 ins. and is used as a hook. Find the maximum stress caused by a load of 1,000 lbs. acting through the centre of curvature of the bar.

Ans. 7,700 *lbs. per sq. in.*

3. Find the maximum bending moments upon a chain link made of $\frac{3}{4}$ in. circular stock, the link being 5 ins. deep and 3 ins. broad. Treat the ends as circular and the sides as straight. The load carried is 3,000 lbs.

Ans. 1,300 *in. lbs.*; 387 *in. lbs.*

4. Find the stress due to centrifugal force in the rim of a cast-iron flywheel 8 ft. in diameter running at 160 revolutions per minute.

Ans. 437 *lbs. per sq. in.*

5. A steel shaft is of 13·4 in. diameter and is 98 ft. long. Find its critical speed when unloaded. *Ans.* 46 *revolutions per minute.*

6. Find the angular velocity at which whirling will start in an unloaded steel shaft 3 ins. in diameter and 11 ft. long. *Ans.* 75 *radians per second.*

7. Find the whirling speed of a shaft carrying a central load of 1,170 lbs. between swivelled bearings 7 ft. apart. The shaft is 3½ ins. in diameter. *Ans.* 725 *revolutions per minute.*

APPENDIX

APPENDIX.

The following Tables give the properties of the British Standard Sections which are usually listed by makers.

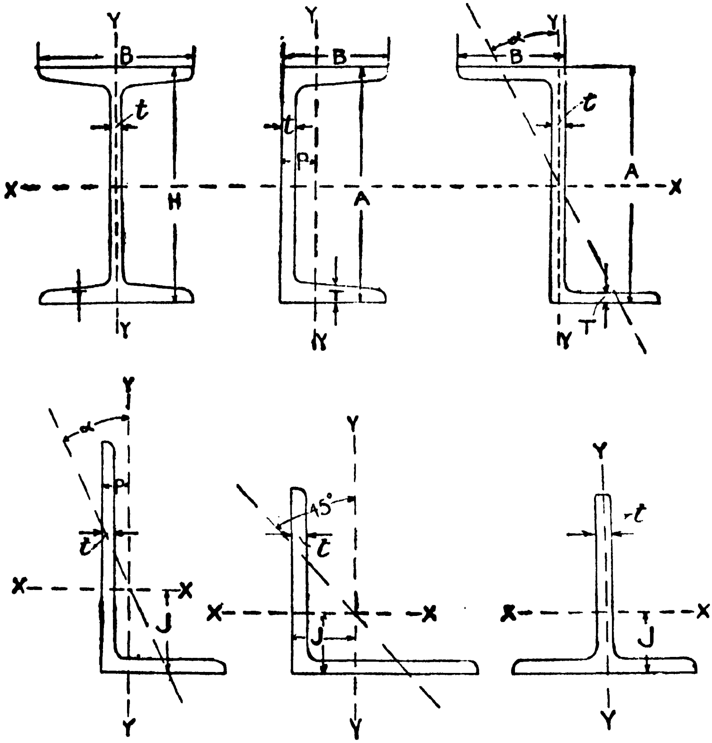

Fig. A.—Properties of British Standard Sections.

Note.—The original British Sections were replaced in 1920 by a revised list which are known as the *New* British Standard Sections; as many of the old **I** beam and channel sections will probably remain in use for some years, we have retained them in this appendix.

BRITISH STANDARD SECTIONS.* (See Fig. A.)

PROPERTIES OF **OLD** BRITISH STANDARD **I** BEAMS.

Size H. B.	Wt. per foot	Thickness		Sectional Area	Moments of Inertia		Section Moduli		Radii of Gyration	
		t	T		XX	YY	XX	YY	XX	YY
inches.	lb.	ins.	ins.	sq. ins.	ins.	ins.	ins.	ins.	ins.	ins.
3 × 1½	4	·16	·248	1·18	1·66	·124	1·11	·165	1·19	·325
3 × 3	8½	·20	·332	2·50	3·79	1·26	2·53	·841	1·23	·710
4 × 1¾	5	·17	·240	1·47	3·67	·194	1·84	·222	1·58	·363
4 × 3	9½	·22	·336	2·80	7·53	1·28	3·76	·854	1·64	·677
4¾ × 1¾	6½	·18	·323	1·91	6·77	·263	2·85	·300	1·88	·371
5 × 3	11	·22	·376	3·24	13·6	1·46	5·45	·974	2·05	·672
5 × 4½	18	·29	·448	5·29	22·7	5·66	9·08	2·51	2·07	1·03
6 × 3	12	·26	·348	3·53	20·2	1·34	6·74	·892	2·40	·616
6 × 4½	20	·37	·431	5·88	34·7	5·41	11·6	2·40	2·43	·959
6 × 5	25	·41	·520	7·35	43·6	9·11	14·5	3·64	2·44	1·11
7 × 4	16	·25	·387	4·71	39·2	3·41	11·2	1·71	2·89	·851
8 × 4	18	·28	·402	5·30	55·7	3·57	13·9	1·79	3·24	·821
8 × 5	28	·35	·575	8·24	89·4	10·3	22·3	4·10	3·29	1·12
8 × 6	35	·44	·597	10·3	111	17·9	27·6	5·98	3·28	1·32
9 × 4	21	·30	·460	6·18	81·1	4·20	18·0	2·10	3·62	·824
9 × 7	58	·55	·924	17·1	230	46·3	51·1	13·2	3·67	1·65
10 × 5	30	·36	·552	8·82	146	9·78	29·1	3·91	4·06	1·05
10 × 6	42	·40	·736	12·4	212	22·9	42·3	7·64	4·14	1·36
10 × 8	70	·60	·970	20·6	345	71·6	69·0	17·9	4·09	1·87
12 × 5	32	·35	·550	9·41	220	9·74	36·7	3·90	4·84	1·02
12 × 6	44	·40	·717	12·9	315	22·3	52·6	7·42	4·94	1·31
12 × 6	54	·50	·883	15·9	376	28·3	62·6	9·43	4·86	1·33
14 × 6	46	·40	·698	13·5	441	21·6	62·9	7·20	5·71	1·26
14 × 6	57	·50	·873	16·8	553	27·9	76·2	9·31	5·64	1·29
15 × 5	42	·42	·647	12·4	428	11·9	57·1	4·78	5·89	·983
15 × 6	59	·50	·880	17·3	629	28·2	83·9	9·40	6·02	1·28
16 × 6	62	·55	·847	18·2	726	27·1	90·7	9·02	6·31	1·22
18 × 7	75	·55	·928	22·1	1150	46·6	128	13·3	7·22	1·45
20 × 7½	89	·60	1·01	26·2	1671	62·6	167	16·7	7·99	1·55
24 × 7½	100	·60	1·07	29·4	2655	66·9	221	17·8	9·50	1·51

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PROPERTIES OF **OLD** BRITISH STANDARD CHANNELS.

Size A × B	Standard Thicknesses		Weight per foot	Area	Dimension p	Moments of Inertia		Section Moduli		Radii of Gyration	
	t	T				About X X	About Y Y	About X X	About Y Y	About X X	About Y Y
ins.	ins.	ins.	lbs.	sq. ins.	ins.	ins.	ins.	ins.	ins.	ins.	ins.
15×4	·525	·630	41·94	12·334	·935	377·0	14·55	50·27	4·748	5·53	1·09
12×4	·525	·625	36·47	10·727	1·031	218·2	13·65	36·36	4·599	4·51	1·13
12×3½	·500	·600	32·88	9·671	·867	190·7	8·922	31·79	3·389	4·44	·960
12×3½	·375	·500	26·10	7·675	·860	158·6	7·572	26·44	2·868	4·55	·993
11×3½	·475	·575	29·82	8·771	·896	148·6	8·421	27·02	3·234	4·12	·980
10×4	·475	·575	30·16	8·871	1·102	130·7	12·02	26·14	4·147	3·84	1·16
10×3½	·475	·575	28·21	8·296	·933	117·9	8·194	23·59	3·192	3·77	·994
10×3½	·375	·500	23·55	6·925	·933	102·6	7·187	20·52	2·800	3·85	1·02
9×3½	·450	·550	25·39	7·469	·971	88·07	7·660	19·57	3·029	3·43	1·01
9×3½	·375	·500	22·27	6·550	·976	79·90	6·963	17·76	2·759	3·49	1·03
9×3	·375	·437	19·37	5·696	·754	65·18	4·021	14·48	1·790	3·38	·840
8×3½	·425	·525	22·72	6·682	1·011	63·76	7·067	15·94	2·839	3·09	1·03
8×3	·375	·500	19·30	5·675	·844	53·43	4·329	13·36	2·008	3·07	·873
7×3½	·400	·500	20·23	5·950	1·061	44·55	6·498	12·73	2·664	2·74	1·04
7×3	·375	·475	17·56	5·166	·874	37·63	4·017	10·75	1·889	2·70	·882
6×3½	·375	·475	17·9	5·266	1·119	29·66	5·907	9·885	2·481	2·36	1·06
6×3	·312	·437	14·49	4·261	·938	24·01	3·503	8·003	1·699	2·37	·907

PROPERTIES OF **OLD** BRITISH STANDARD ZED BARS.

Size A × B	Standard Thicknesses		Area	Weight per foot	Moments of Inertia		Section Moduli		Angle *a* in degrees	Least Radius of Gyration
	t	T			About X X	About Y Y	About X X	About Y Y		
ins.	ins.	ins.	sq. ins.	lbs.	ins.	ins.	ins.	ins.		ins.
10×3½	·475	·575	8·283	28·16	117·865	12·876	23·573	3·947	14	·839
9×3½	·450	·550	7·449	25·33	87·889	12·418	19·531	3·792	16½	·843
8×3½	·425	·525	6·670	22·68	63·729	12·024	15·932	3·657	19½	·845
7×3½	·400	·500	5·948	20·22	44·609	11·618	12·745	3·521	23	·840
6×3½	·375	·475	5·258	17·88	29·660	11·134	9·887	3·361	28½	·821
5×3	·350	·450	4·169	14·17	16·145	6·578	6·458	2·328	29½	·698

NEW BRITISH STANDARD ROLLED STEEL SECTIONS.*

PROPERTIES OF I BEAM SECTIONS.

'A.'—GIRDER SECTIONS.

Size inches	Wt. per foot lbs.	Thickness *t*	Thickness T	Area	Moments of Inertia XX	Moments of Inertia YY	Section Moduli XX	Section Moduli YY	Radii of Gyration XX	Radii of Gyration YY
3 × 1½	4	·16	·249	1·177	1·660	·125	1·107	·167	1·19	·326
4 × 1¾	5	·17	·239	1·470	3·664	·186	1·832	·213	1·58	·356
4½ × 2	7	·19	·322	2·060	6·652	·383	2·957	·383	1 80	·431
5 × 2½	9	·20	·347	2·647	10·91	·789	4·364	·631	2·03	·546
6 × 3	12	·23	·377	3·533	20 99	1·461	6 996	·974	2·44	·643
7 × 3½	15	·25	·398	4·416	35·90	2·408	10· 6	1·376	2·85	·738
8 × 4	18	·28	·398	5·296	55·63	3·506	13·91	1·753	3·24	·814
9 × 4	21	·30	·457	6·177	81·13	4·148	18·03	2·074	3·62	·820
10 × 4½	25	·30	·505	7·354	122 3	6·486	24·47	2·883	4 08	·939
12 × 5	30	·33	·507	8·827	206·9	8·770	34·49	3·508	4·84	·997
13 × 5	35	·35	·604	10·30	283·5	10 8	43·62	4·326	5·25	1·03
14 × 5½	40	·37	·627	11·77	377·1	14·79	53·87	5·377	5·66	1·12
15 × 6	45	·38	·655	13·24	491·9	19 87	65·59	6·624	6·10	1·23
16 × 6	50	·40	·726	14·71	618·1	22·47	77·26	7·489	6·48	1·24
18 × 6	55	·42	·757	16·18	841 8	23·64	93·53	7 878	7·21	1·21
20 × 6½	65	·45	·820	19·12	1226·0	32·56	122·6	10·02	8·01	1·31
22 × 7	75	·50	·834	22·06	1677·0	41·07	152·4	11·73	8·72	1·36
24 × 7½	90	·52	·984	26·47	2443·0	60·44	203·6	16·12	9·61	1·51

'B.'—HEAVY BEAMS AND PILLARS.

Size inches	Wt. per foot lbs.	Thickness *t*	Thickness T	Area	Moments of Inertia XX	Moments of Inertia YY	Section Moduli XX	Section Moduli YY	Radii of Gyration XX	Radii of Gyration YY
18 × 8	80	·50	·950	23·53	1292	69·43	143·6	17·36	7·41	1·72
16 × 8	75	·48	·938	22·06	973·9	68·30	121·7	17·08	6·64	1·76
14 × 8	70	·46	·920	20·59	705·6	66·67	100·8	16·67	5·85	1·80
12 × 8	65	·43	·904	19·12	487·8	65·18	81·30	16·30	5·05	1·85
10 × 8	55	·40	·783	16·18	288·7	54·74	57·74	13·69	4·22	1·84
10 × 6	40	·36	·709	11·77	204·8	21·76	40·96	7·253	4·17	1·36
9 × 7	50	·40	·825	14·71	208·1	40·17	46·25	11·48	3·76	1·65
× 6	35	·35	·648	10·30	115·1	19·54	28·76	6·513	3·34	1·38
6 × 5	25	·33	·561	7·351	45·16	9·876	15·05	3·951	2·48	1·16
5 × 4½	20	·29	·513	5·882	25·03	6·590	10·01	2·929	2·06	1·06
4 × 3	10	·24	·347	2·940	7·786	1·326	3·893	·884	1·63	·67

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NEW BRITISH STANDARD EQUAL ANGLES *(continued).*

Sizes	Area	Weight per foot lb.	J	I_{xx}	Section Modulus about XX	Least Radius of Gyration
$3 \times 3 \times \frac{5}{8}$	3·360	11·42	·97	2·59	1·27	·58
$2\frac{3}{4} \times 2\frac{3}{4} \times \frac{1}{4}$	1·313	4·46	·76	·917	·46	·54
$2\frac{3}{4} \times 2\frac{3}{4} \times \frac{3}{8}$	1·923	6·54	·81	1·30	·67	·53
$2\frac{3}{4} \times 2\frac{3}{4} \times \frac{1}{2}$	2·501	8·50	·86	1·64	·87	·53
$2\frac{1}{2} \times 2\frac{1}{2} \times \frac{1}{4}$	1·188	4·04	·70	·680	·38	·49
$2\frac{1}{2} \times 2\frac{1}{2} \times \frac{5}{16}$	1·465	4·98	·73	·826	·47	·48
$2\frac{1}{2} \times 2\frac{1}{2} \times \frac{3}{8}$	1·735	5·90	·75	·962	·55	·48
$2\frac{1}{2} \times 2\frac{1}{2} \times \frac{1}{2}$	2·250	7·65	·80	1·21	·71	·48
$2\frac{1}{4} \times 2\frac{1}{4} \times \frac{1}{4}$	1·063	3·61	·64	·488	·30	·44
$2\frac{1}{4} \times 2\frac{1}{4} \times \frac{5}{16}$	1·309	4·45	·67	·591	·37	·43
$2\frac{1}{4} \times 2\frac{1}{4} \times \frac{3}{8}$	1·547	5·26	·69	·685	·44	·43
$2 \times 2 \times \frac{3}{16}$	·715	2·43	·56	·260	·18	·39
$2 \times 2 \times \frac{1}{4}$	·937	3·19	·58	·336	·24	·39
$2 \times 2 \times \frac{5}{16}$	1·152	3·92	·61	·405	·29	·38
$2 \times 2 \times \frac{3}{8}$	1·359	4·62	·63	·468	·34	·38
$1\frac{3}{4} \times 1\frac{3}{4} \times \frac{3}{16}$	·621	2·11	·49	·171	·14	·34
$1\frac{3}{4} \times 1\frac{3}{4} \times \frac{1}{4}$	·813	2·76	·52	·219	·18	·34
$1\frac{3}{4} \times 1\frac{3}{4} \times \frac{5}{16}$	·996	3·39	·54	·263	·22	·34
$1\frac{3}{4} \times 1\frac{3}{4} \times \frac{3}{8}$	1·172	3·98	·57	·303	·26	·34
$1\frac{1}{2} \times 1\frac{1}{2} \times \frac{3}{16}$	·527	1·79	·43	·105	·10	·29
$1\frac{1}{2} \times 1\frac{1}{2} \times \frac{1}{4}$	·687	2·34	·46	·134	·13	·29
$1\frac{1}{4} \times 1\frac{1}{4} \times \frac{3}{16}$	·434	1·48	·37	·059	·07	·24
$1\frac{1}{4} \times 1\frac{1}{4} \times \frac{1}{4}$	·563	1·91	·40	·074	·09	·24

PROPERTIES OF NEW BRITISH STANDARD UNEQUAL ANGLES.

Size and Thickness	Area	Weight per foot lbs.	Dimensions		Moments of Inertia		Section Moduli		Angle α in degrees	Least Radius of Gyration
			J	P	X X	Y Y	X X	Y Y		
10 × 4 × 7/16	5·934	20·18	3·70	·73	61·84	6·00	9·81	1·83	10½	·83
" " 9/16	7·599	25·70	3·76	·78	78·06	7·50	12·50	2·33	10	·82
" " 11/16	9·153	31·12	3·81	·83	93·57	8·90	15·12	2·81	10	·82
9 × 4 × 7/16	5·496	18·69	3·24	·77	46·36	5·90	8·05	1·83	12	·84
" " 5/8	7·735	26·30	3·33	·85	64·23	8·06	11·33	2·56	12	·83
" " 13/16	9·903	33·67	3·41	·92	80·86	9·99	14·46	3·25	11½	·83
8 × 6 × ½	6·751	22·95	2·44	1·45	43·47	21·08	7·82	4·63	29	1·29
" " ⅝	8·360	28·42	2·49	1·50	53·27	25·74	9·67	5·72	29	1·28
" " ¾	9·938	33·79	2·54	1·55	62·60	30·14	11·47	6·77	28½	1·28
" " 1	13·001	44·20	2·64	1·64	79·97	38·21	14·91	8·77	28½	1·27
8 × 4 × 7/16	5·058	17·20	2·80	·82	33·57	5·78	6·46	1·82	15	·86
" " ⅝	7·109	24·17	2·88	·90	46·37	7·87	9·06	2·54	14½	·85
" " ¾	8·437	28·69	2·93	·94	54·36	9·14	10·73	2·99	14½	·84
7 × 3½ × ⅜	3·797	12·91	2·44	·71	19·28	3·31	4·23	1·19	15	·75
" " ½	5·000	17·00	2·50	·76	25·07	4·27	5·57	1·56	14½	·74
" " ⅝	6·172	20·98	2·55	·81	30·53	5·14	6·86	1·91	14½	·74
" " ¾	7·313	24·86	2·60	·86	35·66	5·94	8·11	2·25	14	·73
6 × 4 × ⅜	3·611	12·28	1·91	·92	13·21	4·74	3·23	1·54	23½	·87
" " ½	4·752	16·16	1·97	·97	17·14	6·11	4·25	2·02	23½	·86
" " ⅝	5·861	19·93	2·02	1·02	20·82	7·37	5·23	2·48	23½	·86
" " ¾	6·939	23·59	2·06	1·07	24·26	8·53	6·16	2·91	23	·85
6 × 3½ × ⅜	3·422	11·63	2·01	·77	12·62	3·21	3·16	1·18	19	·76
" " ½	4·500	15·30	2·06	·82	16·36	4·13	4·15	1·54	19	·75
" " ⅝	5·547	18·86	2·11	·87	19·85	4·96	5·11	1·89	18½	·75
" " ¾	6·562	22·31	2·16	·92	23·11	5·73	6·02	2·22	18½	·74
6 × 3 × 5/16	2·716	9·23	2·09	·61	10·13	1·74	2·59	·73	15	·64
" " ⅜	3·236	11·00	2·12	·63	11·99	2·05	3·09	·87	14½	·64
" " ½	4·251	14·45	2·17	·68	15·51	2·62	4·05	1·13	14½	·63
" " ⅝	5·236	17·80	2·22	·73	18·80	3·14	4·98	1·38	14	·63
5 × 4 × 5/16	2·716	9·23	1·48	·99	6·74	3·84	1·91	1·27	32	·85
" " ⅜	3·236	11·00	1·51	1·01	7·97	4·53	2·28	1·52	32	·85
" " ½	4·251	14·45	1·56	1·06	10·29	5·83	2·99	1·98	32	·84
" " ⅝	5·236	17·80	1·61	1·11	12·44	7·02	3·67	2·43	31½	·84

PROPERTIES OF NEW BRITISH STANDARD UNEQUAL ANGLES (*continued*).

Size and Thickness	Area	Weight per foot lbs.	Dimensions		Moments of Inertia		Section Moduli		Angle a in degrees	Least Radius of Gyration
			J	P	X X	Y Y	X X	Y Y		
5 × 3½ × $\frac{5}{16}$	2·561	8·71	1·56	·82	6·46	2·62	1·88	·98	26	·76
„ „ $\frac{3}{8}$	3·049	10·37	1·59	·85	7·63	3·09	2·24	1·16	25½	·75
„ „ $\frac{1}{2}$	4·002	13·61	1·64	·90	9·84	3·95	2·93	1·52	25½	·75
„ „ $\frac{5}{8}$	4·924	16·74	1·69	·94	11·89	4·74	3·59	1·86	25	·74
5 × 3 × $\frac{5}{16}$	2·403	8·17	1·66	·67	6·14	1·68	1·83	·72	20	·65
„ „ $\frac{3}{8}$	2·860	9·72	1·68	·69	7·25	1·97	2·18	·85	19½	·65
„ „ $\frac{1}{2}$	3·751	12·75	1·73	·74	9·33	2·51	2·86	1·11	19½	·64
„ „ $\frac{5}{8}$	4·610	15·67	1·78	·79	11·25	3·00	3·50	1·36	19	·64
4 × 3½ × $\frac{3}{8}$	2·673	9·09	1·19	·94	4·09	2·91	1·45	1·14	37	·72
„ „ $\frac{1}{2}$	3·502	11·91	1·24	·99	5·24	3·72	1·90	1·48	37	·71
„ „ $\frac{5}{8}$	4·298	14·61	1·29	1·04	6·28	4·45	2·31	1·[illegible]1	36½	·71
4 × 3 × $\frac{5}{16}$	2·091	7·11	1·24	·75	3·30	1·59	1·20	·71	28½	·64
„ „ $\frac{3}{8}$	2·485	8·45	1·27	·77	3·89	1·87	1·42	·84	28½	·64
„ „ $\frac{1}{2}$	3·251	11·05	1·32	·82	4·97	2·37	1·85	1·09	28½	·63
„ „ $\frac{5}{8}$	3·985	13·55	1·36	·87	5·96	2·83	2·26	1·32	28	·63
3½ × 3 × $\frac{1}{4}$	1·564	5·32	1·01	·77	1·86	1·26	·75	·56	36	·62
„ „ $\frac{5}{16}$	1·935	6·58	1·04	·79	2·27	1·54	·92	·70	35½	·62
„ „ $\frac{3}{8}$	2·298	7·81	1·07	·82	2·67	1·80	1·10	·83	35½	·62
„ „ $\frac{1}{2}$	3·001	10·20	1·12	·87	3·40	2·28	1·43	1·07	35½	·61
„ „ $\frac{5}{8}$	3·673	12·49	1·16	·91	4·06	2·72	1·73	1·30	35	·61
3½ × 2½ × $\frac{1}{4}$	1·438	4·89	1·09	·60	1·75	·745	·73	·39	26½	·54
„ „ $\frac{5}{16}$	1·778	6·05	1·12	·63	2·14	·907	·90	·48	26½	·53
„ „ $\frac{3}{8}$	2·110	7·17	1·15	·65	2·51	1·06	1·07	·57	26	·53
„ „ $\frac{1}{2}$	2·750	9·35	1·19	·70	3·19	1·33	1·39	·74	26	·53
3 × 2½ × $\frac{1}{4}$	1·313	4·46	·89	·65	1·14	·716	·54	·39	34	·52
„ „ $\frac{5}{16}$	1·622	5·51	·92	·67	1·39	·871	·67	·48	34	·52
„ „ $\frac{3}{8}$	1·923	6·54	·94	·70	1·62	1·02	·79	·56	34	·52
„ „ $\frac{1}{2}$	2·501	8·50	·99	·74	2·05	1·28	1·02	·73	33½	·52
3 × 2 × $\frac{1}{4}$	1·188	4·04	·98	·48	1·06	·375	·52	·25	23½	·43
„ „ $\frac{5}{16}$	1·465	4·98	1·00	·51	1·29	·454	·65	·30	23	·43
„ „ $\frac{3}{8}$	1·735	5·90	1·03	·53	1·50	·527	·76	·36	23	·42
„ „ $\frac{1}{2}$	2·250	7·65	1·08	·58	1·90	·657	·98	·46	22½	·42
2½ × 2 × $\frac{1}{4}$	1·063	3·61	·77	·53	·635	·358	·37	·24	32	·42

PROPERTIES OF NEW BRITISH STANDARD UNEQUAL ANGLES (*continued*).

Size and Thickness	Area	Weight per foot lbs.	Dimensions		Moments of Inertia		Section Moduli		Angle α in degrees	Least Radius of Gyration
			J	P	X X	Y Y	X X	Y Y		
2½ × 2 × 5/16	1·309	4·45	·80	·55	·770	·433	·45	·30	31½	·42
,, ,, 3/8	1·547	5·26	·82	·57	·894	·501	·53	·35	31½	·41
2 × 1½ × 3/16	·715	2·43	·83	·34	·446	·120	·27	·10	19½	·32
,, ,, 5/16	1·152	3·92	·89	·39	·697	1·84	·43	·17	19	·32
,, ,, 3/8	1·359	4·62	·91	·42	·808	·212	·51	·20	19	·31
2 × 1½ × 3/16	·621	2·11	·63	·38	·239	·114	·17	·10	28½	·32
,, ,, ¼	·813	2·76	·65	·41	·307	·145	·23	·13	28	·31
,, ,, 5/16	·996	3·39	·68	·43	·368	·174	·28	·16	28	·31
,, ,, 3/8	1·172	3·98	·70	·45	·425	·200	·33	·19	28	·31

Properties on New British Standard Tees.

Sizes	Area	Weight per foot lbs.	J	Moments of Inertia		Section Moduli		Radii of Gyration	
				X X	Y Y	X X	Y Y	X X	Y Y
$6 \times 6 \times \frac{1}{2}$	5·772	19·62	1·630	19·043	8·556	4·357	2·852	1·816	1·218
„ „ $\frac{5}{8}$	7·126	24·23	1·687	23·309	10·866	5·404	3·622	1·809	1·235
$6 \times 4 \times \frac{1}{2}$	4·772	16·22	·968	6·070	8·642	2·002	2·881	1·128	1·346
„ „ $\frac{5}{8}$	5·878	19·99	1·018	7·331	10·932	2·459	3·644	1·117	1·364
$5 \times 4 \times \frac{1}{2}$	4·266	14·50	1·052	5·773	5·024	1·958	2·009	1·163	1·085
„ „ $\frac{5}{8}$	5·248	17·84	1·102	6·963	6·363	2·403	2·545	1·152	1·101
$5 \times 3 \times \frac{3}{8}$	2·878	9·79	·691	1·973	3·716	·855	1·487	·828	1·136
„ „ $\frac{1}{2}$	3·766	12·80	·741	2·515	5·037	1·113	2·015	·817	1·157
$4 \times 3 \times \frac{3}{8}$	2·498	8·49	·767	1·860	1·911	·833	·956	·863	·875
„ „ $\frac{1}{2}$	3·262	11·09	·816	2·366	2·595	1·083	1·298	·852	·892
$3 \times 3 \times \frac{5}{16}$	1·786	6·07	·842	1·455	·666	·674	·444	·903	·611
„ „ $\frac{3}{8}$	2·119	7·20	·869	1·708	·813	·801	·542	·898	·619
„ „ $\frac{7}{16}$	2·440	8·30	·894	1·942	·959	·922	·640	·892	·627
$2\frac{1}{2} \times 2\frac{1}{2} \times \frac{1}{4}$	1·197	4·07	·697	·678	·305	·376	·244	·752	·504
„ „ $\frac{5}{16}$	1·471	5·00	·724	·822	·388	·463	·311	·748	·514
„ „ $\frac{3}{8}$	1·742	5·92	·749	·959	·475	·548	·380	·742	·522
$2 \times 2 \times \frac{1}{4}$	·945	3·21	·579	·337	·157	·237	·157	·597	·408
„ „ $\frac{5}{16}$	1·158	3·94	·604	·406	·201	·291	·201	·592	·417
„ „ $\frac{3}{8}$	1·366	4·64	·628	·469	·246	·342	·246	·586	·425
$1\frac{1}{2} \times 1\frac{1}{2} \times \frac{3}{16}$	·533	1·81	·434	·106	·048	·100	·065	·447	·302
„ „ $\frac{1}{4}$	·693	2·36	·459	·135	·067	·130	·090	·442	·312

Angle		Chord.	Sine.	Tangent.	Co-tangent.	Cosine.			
Degrees	Radians.								
0°	0	0	0	0	∞	1	1·414	1·5708	90°
1	·0175	·017	·0175	·0175	57·2900	·9998	1·402	1·5533	89
2	·0349	·035	·0349	·0349	28·6363	·9994	1·389	1·5359	88
3	·0524	·052	·0523	·0524	19·0811	·9986	1·377	1·5184	87
4	·0698	·070	·0698	·0699	14·3007	·9976	1·364	1·5010	86
5	·0873	·087	·0872	·0875	11·4301	·9962	1·351	1·4835	85
6	·1047	·105	·1045	·1051	9·5144	·9945	1·338	1·4661	84
7	·1222	·122	·1219	·1228	8·1443	·9925	1·325	1·4486	83
8	·1396	·140	·1392	·1405	7·1154	·9903	1·312	1·4312	82
9	·1571	·157	·1564	·1584	6·3138	·9877	1·299	1·4137	81
10	·1745	·174	·1736	·1763	5·6713	·9848	1·286	1·3963	80
11	·1920	·192	·1908	·1944	5·1446	·9816	1·272	1·3788	79
12	·2094	·209	·2079	·2126	4·7046	·9781	1·259	1·3614	78
13	·2269	·226	·2250	·2309	4·3315	·9744	1·245	1·3439	77
14	·2443	·244	·2419	·2493	4·0108	·9703	1·231	1·3265	76
15	·2618	·261	·2588	·2679	3·7321	·9659	1·218	1·3090	75
16	·2793	·278	·2756	·2867	3·4874	·9613	1·204	1·2915	74
17	·2967	·296	·2924	·3057	3·2709	·9563	1·190	1·2741	73
18	·3142	·313	·3090	·3249	3·0777	·9511	1·176	1·2566	72
19	·3316	·330	·3256	·3443	2·9042	·9455	1·161	1·2392	71
20	·3491	·347	·3420	·3640	2·7475	·9397	1·147	1·2217	70
21	·3665	·364	·3584	·3839	2·6051	·9336	1·133	1·2043	69
22	·3840	·382	·3746	·4040	2·4751	·9272	1·118	1·1868	68
23	·4014	·399	·3907	·4245	2·3559	·9205	1·104	1·1694	67
24	·4189	·416	·4067	·4452	2·2460	·9135	1·089	1·1519	66
25	·4363	·433	·4226	·4663	2·1445	·9063	1·075	1·1345	65
26	·4538	·450	·4384	·4877	2·0503	·8988	1·060	1·1170	64
27	·4712	·467	·4540	·5095	1·9626	·8910	1·045	1·0996	63
28	·4887	·484	·4695	·5317	1·8807	·8829	1·030	1·0821	62
29	·5061	·501	·4848	·5543	1·8040	·8746	1·015	1·0647	61
30	·5236	·518	·5000	·5774	1·7321	·8660	1·000	1·0472	60
31	·5411	·534	·5150	·6009	1·6643	·8572	·985	1·0297	59
32	·5585	·551	·5299	·6249	1·6003	·8480	·970	1·0123	58
33	·5760	·568	·5446	·6494	1·5399	·8387	·954	·9948	57
34	·5934	·585	·5592	·6745	1·4826	·8290	·939	·9774	56
35	·6109	·601	·5736	·7002	1·4281	·8192	·923	·9599	55
36	·6283	·618	·5878	·7265	1·3764	·8090	·908	·9425	54
37	·6458	·635	·6018	·7536	1·3270	·7986	·892	·9250	53
38	·6632	·651	·6157	·7813	1·2799	·7880	·877	·9076	52
39	·6807	·668	·6293	·8098	1·2349	·7771	·861	·8901	51
40	·6981	·684	·6428	·8391	1·1918	·7660	·845	·8727	50
41	·7156	·700	·6561	·8693	1·1504	·7547	·829	·8552	49
42	·7330	·717	·6691	·9004	1·1106	·7431	·813	·8378	48
43	·7505	·733	·6820	·9325	1·0724	·7314	·797	·8203	47
44	·7679	·749	·6947	·9657	1·0355	·7193	·781	·8029	46
45°	·7854	·765	·7071	1·0000	1·0000	·7071	·765	·7854	45°
			Cosine	Co-tangent	Tangent	Sine	Chord	Radians	Degrees
								Angle	

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 4	9 8	13 12	17 16	21 20	26 24	30 28	34 32	38 37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 4	8 7	12 11	15 15	19 19	23 22	27 26	31 30	35 33
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 3	7 7	11 10	14 14	18 17	21 20	25 24	28 27	32 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 3	7 7	10 10	13 12	16 16	20 19	23 22	26 25	30 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 3	6 6	9 9	12 12	15 15	18 17	21 20	24 23	28 26
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 3	6 5	9 8	11 11	14 14	17 16	20 19	23 22	26 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 3	5 5	8 8	11 10	14 13	16 15	19 18	22 21	24 23
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	3 2	5 5	8 7	10 10	13 12	15 15	18 17	20 19	23 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 2	5 5	7 7	9 9	12 11	14 14	16 16	19 18	21 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 2	4 4	7 6	9 8	11 11	13 13	16 15	18 17	20 19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

ANTILOGARITHMS

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·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
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·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
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·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
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·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

INDEX

ABRUPT change of section, 56, 90
Adams's experiments on marble, 42
Adhesion between concrete and steel, 80
Alignment charts for springs, 342
Aluminium, 62
Andrews-Pearson formula for curved beams, 545
Areas—
  mathematical determination, 161
  Parmontier's rule, 164
  Simpson's rule, 164
  sum curves, 162
  table of various sections, 185
Arnold's testing machine, 399
Autographic recorders, 379
Avery's reverse torsion machine, 385

Bach on plates and slabs, 488–509
Bairstow, 59, 88
Baker, B.—
  abrupt change of section experiments, 56
  repetitions of stress, 86
Basquin, 307
Bauschinger on strength of stone, 72
Beam factor, 197
Beams. *See* Bending moments; deflections; inclined beams; shear; stresses, bending.
Bending moments—
  cantilevers, 122–127
  continuous beams, 430–458
  fixed beams, 414–438, 458
  simply supported beams, 127–160
Bernoulli's assumption, 194
Bolts, coupling, 311
Brinell hardness test, 402
Buckton testing machines, 364, 368, 381
Built-in beams, 416–438, 458
Bouton experiments on compression, 47
Bracing of columns, 294
Brass, 62, 83
Brittleness, 41
Bronze, 62, 83
Buchanan on columns, 287
Buckling factor, 279
Bulk modulus, 7

Cantilevers, 122–127
Cast iron, 49, 209, 305
Centroid, 165
Chain links, 546–551
Cleat, stresses in rivets, 315
Coker—
  thermal and optical testing, 409, 411
  torsion testing, 388
Collapsing pressure for pipes, 118
Columns—
  centrally loaded, 279–301
  eccentrically loaded, 301–310
Compressive strength, *see* various materials.
Concrete—
  compressive strength, 69–77, 82
  reinforced, 215–235, 292, 479
  shear strength, 78
  tensile strength, 77
Considère, 42
Continuous beams, 438–458
Critical speed of shafts, 562–569
Curvature of beams, 250
Curved beams—
  Andrews-Pearson formula, 545
  correction coefficients, 543
  general conditions of strain, 529
  Résal's construction, 538

Curved beams (*continued*)—
  rings and chain links, 546
  Winkler's formula, 533
Cylinders, *see* Pipes.

Darwin's extensometer, 376
Deflections—
  beams, simply supported and cantilever, 248–278
  fixed beams, 419, 420, 460
  shear, due to, 483
Diagonal square beam section, 210
Disks, rotating, 552–562
Distribution of shear stress in beams, 470–487
Dixon and Hummel testing machine, 368
Drums, rotating, 552
Ductility, 41
Dunkerley on whirling of shafts, 568
Dynamic stress and strain, 33
Dyson on internal friction, 47

Eden on stress repetition, 89
Elastic bodies, 1
Elastic moduli, 7
  relation between, 8
Ellipse—
  of inertia, 169
  of stress, 15
Elongation and gauge length, 54
Encastré beams, 416–438, 458
Euler's column formula, 280, 306
Ewing's extensometer, 374

Failure, cause of, 42
Fairbairn, 84, 119
Fatigue of metals, 89
Fidler column formula, 289
Filon on optical testing, 415
Fitchett, F., on springs, 343
Fixed beams, 416–438, 458
Flitched beams, 203
Foster on stress repetition, 89

Gadd on cement tests, 408
Glass, 83
Goodenough on chain links, 551
Goodman extensometer, 371
Gordon column formula, 288
Grashof on plates and slabs, 494
Greenwood and Batley testing machine, 367
Grips for test-pieces, 370
Guest theory of stress, 44, 328

Hardness, 41, 402
Hartnell, W., on springs, 341
Heterogeneous sections—
  direct stress, 38
  geometrical properties, 183
Hooke's Law, 2
Horse-power of shafting, 323, 335
Hysteresis, mechanical, 59, 89

Illinois experiments, 119, 297, 551
Impact, strain and stress due to, 35
Impact testing, 400
Inclined beams, 155–160
Inertia, moment of, 169–191
Internal friction in materials, 45
Izod—
  experiments on shear, 64, 67
  impact testing machine, 401

Johnson—
  column formula, 288
  eccentric loading, 309

Kennedy's extensometer, 372
Keyways in shafting, 334

Lamé's theory for thick pipes, 516
Lilly—
  column formula, 290
  torsion testing machine, 386
Live loads, 100
Lüder's Lines, 54

Macklow-Smith torsion meter, 392
Malleability, 41
Malleable cast iron, 51
Modulus—
  elastic, 7
  section, 197
Mohr, 181, 252
Moment of inertia, 167–191
Moment of resistance, 196, 213, 222
Momental ellipse, 167
Moore, 297, 334, 551
Morley, 3
  on columns, 310
  on curved beams, 545
  on slabs, 494
Muir on overstrain, 59

Navier internal friction theory, 45
Non-circular shafts, 333

Oblique loading on beams, 241
Overstrain, 57

Permanent set, 1
Perry, 70, 192
Pipes and cylinders—
  collapse of, 118
  initial pressure in, 524
  Lamé's theory, 510
  shear stresses in, 516
  thick, 510–538
  thin, 115
Piston rings, 355
Plastic bodies, 1
Plates—
  Bach theory, 488–493, 499–509
  circular, 488
  Grashof and Rankine, 494
  oval, 492
  square and rectangular, 494
Poisson's ratio, 3
Polar moment of inertia, 173
Portland cement, strength of, 68–79, 404
Principal stresses, 13

Quality factor, 62

Radius of gyration, 169–191
Rankine—
  column formula, 285
  combined stress theory, 43, 320
  slab formula, 494
  stress lines, 213
Reinforced concrete—
  beams, 215–235
  columns, 292
  shear in beams, 479
Repetition of stress, 84
Résal's construction, 538
Resilience—
  definition, 33
  in bending, 272
  in torsion, 331
  summary for various springs, 363
Rigidity modulus, 7
Rings, 546
Riveted joints, 102–113
Rotating drums and disks, 550–562
Rotating shafts, 562–569

St. Venant, 44, 328, 333
Sankey bending machine, 396
Scoble on optical testing, 404
Secondroid, 168
Section modulus, 197
Shaft coupling, stresses in, 311
Shafting, stresses in, 316–335
Shear—
  diagrams for cantilevers, 122–127
  diagrams for simply supported beams, 127–160
  diagrams, steps in, 144
  strain and stress, 3
  stress and strain equivalent to complex stresses, 19, 30
  stress in beams, 470
Shrinkage stresses in pipes, 525
Slabs, *see* Plates.
Slate, 83
Slenderness ratio, 279
Smith, C. A. M., 48, 517
Smith, J. H., repetition testing machine, 392
Smith, R. H., construction for combined stresses, 23
Springs—
  closed-coiled helical, 337, 362
  leaf or plate, 351
  open-coiled helical, 346
  piston rings, 355
  plane spiral, 360
  summary, 363
  time of vibration, 337
Steel, strength of, 52–63
Stewart on collapse of pipes, 119
Stone, compressive strength, 68, 82
Straight line column formula, 287
Strain—
  definition and kinds, 1–3
  in different directions, 12
  maximum strain equivalent to combined strains, 25
  transverse, 3
Stress—
  bending, 192–247
  cause of failure under, 42

Stress (*continued*)—
  combined bending and direct, 235–240, 328
  complex, 13
  definitions and kinds, 1–3
  dynamic, 33
  ellipse of, 15
  impact, 35
  principal, 13, 19
  repetition of, 84
  shear, 3, 212
  shear in beams, 470
  temperature, 37
  working, 93–100
Stress-strain diagrams, 5, 49, 53, 60, 62, 65
Struts, *see* Columns.
Sum curve, 162
Superposition, principle of, 244

T beams, reinforced concrete, 229
Temperature stresses, 37
  effect on strength, 63
Tensile strength, real and apparent, 52
  of materials, *see* various materials.
Testing—
  calibration of machines, 369
  cement and concrete, 404
  extensometers, 371–380
  grips and forms of test-pieces, 370
  impact, hardness and ductility, 396
  machines (general), 364–369
Testing (*continued*)—
  optical, 411
  thermal, 409
  torsion machines, 382–393
Theorem of three moments, 445
Thick pipes, *see* Pipes.
Thurston testing machine, 383–386
Timber, 65, 82
Torsion, 311–335
Turbine shaft, settling down, 564
Twisting, 311–335

Unit section modulus, 197
Unital strain, 3
Unwin—
  elongation formula, 55
  extensometer, 379
  on piston rings, 359
  on stone cubes, 70

Waist in tensile fracture, 54, 56
Werder testing machine, 366
Whirling of shafts, 562–569
Wicksteed-Buckton testing-machines, 364, 368, 381
Winkler's formula for curved beams, 533
Wöhler's experiments, 84
Woolson experiments on concrete, 43
Working stresses, 93–99
Wrought iron, 82

Yield point, 4
Young's Modulus, 7

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